Timing Channel Capacity for Uniform and Gaussian Servers
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Abstract—It is well known that queues with exponentially distributed service time have the smallest Shannon capacity among all single-server queues with the same service rate. However, the capacity of other service disciplines such as Gaussian service timing channels (GSTC) remains unknown. In this paper, we outline a preliminary investigation into timing channel capacity, when the service distributions are uniform, Gaussian, and truncated Gaussian (TGSTC). In particular, we observe that the capacity per service time increases as the ratio of the standard deviation to the mean service time of the server decreases.

I. INTRODUCTION

A timing channel is a non-conventional communication channel, in which a message is encoded in terms of the arrival times of bits. The receiver observes the time intervals between the departing bits and decodes the message. It has been shown in [1] that the Exponential Service Timing Channel (ESTC) has a capacity of $e^{-1} \mu$ nats/sec, where $\mu$ is the service rate. Its capacity is the lowest among all the servers with the same service rate. Most of the existing work such as [1], [2], [3], [4], [5], [6] has focused on ESTC. The discrete-time counterpart has been studied in [7], [8].

We also know that the Deterministic Service Timing Channel (DSTC) has infinite capacity. A question arises naturally from the above two extremes: if the service time distribution is just a little "short" of being deterministic, how large is the timing capacity? We attempt to answer this question by investigating the capacities of three timing channels, in which the service time distributions are uniform (USTC), Gaussian (GSTC), and truncated Gaussian (TGSTC). In particular, we are interested in characterizing the capacities of these channels when $\sigma/T$ is small, where $\sigma$ is the standard deviation of the service time, and $T$ is the mean service time. We show the capacity of these timing channels increases when $\sigma/T$ decreases. The lower bound in each case tends to $\infty$ as $\sigma/T \to 0$, which agrees with the fact that the DSTC has infinite capacity.

II. BOUNDS ON TIMING CHANNEL CAPACITIES

We will present our results for the lower bounds (denoted as $C_L$) and upper bounds (denoted as $C_U$) on the timing channel capacities for the USTC, GSTC, and TGSTC in this section. The upper bounds $C_U$ in all cases are obtained by using Theorem 5 in [1].

We first consider the USTC in which the service times $S_1, S_2, \ldots, S_n$ are i.i.d. uniform random variables $U(T - \epsilon, T + \epsilon)$. The bounds are provided below:

$$C_L = \mu \max_{0 < p < 1} \left[ \frac{H(p)}{p + (\epsilon/T)(4 - 3p)} \right] \text{ nats/sec} \quad (1)$$

$$C_U = \mu (\ln(\frac{T}{2\epsilon}) + 1) \text{ nats/sec} \quad (2)$$

where $H(p) = -p \ln(p) - (1-p) \ln(1-p)$.

We obtain $C_L$ by choosing the inter-arrival times $A_1, A_2, \ldots$ to be i.i.d. geometric random variables, having discrete density function

$$P[A_i = T + \epsilon + 4k\epsilon] = p(1-p)^k, \quad k = 0, 1, \ldots$$

This encoding scheme ensures no queueing and also allows the receiver to decode a message error-free.

Next, we consider the GSTC in which the service times $S_1, \ldots, S_n$ are i.i.d. Gaussian random variables $N(T, \sigma^2)$ and $\sigma/T \leq 1/\sqrt{2\pi}e \approx 0.24$. The bounds are estimated to be:

$$C_L \approx \mu \max_{Y \geq 0} \left[ \frac{\ln \frac{T}{Y} + \ln Y}{1 + 3\sqrt{2Y}} \right] \text{ nats/sec} \quad (3)$$

$$C_U \approx \mu \ln(\sqrt{\frac{e}{2\pi \sigma}}) \text{ nats/sec} \quad (4)$$

We obtain $C_L$ for the GSTC by choosing the inter-arrival times $A_1, A_2, \ldots$ to be i.i.d. Gaussian $N(T_a, \sigma_a^2)$, and independent of the service times. We require

$$T_a = T + 3\sqrt{\sigma_a^2 + \sigma^2} \quad (5)$$

Condition (5) minimizes queueing effect. This is because $A_i - S_j$ is Gaussian $N(T_a - T; \sigma^2 + \sigma_a^2)$ and
Finally, we consider TGSTC in which the service times $S_1, S_2, \cdots$ are i.i.d. with density function:

$$f(x) = \frac{1}{K \sqrt{2\pi}\sigma} \exp\left(-\frac{(x-T)^2}{2\sigma^2}\right) I(T - 3\sigma, T + 3\sigma)$$

where $T > 3\sigma$ and $K = \int_{-\infty}^{\infty} f(x)dx = 0.997$

Its upper bound $C_U$ is estimated to be the same as that of GSTC given by (4). Its lower bound $C_L$ is estimated to be:

$$C_L \approx \mu \max_y \left\{ 0.5 \ln(1 + 0.5Y^2) \right\} \frac{1}{1 + 3\pi(Y + 1)} \text{ nats/sec} (6)$$

Similar to GSTC, we obtain $C_L$ by choosing the inter-arrival times $A_1, A_2, \cdots$ to be i.i.d. truncated Gaussian with density function:

$$f_a(x) = \frac{1}{K \sqrt{2\pi}\sigma_a} \exp\left(-\frac{(x-T_a)^2}{2\sigma_a^2}\right) I(T_a - 3\sigma_a, T_a + 3\sigma_a)$$

where $T_a > 3\sigma_a$ and $K = \int_{-\infty}^{\infty} f_a(x)dx = 0.997$

To ensure $P(\text{no queueing}) = 1$, we require

$$T_a = T + 3(\sigma_a + \sigma)$$

Figure 1 illustrates the capacity lower bounds as functions of the load factor $\lambda/\mu$ for the USTC, GSTC, and TGSTC; Table 1 shows the numerical values of $C_L$ for these timing channels. When $\sigma/T = 10^{-2}$, $C_L$ is more than twice the service rate.

We observe that in all three types of timing channels, both $C_L$ and $C_U$ increase when $\frac{1}{T}$ decreases, and $C_L \to \infty$ as $\frac{1}{T} \to 0$. Moreover, the capacity $C$ of the channels satisfies $C \sim \Theta(\ln(\frac{3}{Y}))$.

![Fig. 1. The Capacity Lower Bounds (in bits per average service time) vs the Load Factor $\lambda/\mu$](image)

### Table I

<table>
<thead>
<tr>
<th>$\frac{1}{T}$</th>
<th>GSTC</th>
<th>TGSTC</th>
<th>USTC</th>
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<td>$10^{-3}$</td>
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</tr>
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</table>

### III. Ongoing Research

We are finding tighter bounds for GSTC and TGSTC. We are studying the capacities of covert network timing channels and implementing efficient and good coding schemes.

### References


