Problem 1. Fred is giving out samples of canned dog food. He makes calls door to door, but he leaves a sample (one can) only on those calls for which the door is answered and a dog is in residence. On any call the probability of the door being answered is $\frac{3}{4}$, and the probability that any household has a dog is $\frac{2}{3}$. Assume that the events “Door answered” and “A dog lives here” are independent and also that the outcomes of all calls are independent.

(a) Determine the probability that Fred gives away his first sample on his third call.

**Solution.** This is a Bernoulli process: each call is a Bernoulli trial, and the calls are independent. The probability of success $p$ is:

$$p = P\{ \text{the door is answered} \} \cap \{ \text{a dog is in residence} \}$$

$$= P\{ \text{the door is answered} \} \cdot P\{ \text{a dog is in residence} \}$$

$$= \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}.$$  

$Y_1$, the time of the first success is a geometric random variable with parameter $p$, therefore we have:

$$P(Y_1 = 3) = pY_1(3) = (1 - 1/2)^3 - 1(1/2) = 1/8.$$

(b) Given that he has given away exactly four samples on his first eight calls, determine the conditional probability that Fred will give away his fifth sample on his eleventh call.

**Solution.** Due to the fresh-start property of the Bernoulli process, what will happen from the ninth call and thereafter will also be a Bernoulli process. So the answer to this problem
is equivalent to the probability that he will give out the first can on his third trial in the new process, which is exactly the same as Part (a), i.e. 1/8.

(c) Determine the probability that he gives away his second sample on his fifth call.

Solution. Recall that $Y_k$, the arrival time of the $k$-th success in a Bernoulli process is a random variable with the following PMF:

$$p_{Y_k}(t) = \begin{cases} \binom{t-1}{k-1} p^k (1-p)^{t-k}, & \text{if } t \geq k; \\ 0 & \text{if } t < k. \end{cases}$$

Here we have $k = 2$, $t = 5$, and hence the answer is:

$$P\left(\{Y_2 = 5\}\right) = p_{Y_2}(5) = \binom{4}{1} \left(\frac{1}{2}\right)^2 \left(1 - \frac{1}{2}\right)^3 = 4 \left(\frac{1}{2}\right)^5 = \frac{1}{8}.$$

(d) Given that he did not give away his second sample on his second call, determine the conditional probability that he will give away his second sample on his fifth call.

Solution. This question can be formulated to:

$$P(\{Y_2 = 5\} | \{Y_2 > 2\}) = \frac{P(\{Y_2 = 5\} \cap \{Y_2 > 2\})}{P(\{Y_2 > 2\})} = \frac{P(\{Y_2 = 5\})}{P(\{Y_2 > 2\})}.$$ 

Using the normalization axiom and the fact that $Y_2$ cannot be less than 2, we have:

$$P(\{Y_2 > 2\}) = 1 - P(\{Y_2 \leq 2\}) = 1 - P(\{Y_2 = 2\}).$$

Apply the PMF of $Y_2$ to get:

$$P(\{Y_2 = 5\} | \{Y_2 > 2\}) = \frac{1/8}{1 - \left(\frac{1}{1}\right) \left(\frac{1}{2}\right)^2} = \frac{1}{6}.$$

(e) We shall say that Fred “needs a new supply” immediately after the call on which he gives away his last can. If he starts out with two cans, determine the probability that he completes at least five calls before he needs a new supply.

Solution. This problem is equivalent to determine the probability that the second success arrives on or after the fifth trial. It can be formulated as:

$$P(\{Y_2 \geq 5\}) = 1 - P(\{Y_2 = 2\}) - P(\{Y_2 = 3\}) - P(\{Y_2 = 4\})$$

$$= 1 - p_{Y_2}(2) - p_{Y_2}(3) - p_{Y_2}(4)$$

$$= 1 - \left(\frac{1}{1}\right) \left(\frac{1}{2}\right)^2 - \left(\frac{2}{1}\right) \left(\frac{1}{2}\right)^3 - \left(\frac{3}{1}\right) \left(\frac{1}{2}\right)^4$$

$$= 1 - \frac{1}{4} - \frac{2}{8} - \frac{3}{16} = \frac{5}{16}.$$
Problem 2. The PDF of the duration of the (independent) interarrival times between successive cars on the Trans-Australian Highway is given by

\[ f_T(t) = \begin{cases} \frac{1}{12} e^{-\frac{t}{12}}, & t \geq 0, \\ 0, & t < 0, \end{cases} \]

where these durations are measured in seconds.

