Extension of compressible ideal-gas rapid distortion theory to general mean velocity gradients

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The homogeneity condition in compressible flows requires that mean velocity gradient and mean thermodynamic variables must be spatially invariant. This has restricted the use of rapid distortion theory (RDT) for compressible flows to a small set of mean-velocity gradients. By introducing an appropriate body force, we show that the homogeneity condition can be satisfied for a large class of compressible turbulence. We proceed to derive RDT spectral covariance equations of all relevant moments and recover the limiting behavior at vanishing and infinite (pressure-release) Mach numbers for homogeneous shear, plain-strain, axisymmetric expansion, and contraction cases.
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Rapid distortion theory (RDT)[1] is an important analytical tool for investigating incompressible turbulence physics in the limit of mean flow strain vastly exceeding that of the fluctuating field. In this linear limit, the statistical evolution of incompressible homogeneous turbulence can be described completely in terms of closed spectral covariance equations (see Refs. 2–4 and references therein). The utility of RDT has been very limited in compressible flows. The homogeneity condition in compressible turbulence is far more restrictive requiring that mean velocity gradients and mean thermodynamic variables be spatially invariant. The requisite homogeneity conditions are naturally satisfied only in a limited class of compressible flows due to the coupling between velocity and thermodynamic variables.[5–11] While these studies have yielded very valuable physical insight and closure-model suggestions, they all invoke the isentropic flow condition further restricting their overall utility. For the special case of homogeneous shear flow, Livescu and Madnia[12] perform RDT analysis with the ideal-gas state equation. In all the above works, the fluctuation equations are solved directly to infer turbulence physics. The merits of solving RDT spectral covariance equations [particle representation method (PRM)], rather than the primitive fluctuation equations, are well documented.[4] The benefits are (i) reduced statistical error (as some of the averaging is performed analytically) and (ii) better closure-model suggestion as the interaction between the various moments become more apparent. To date, none of the compressible RDT studies attempt to derive closed spectral covariance (PRM) equations for the statistical moments.

In this Letter, we seek to take an important step in advancing compressible RDT to the same level of sophistication and broad applicability as incompressible RDT. Toward that end, we aim to (i) extend the range of ideal-gas compressible RDT to a broader class of mean flows, and (ii) derive a fully closed set of equations for all key spectral covariance. By introducing an appropriate body force, we decouple the mean flow velocity field from the mean thermodynamic variables (density, pressure, and temperature). Then we derive the linearized fluctuating field equations for general mean flows. Finally, we derive the closed set of covariance moment equations. To demonstrate the validity of the derived equations, we compare the current results with incompressible RDT at the zero Mach number limit and pressure-release turbulence at the infinite Mach number limit for a variety of mean flows.

The governing equations consist of mass, momentum, energy equations, and the ideal-gas state equation. We start from the normalized compressible Euler equations with an added body force $(\vec{f})$ to be specified later,

$$\partial_t\rho + U_j\partial_{x_j}\rho = -\rho\partial_{x_j}U_j, \tag{1a}$$

$$\partial_t U_i + U_j\partial_{x_j}U_i = -\alpha_0/(M_{a0}^2\rho)\partial_{x_i}P + f_i, \tag{1b}$$

$$\partial_t T + U_j\partial_{x_j}T = -(\gamma-1)T\partial_{x_j}U_j. \tag{1c}$$

In the above γ and M_{a0} are the ratio of specific heats and reference Mach number defined by the ratio of reference velocity to reference sound speed. Parameter $\alpha_0=1/\gamma$ (for the isentropic process: $P\sim\rho^\gamma$) or 1 (for the isothermal process: $P\sim\rho$). The normalized state equation of ideal gas is $P=\rho T$. The body force is a deterministic function of space and time and contains no fluctuation component.

We employ Reynolds decomposition to partition all variables into mean and fluctuation parts as $U_i=\bar{U}_i+u_i'$, $\rho=\bar{\rho}+\rho'$, $T=\bar{T}+T'$, $P=\bar{P}+P'$, and neglect terms involving the product of fluctuations for linearization. Thus, we obtain the mean flow equations,

$$\partial_t\bar{\rho} + \bar{U}_j\partial_{x_j}\bar{\rho} = -\bar{\rho}\partial_{x_j}\bar{U}_j, \tag{2a}$$

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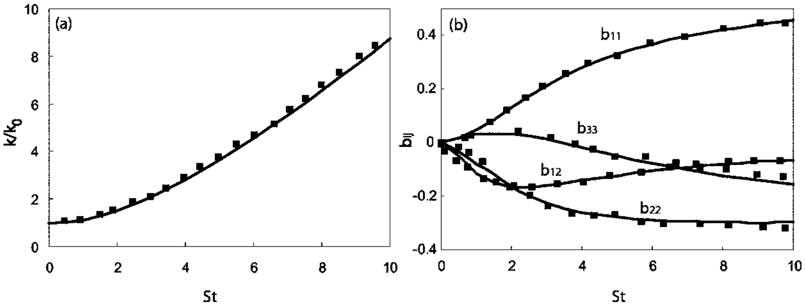

