1. **Binomial proofs**

Let $X$ denote a binomial random variable with parameters $(N, p)$.

(a) Show that $Y = N - X$ is a binomial random variable with parameters $(N, 1 - p)$

(b) What is $P\{X \text{ is even}\}$? Hint: Use the binomial theorem to write an expression for $(x + y)^n + (x - y)^n$ and then set $x = 1 - p, y = p$. If you are not familiar with the binomial theorem, it should be easy to look up.

**Solution:**

(a) We have that

$$P(X = k) = \binom{N}{k} p^k (1 - p)^{N-k}.$$

$Y = N - X \Rightarrow X = N - Y$, so that

$$P(Y = k) = P(X = N - k) = \binom{N}{N - k} p^{N-k}(1 - p)^{N-(N-k)}$$

$$= \binom{N}{N - k} q^k q^{N-k},$$

where $q \triangleq 1 - p$. We also have that

$$\binom{N}{N - k} \triangleq \frac{N!}{(N - (N - k))!(N - k)!} = \frac{N!}{k!(N - k)!} = \binom{N}{k}.$$
Therefore
\[ p(Y = k) = \binom{N}{k} q^k q^{N-k}, \]
so that find that \( Y \) has a binomial distribution with parameters \((N, q) = (N, 1 - p)\).

(b) We have
\[
P\{X \text{ is even}\} = \sum_{k \text{ even} \in [0, N]} P(X = k)
\]
\[
= \sum_{k \text{ even} \in [0, N]} \binom{N}{k} p^k (1 - p)^{N-k}.
\]

Using the hint, we first find using the Binomial theorem that
\[
(p + (1 - p))^N = \sum_{k=0}^{N} \binom{N}{k} p^k (1 - p)^{N-k}
\]
and \((-p + (1 - p))^N = \sum_{k=0}^{N} \binom{N}{k} (-p)^k (1 - p)^{N-k}.
\]

Adding these two expressions, we have
\[
(p + (1 - p))^N + (-p + (1 - p))^N = \sum_{k=0}^{N} \binom{N}{k} (1 - p)^{N-k}(p^k + (-p)^k)
\]
\[
= \sum_{k=0}^{N} \binom{N}{k} p^k (1 - p)^{N-k}(1 + (-1)^k).
\]

Since \(1 + (-1)^k\) is zero for \(k\) odd, and 2 for \(k\) even, we find that
\[
(p + (1 - p))^N + (-p + (1 - p))^N = \sum_{k \text{ even} \in [0, N]} 2 \binom{N}{k} p^k (1 - p)^{N-k}
\]
\[
= 2P\{X \text{ is even}\}.
\]

Reducing the left-hand side, and rewriting, we have
\[
P\{X \text{ is even}\} = \frac{1}{2} \left(1 + (1 - 2p)^N\right)
\]
2. **Locked doors**

An absent-minded professor has \( n \) keys in his pocket of which only one (he does not remember which one) fits his office door. He picks a key at random and tries it on his door. If that does not work, he picks a key again to try, and so on until the door unlocks. Let \( X \) denote the number of keys that he tries. Find \( E[X] \) in the following two cases.

(a) A key that does not work is put back in his pocket so that when he picks another key, all \( n \) keys are equally likely to be picked (sampling with replacement).

(b) A key that does not work is put in his briefcase so that when he picks another key, he picks at random from those remaining in his pocket (sampling without replacement).

**Solution:**

(a) We have that \( P(X = k) = P(\text{first } k-1 \text{ guesses are wrong})P(k-\text{th guess is correct}) \), so that

\[
P(X = k) = \left( \frac{n-1}{n} \right)^{k-1} \left( \frac{1}{n} \right),
\]

for \( k = 1, 2, 3, \ldots \). We then have that

\[
E[X] = \sum_{k=1}^{\infty} k P(X = k) = \sum_{k=1}^{\infty} k \left( \frac{n-1}{n} \right)^{k-1} \left( \frac{1}{n} \right).
\]

Recall that

\[
\sum_{k=1}^{K} k \alpha^{k-1} = \frac{K \alpha^{K+1} - (K + 1) \alpha^K + 1}{(1 - \alpha)^2}.
\]

\[
\Rightarrow E[X] = \lim_{K \to \infty} \left( \frac{1}{n} \right) \frac{K \alpha^{K+1} - (K + 1) \alpha^K + 1}{(1 - \alpha)^2}
\]

(\text{where } \alpha = \frac{n-1}{n} < 1)

\[
= \left( \frac{1}{n} \right) \frac{1}{(1 - \frac{n-1}{n})^2} = n.
\]

Thus, \( E[X] = n \).
(b) We again have \( P(X = k) = P(\text{first } k-1 \text{ guesses are wrong})P(\text{k-th guess is correct}) \), for \( k = 1, 2, \ldots, n \). We now have that this expression is

\[
P(X = k) = \frac{(n-1)!/(n-k)!}{n!/(n-k)!} = \frac{1}{n}.
\]

We then have that

\[
E[X] = \sum_{k=1}^{n} k P(X = k) = \sum_{k=1}^{n} \frac{k}{n} = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{n^2 + n}{2} \right) = \frac{n+1}{2}.
\]

We now have solid proof that it would be better to not try the same wrong keys over and over!