(a) An old wombat (on his way to the wombat meeting of Homework 9) requires 12 seconds to cross the highway, and he starts out immediately after a car goes by. What is the probability that he will survive?

Solution. Notice that the arrival of cars is a Poisson process with arrival rate \( \lambda = 1/12 \) cars per second (since the interarrival times are i.i.d. exponential random variables with parameter 1/12). Therefore, the number \( N(t) \) of arrivals during a time interval of duration \( t \), is a Poisson random variable with parameter \( t/12 \). Note that, due to the fresh-start property, no matter when the wombat starts crossing the highway, what will happen afterwards is still a Poisson process with the same rate.

Therefore, this wombat will survive if and only if there is no cars during the 12 seconds it takes for him to cross the highway: With the given PDF, we have:

\[ P(\text{wombat survives}) = P(0 \text{ cars in 12 seconds}) = e^{-12 \cdot \frac{1}{12} \cdot (12 \cdot \frac{1}{12})^0} = e^{-1}. \]

Equivalently, he will survive if it takes more than 12 seconds for the first car to arrive:

\[ P(\text{wombat survives}) = P(T > 12) = \int_{12}^{\infty} f_T(t) dt = e^{-\frac{12}{12}} = e^{-1}. \]

(b) Another old wombat, slower but tougher, requires 24 seconds to cross the road, but it takes two cars to kill him. (A single car won’t even slow him down.) If he starts out at an arbitrary time, determine the probability that he survives.

Solution. This wombat will survive if and only if there are fewer than two cars during the 24 seconds it takes to cross. The probability of this is:

\[ P(N(24) \leq 1) = P(N(24) = 0) + P(N(24) = 1) = e^{-24 \cdot \frac{1}{12} \cdot (24 \cdot \frac{1}{12})^0} + e^{-24 \cdot \frac{1}{12} \cdot (24 \cdot \frac{1}{12})^1} = 3e^{-2}. \]

Equivalently, the wombat will survive if and only if \( Y_2 \), the time of the second arrival, is greater than 24 seconds:

\[ P(\{\text{wombat survives}\}) = P(\{Y_2 > 24\}) = \int_{24}^{\infty} f_{Y_2}(t) dt = \int_{24}^{\infty} \lambda^2 t e^{-\lambda t} dt = 3e^{-2}. \]
(c) If both these wombats start out at the same time, immediately after a car goes by, what is the probability that exactly one of them survives? (Hint. Consider random variables $N_1 = \text{the number of cars in the first 12 seconds and } N_2 = \text{the number of cars in the second 12 seconds.})

Solution. Observe the following:

$\{\text{exactly one survives}\} = \{\text{only the 1st survives}\} \cup \{\text{only the second survives}\}$

$= \{N_1 = 0 \text{ and } N_2 > 1\} \cup \{N_1 = 1 \text{ and } N_2 = 0\}$

Due to the memorylessness of Poisson process, $N_1$ and $N_2$ are independent, and they both Poisson random variables with parameter $\lambda \cdot 12 = 1$. Therefore we have:

$P(\{\text{exactly one survives}\}) = P(\{N_1 = 0 \text{ and } N_2 > 1\}) + P(\{N_1 = 1 \text{ and } N_2 = 0\})$

$= p_{N_1}(0)(1 - p_{N_2}(0) - p_{N_2}(1)) + p_{N_1}(1)p_{N_2}(0)$

$= e^{-1}(1 - e^{-1} - e^{-1}) + e^{-1}e^{-1} = e^{-1}(1 - e^{-1})$.

Problem 3. Eight light bulbs are turned on at $t = 0$. The lifetime of any particular bulb is independent of the lifetimes of all other bulbs and is described by the exponential PDF with parameter $\lambda$. Let $Y$ be the time until the third failure.

(a) Find $E[Y]$.

Solution. The failure of a bulb can be thought of as the first arrival of a Poisson process with parameter $\lambda$. Since the eight bulbs are independent, we have eight independent Poisson process running in parallel at the very beginning. The first burnout can be viewed as the first arrival of the merged process, which is also a Poisson process, with parameter $8\lambda$. So $T_1$, the time of the first burnout is an exponential random variable with parameter $8\lambda$. After that, due to the memorylessness of exponential distribution, the remaining lifetimes of the other 7 bulbs are still independent exponential random variables. Therefore we have 7 independent Poisson process running. The remaining time $T_2$ of the next burnout is an exponential random variable with parameter $7\lambda$. By the same argument, the time between the second and the third burnout is an exponential random variable with parameter $6\lambda$. Now we can represent $Y$ as:

$Y = T_1 + T_2 + T_3$.