FIG. 1. The evolution of (a) normalized turbulent kinetic energy and (b) the Reynolds-stress anisotropies with HS rapid distortion at incompressible limit ($\mathrm{Ma}_0 = 0.1$). Symbols are incompressible RDT data from Pope (Ref. 14).

$$\partial_t \bar{U}_i + \bar{U}_j \partial_{x_j} \bar{U}_i = - a \partial_{x_i} \bar{P} + f_i, \tag{2b}$$

$$\partial_t \bar{T} + \bar{U}_j \partial_{x_j} \bar{T} = - c \partial_{x_j} \bar{U}_j, \tag{2c}$$

$$\bar{P} = \bar{\rho} \bar{T}, \tag{2d}$$

and the fluctuating field equations

$$\partial_t \rho' + \bar{U}_k \partial_{x_k} \rho' = - u_k' \partial_{x_k} \bar{\rho} - \bar{\rho} \partial_{x_k} u_k' - \rho' \partial_{x_k} \bar{U}_k, \tag{3a}$$

$$\partial_t u_i' + \bar{U}_k \partial_{x_k} u_i' = - u_k' \partial_{x_k} \bar{U}_i - a \partial_{x_i} (\bar{\rho} T')/\bar{\rho} - b \partial_{x_i} (\bar{T} \rho')/\bar{T} + a \partial_{x_i} (\bar{\rho} \bar{T}) \rho'/\bar{\rho}^2, \tag{3b}$$

$$\partial_t T' + \bar{U}_k \partial_{x_k} T' = - u_k' \partial_{x_k} \bar{T} - c \partial_{x_k} u_k' - (\gamma - 1) T' \partial_{x_k} \bar{U}_k, \tag{3c}$$

with $a = \alpha_0 / M_{a0}^2$, $b = \alpha_0 \bar{T} / M a_{a0}^2 \bar{\rho}$, and $c = (\gamma - 1)\bar{T}$. Here we have eliminated P' by applying the state equation. The difficulty with the compressible flow homogeneity requirement is evident from the fluctuation equations. Due to the explicit appearance of mean density and temperature in the equations, these quantities must be spatially invariant for the flow to be homogeneous. The mean pressure must be spatially uniform to satisfy the state equation of ideal gas. This is the more stringent requirement for homogeneity in compressible turbulence as discussed in Durbin and Zeman in Ref. 5. This homogeneity requirement restricts the use of compressible RDT to a small class of mean velocity gradients.

Even in incompressible flows, body force has been used in the past[13] to generate complex mean-velocity gradients that cannot be produced by the action of the mean pressure field alone. Following a similar line, we suggest that the mean-flow acceleration be entirely balanced by the body force,

$$\partial_t \bar{U}_i + \bar{U}_j \partial_{x_j} \bar{U}_i = f_i \quad \text{so that } \partial_{x_i} \bar{P} = 0. \tag{4}$$

Thus mean pressure is decoupled from the mean velocity and becomes spatially invariant. The above equation does not preclude time-dependent mean velocity gradients, so long as the body force is correspondingly time-varying. In principle, any mean velocity gradient is permitted even for compressible flows provided the body force is chosen as specified by the above equation. While the mean flow and thermodynamic variables are decoupled, the fluctuating components are related as dictated by conservation equations,

$$\partial_t \rho' + \bar{U}_k \partial_{x_k} \rho' = - \bar{\rho} \partial_{x_k} u_k' - \rho' \partial_{x_k} \bar{U}_k, \tag{5a}$$

$$\partial_t u_i' + \bar{U}_k \partial_{x_k} u_i' = - u_k' \partial_{x_k} \bar{U}_i - a \partial_{x_i} T' - b \partial_{x_i} \rho', \tag{5b}$$

$$\partial_t T' + \bar{U}_k \partial_{x_k} T' = - c \partial_{x_k} u_k' - (\gamma - 1) T' \partial_{x_k} \bar{U}_k. \tag{5c}$$

These equations form the basis of the present RDT study.