3. It pays to study

There are \( n \) multiple-choice questions in an exam, each with 5 choices. The student knows the correct answer to \( k \) of them, and for the remaining \( n - k \) guesses one of the 5 randomly. Let \( C \) be the number of correct answers, and \( W \) be the number of wrong answers.

(a) What is the distribution of \( W \)? Is \( W \) one of the common random variables we have seen in class?

(b) What is the distribution of \( C \)? What is its mean, \( E[C] \)?

Solution:

(a) Assuming the student does not intentionally answer any of the questions he knows incorrectly, we have that \( W \) indicates the number of “successes” of a Benoulli random variable, so that

\[
P(W = l) = \binom{n-k}{l} \left( \frac{4}{5} \right)^l \left( \frac{1}{5} \right)^{n-k-l}.
\]
We have, then, that $W$ is distributed as a Binomial random variable with parameters $(n - k, 4/5)$.

(b) We have that $C = n - W \Rightarrow W = n - C$. It follows, then, that

$$P(C = l) = P(W = n - l) = \binom{n - k}{n - l} \left(\frac{4}{5}\right)^{n-l} \left(\frac{1}{5}\right)^{n-k-(n-l)} = \binom{n - k}{n - l} \left(\frac{4}{5}\right)^{n-l} \left(\frac{1}{5}\right)^{l-k}.$$ 

Now, $E[C] = E[n - W] = n - E[W]$. Since $W$ is a binomial random variable, we have that $E[W] = (n-k)\frac{4}{5}$. Then $E[C] = n - \frac{4}{5}(n - k) = \frac{1}{5}(n + 4k)$.

4. No Deal

In "Deal or No Deal" (the most ridiculous game show on TV), there are 5 suitcases. The suitcases contain $1, $10, $100, $1,000 and $10,000 respectively. There is a "banker" who offers the contestant a dollar amount that he can take and go home, right then and there. If the contestant does not use the banker’s offer, he can choose one of the suitcases and "eliminate" it by removing it from play. Then he plays the next round with the remaining suitcases.

(a) The banker wants to offer an amount equal to the average of what will REMAIN, after the choice is made. (for example, if 1000 is chosen, then the average of what will remain is $(1 + 10 + 100 + 10000)/4$.) Of course, the banker has to make an offer before the choice is made. What amount should the banker offer?

(b) The contestant has nerves of steel, and never takes up the banker’s offer in any round. He thus goes home with one of the 5 suitcases. However, he has to pay a 30% tax on the amount he takes home. How much will he be left with on average, after taxes?

Solution: Let $V$ be a random variable indicating the value of a random suitcase. We have that $V \in \{10^0, 10^1, \ldots, 10^4\}$, so that

$$E[V] = \sum_{k=0}^{4} 10^k P[V = 10^k] = \sum_{k=0}^{4} \frac{1}{5} 10^k = \frac{1}{5} 10^5 - 1 = \frac{10^5 - 1}{5} = $2,222.20$$
(a) The sum, $S_4$, of the four remaining suitcases is equal to the sum of all five suitcases less the value, $V$, of the chosen suitcase. i.e. $S_4 = \sum_{k=0}^{4} 10^k - V$. The banker should offer $E[S_4/4]$, where

$$E[S_4/4] = E \left[ \frac{1}{4} \left( \sum_{k=0}^{4} 10^k - V \right) \right] = \frac{1}{4} \sum_{k=0}^{4} 10^k - \frac{1}{4} E[V]$$

$$= \frac{1}{4} \left( \frac{10^5 - 1}{10 - 1} - 2,222.2 \right) = \$2,222.20$$

(b) After all the rounds are over, the contestant is left with a single suitcase of value $V$, so that after taxes the contestant is left with $0.7V$. The probability of the final suitcase being any one of the original five is equal to $1/5$. Thus, we have that the average take-home earnings of the contestant is $E[0.7V] = 0.7E[V] = (0.7)(2222.2) = \$1,555.54$. 