Because linearity of expectation, we have:

$E[Y] = E[T_1] + E[T_2] + E[T_3] = \frac{1}{8\lambda} + \frac{1}{7\lambda} + \frac{1}{6\lambda}$.

(b) Find $\text{var}(Y)$.

Solution. Due to the memorylessness of Poisson process, $T_1$, $T_2$ and $T_3$ are independent. Therefore:

$\text{var}(Y) = \text{var}(T_1) + \text{var}(T_2) + \text{var}(T_3)$

$= \left(\frac{1}{8\lambda}\right)^2 + \left(\frac{1}{7\lambda}\right)^2 + \left(\frac{1}{6\lambda}\right)^2$

$= \frac{1}{64\lambda^2} + \frac{1}{49\lambda^2} + \frac{1}{36\lambda^2}$.
(c) Find the transform associated with $Y$.

**Solution.** Because of independence, we have:

$$M_Y(s) = M_{T_1}(s)M_{T_2}(s)M_{T_3}(s) = \frac{8\lambda}{8\lambda - s} \cdot \frac{7\lambda}{7\lambda - s} \cdot \frac{6\lambda}{6\lambda - s}.$$  

**Problem 4.** Arrivals of certain events at points in time are known to constitute a Poisson process, but it is not known which of two possible values of $\lambda$, the average arrival rate, describes the process. Our a priori guess is that $\lambda = 2$ or $\lambda = 4$ with equal probability.

We observe the process for $t$ units of time and observe exactly $k$ arrivals. Given this information, determine the conditional probability that $\lambda = 2$. Check to see whether your answer is reasonable for some simple limiting values for $k$ and $t$.

**Solution.** By formulating what we need to find and applying Bayes’ rule, we get:

$$P(\lambda = 2 | N_t = k) = \frac{P(N_t = k | \lambda = 2)P(\lambda = 2)}{P(N_t = k | \lambda = 2)P(\lambda = 2) + P(N_t = k | \lambda = 4)P(\lambda = 4)}$$  

(1)

Recall that the number of arrivals of a Poisson process of parameter $\lambda$ within a time of length $t$ is a Poisson random variable with parameter $\lambda t$. The PMF is:

$$P(N_t = k) = p_{N_t}(k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Plug this into Eq. (1) to get:

$$P(\lambda = 2 | N_t = k) = \frac{\frac{(2t)^k}{k!} e^{-2t}}{\frac{(2t)^k}{k!} e^{-2t} + \frac{(4t)^k}{k!} e^{-4t}}$$  

(2)

$$= \frac{(2t)^k}{(2t)^k + (4t)^k e^{-2t}}$$  

(3)

$$= \frac{1}{1 + 2^k e^{-2t}}$$  

(4)

Suppose $\alpha = k/t$, $\alpha$ can be thought of as the observed rate. Then Eq. (4) can be rewritten as:

$$P(\lambda = 2 | N_t = k) = \frac{1}{1 + 2^\alpha e^{-2t}} = \frac{1}{1 + \left(\frac{2^\alpha}{e^2}\right)^t}$$

When $\alpha$ is very close to 2, $2^\alpha \approx 2^2 < e^2$, so we have:

$$\lim_{t \to \infty} P(\lambda = 2 | N_t = k) = 1$$

When $\alpha$ is very close to 4, $2^\alpha \approx 2^4 > e^2$ and we have:

$$\lim_{t \to \infty} P(\lambda = 2 | N_t = k) = 0$$
These limiting cases agree with our intuition. So Eq. (4) seems to be a reasonable answer.

**Problem 5.** In the diagram below, each || represents a communication link. Under the present maintenance policy, link failures may be considered independent events, and one can assume that, at any time, the probability that any link is working properly is $p$.

(a) What is the probability that a total of exactly two links are operating properly at a particular time?

**Solution.** From the setup of the problem, $N$, the number of links working properly is a binomial random variable with parameters $(8, p)$. Where 8 is the total number of links. Therefore:

$$P(N = 2) = \binom{8}{2} p^2 (1-p)^6 = 28p^2(1-p)^6.$$ 

(b) What is the probability that link $g$ and exactly one other link are operating properly at a particular time?

**Solution.** Because of independence between the links, the solution can be represented as:

$$P(g \text{ and another link are working}) = P(g \text{ is working})P(1 \text{ out of other } 7 \text{ is working}) = p \binom{7}{1} p^1 (1-p)^6 = 7p^2(1-p)^6.$$ 

(c) Given that exactly six links are not operating properly at a particular time, what is the probability that $A$ can communicate with $B$?