Since the fluctuation field equations are homogeneous, we can solve for them in spectral or Fourier space.[14] Each fluctuating variable can be written in terms of its Fourier components as $\vec{u}'(\vec{x},t) = \Sigma_{\vec{\kappa}} \hat{\vec{u}}(t) e^{i\vec{\kappa}(t)\cdot\vec{x}}$, $\rho'(\vec{x},t) = \Sigma_{\vec{\kappa}} \hat{\rho}(t) e^{i\vec{\kappa}(t)\cdot\vec{x}}$, and $T'(\vec{x},t) = \Sigma_{\vec{\kappa}} \hat{T}(t) e^{i\vec{\kappa}(t)\cdot\vec{x}}$, where $\vec{\kappa}(t)$ is the wave-number vector and $\hat{\vec{u}}$, $\hat{\rho}$, and $\hat{T}$ are the Fourier coefficients of the velocity, density, and temperature fluctuations, respectively. Equations (5) are then transformed to spectral space,

$$d_t \hat{\rho} = - i \bar{\rho} \kappa_k \hat{u}_k - \partial_{x_k} \bar{U}_k \hat{\rho}, \tag{6a}$$

$$d_t \hat{u}_i = - \hat{u}_k \partial_{x_k} \bar{U}_i - i (a \hat{T} + b \hat{\rho}) \kappa_i, \tag{6b}$$

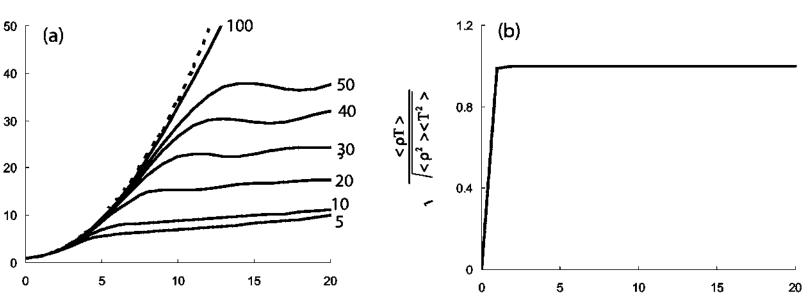

FIG. 2. (a) The evolutions of normalized turbulent kinetic energy at different Mach numbers as indicated. Dashed line is computed from Eq. (9) at the high Mach-number limit. (b) Evolution of normalized density-temperature correlation for all M_{a0} in the range 0.05–100.

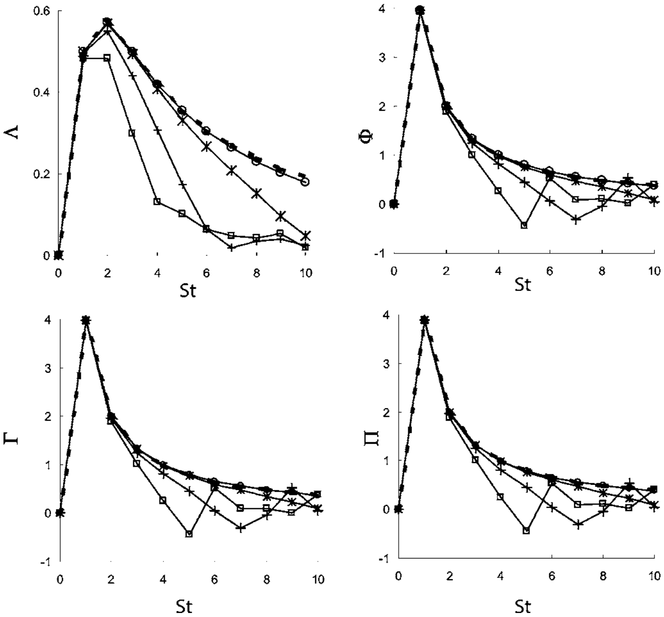

FIG. 3. The evolution of nondimensional growth rates at different Mach numbers. □: $M_{a0}=5$; +: $M_{a0}=10$; *: $M_{a0}=30$; ○: $M_{a0}=100$. Dashed lines are the corresponding high Mach limits.

$$d_t\hat{T} = -ic\kappa_k\hat{u}_k - (\gamma-1)\partial_{x_k}\bar{U}_k\hat{T}, \quad (6c)$$

where nonsubscript i is the imaginary unit equal to $\sqrt{-1}$. The wave-number vector evolves as[14]

$$d_t\kappa_i + \kappa_k\partial_{x_i}\bar{U}_k = 0. \quad (7)$$

For given initial conditions, Eqs. (6) and (7) can be solved directly. Then, the covariance in the Fourier space (spectral covariance), defined as $\hat{R}_{ij}(\vec{\kappa},t)=\langle\hat{u}_i^*(\vec{\kappa},t)\hat{u}_j(\vec{\kappa},t)\rangle$, $\hat{L}_i(\vec{\kappa},t)=\langle\hat{u}_i(\vec{\kappa},t)\hat{T}^*(\vec{\kappa},t)\rangle$, $\hat{M}_i(\vec{\kappa},t)=\langle\hat{u}_i(\vec{\kappa},t)\hat{\rho}^*(\vec{\kappa},t)\rangle$, $\hat{A}(\vec{\kappa},t)=\langle\hat{\rho}^*(\vec{\kappa},t)\hat{T}(\vec{\kappa},t)\rangle$, $\hat{B}(\vec{\kappa},t)=\langle\hat{T}^*(\vec{\kappa},t)\hat{T}(\vec{\kappa},t)\rangle$, and $\hat{C}(\vec{\kappa},t)=\langle\hat{\rho}^*(\vec{\kappa},t)\hat{\rho}(\vec{\kappa},t)\rangle$, is obtained for each given $\vec{\kappa}(t)$.