**Solution.** There are $\binom{8}{2} = 28$ equally likely combinations for the two links that are working. Out of these possibilities, only $(c,h), (b,g)$ can form a channel between $A$ and $B$. So the answer is $2/28 = 1/14$. 


(d) Under a new maintenance policy, the system was put into operation in perfect condition at $t = 0$, and the PDF for the time until failure of any link is an exponential random variable with parameter $\lambda$. Link failures are independent, but no repairs are to be made until the third failure occurs. At the time of this third failure, the system is shut down, fully serviced, and then “restarted” in perfect order. The downtime for the service operation is a random variable with the following PDF:

$$f_X(x) = \begin{cases} \mu^2 x e^{-\mu x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

where $\mu$ is a positive parameter.

(i) What is the probability that link $g$ will fail before the first service operation?

**Solution.** Approach 1: Since all the links have the same distribution of failure time, each is equally likely to fail first, and therefore the probability that $g$ fails first is $1/8$. Similarly, the probability that $g$ fails second is $1/8$, and the probability that $g$ fails third is $1/8$. Therefore the probability that $g$ is among the first three to fail is $3/8$.

Approach 2: There are a total of $\binom{8}{3} = 56$ different combinations for the three links that fail before the service. Since all the failure times have the same distribution, the 56 possibilities are equally likely. There are $\binom{7}{2} = 21$ out of the 56 combinations contain $g$. Therefore the probability that $g$ is among the first 3 failures is:

$$\frac{21}{56} = \frac{3}{8}.$$

(ii) Determine the PDF for random variable $Y$, the time until first link failure after $t = 0$.

**Solution.** Just like Problem 3, the first failure can be viewed as the first arrival of a merged Poisson process with parameter $8\lambda$. Therefore the time of the first failure is an Exponential Random variable with parameter $8\lambda$ and we have:

$$f_{Y_1}(y) = \begin{cases} 8\lambda e^{-8\lambda y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

(iii) Determine the mean, variance, and the transform for $W$, the time from $t = 0$ until the end of the first service operation.

**Solution.** Suppose $Y_3$ is the time of the third failure. By the reasoning in the solution to Problem 3, we know that $Y_3$ can be represented as a sum of three independent exponential random variables with parameters $8\lambda$, $7\lambda$, $6\lambda$ respectively, i.e.:

$$Y_3 = T_1 + T_2 + T_3$$

$$E[Y_3] = E[T_1] + E[T_2] + E[T_3] = \frac{1}{8\lambda} + \frac{1}{7\lambda} + \frac{1}{6\lambda}$$

$$\text{var}(Y_3) = \text{var}(T_1) + \text{var}(T_2) + \text{var}(T_3)$$

$$= \frac{1}{64\lambda^2} + \frac{1}{49\lambda^2} + \frac{1}{36\lambda^2}$$

$$M_{Y_3}(s) = M_{T_1}(s)M_{T_2}(s)M_{T_3}(s)$$

$$= \frac{336\lambda^3}{(8\lambda - s)(7\lambda - s)(6\lambda - s)}$$

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Notice that the PDF of the service time \( X \) is a second order Erlang PDF with parameter \( \mu \). So \( X \) can be modeled as a sum of two independent exponential random variables \( X_1, X_2 \) with common parameter \( \mu \), i.e.

\[
X = X_1 + X_2
\]

\[
E[X] = E[X_1] + E[X_2] = \frac{1}{\mu} + \frac{1}{\mu} = \frac{2}{\mu}
\]

\[
\text{var}(X) = \text{var}(X_1) + \text{var}(X_2) = \frac{1}{\mu^2} + \frac{1}{\mu^2} = \frac{2}{\mu^2}
\]

\[
M_X(s) = M_{X_1}(s)M_{X_2}(s) = \left(\frac{\mu}{\mu - s}\right)^2
\]

The time of the end of the service is \( Y = X + Y_3 \). Since \( X \) and \( Y_3 \) are independent, we have:

\[
E[Y] = E[X] + E[Y_3] = \frac{2}{\mu} + \frac{1}{\lambda}\left(\frac{1}{8} + \frac{1}{7} + \frac{1}{6}\right)
\]

\[
\text{var}(Y) = \text{var}(X) + \text{var}(Y_3) = \frac{2}{\mu^2} + \frac{1}{\lambda^2}\left(\frac{1}{64} + \frac{1}{49} + \frac{1}{36}\right)
\]

\[
M_Y(s) = M_X(s)M_{Y_3}(s) = \left(\frac{\mu}{\mu - s}\right)^2 \frac{336\lambda^3}{(8\lambda - s)(7\lambda - s)(6\lambda - s)}
\]