As mentioned above, rather than solving the primitive fluctuation equations (6) directly, we solve spectral covariance evolution (PRM) equations.[4] The full closed set of covariance equations are

$$d_t\hat{R}_{ij} = -\partial_{x_k}\bar{U}_i\hat{R}_{kj} - \partial_{x_k}\bar{U}_j\hat{R}_{ik} + ia(\hat{L}_j\kappa_i - \hat{L}_i^*\kappa_j) + ib(\hat{M}_j\kappa_i - \hat{M}_i^*\kappa_j), \quad (8a)$$

$$d_t\hat{M}_i = i\bar{\rho}\kappa_k\hat{R}_{ki} - \partial_{x_k}\bar{U}_k\hat{M}_i - \partial_{x_k}\bar{U}_i\hat{M}_k - ia\hat{A}\kappa_i - ib\hat{C}\kappa_i, \quad (8b)$$

$$d_t\hat{L}_i = ic\kappa_k\hat{R}_{ki} - (\gamma-1)\partial_{x_k}\bar{U}_k\hat{L}_i - \partial_{x_k}\bar{U}_i\hat{L}_k - ia\hat{B}\kappa_i - ib\hat{A}^*\kappa_i, \quad (8c)$$

$$d_t\hat{A} = i\bar{\rho}\kappa_k\hat{L}_k^* - ic\hat{M}_k\kappa_k - \gamma\partial_{x_k}\bar{U}_k\hat{A}, \quad (8d)$$

$$d_t\hat{B} = -2(\gamma-1)\partial_{x_k}\bar{U}_k\hat{B} + ic\kappa_k(\hat{L}_k^* - \hat{L}_k), \quad (8e)$$

$$d_t\hat{C} = -2\partial_{x_k}\bar{U}_k\hat{C} + i\bar{\rho}\kappa_k(\hat{M}_k^* - \hat{M}_k). \quad (8f)$$

Note that each covariance consists of a real part and an imaginary part. The evolution of the wave-number vector is again given by Eq. (7). Summations of these covariances over all wave-number vectors give the second moments in the physical space $\langle u_iu_j\rangle=\Sigma_{\vec{\kappa}}\hat{R}_{ij}(\vec{\kappa},t)$, $\langle u_iT\rangle=\Sigma_{\vec{\kappa}}\hat{L}_i(\vec{\kappa},t)$, $\langle u_i\rho\rangle=\Sigma_{\vec{\kappa}}\hat{M}_i(\vec{\kappa},t)$, $\langle\rho T\rangle=\Sigma_{\vec{\kappa}}\hat{A}(\vec{\kappa},t)$, $\langle T^2\rangle=\Sigma_{\vec{\kappa}}\hat{B}(\vec{\kappa},t)$, and $\langle\rho^2\rangle=\Sigma_{\vec{\kappa}}\hat{C}(\vec{\kappa},t)$, respectively.

In our computations, we employ a fourth-order Runge-Kutta scheme to solve the ODEs in Fourier space. Without loss of generality, we set the normalized mean density and temperature to unity, $\bar{\rho}=\bar{T}=1.0$. The initial density and temperature fluctuations are uniformly distributed over all wave numbers leading to an overall intensity of about 5%. The initial turbulence field is isotropic with $N(=n_\phi\times n_\theta)$ wave-number vectors distributed uniformly on a unit sphere ($r=1$, $0\leqslant\phi\leqslant\pi$, and $0\leqslant\theta\leqslant 2\pi$). The details are as given in Ref. 15, and we briefly summarize the procedure here. Uniform wave-vector distribution is obtained by requiring $\sin\phi d\phi d\theta=4\pi/N$, where $d\phi=\pi/n_\phi$ and $d\theta=2\pi/n_\theta$. The initial velocity covariance for each wave vector $\hat{R}_{ij}$ is specified as follows. If incompressibility is to be imposed, we first consider a 2D velocity vector field on the plane perpendicular to $\vec{k}$. All Fourier coefficients on this plane are taken to be equally probable. The average velocity covariance on this plane $\hat{R}'_{ij}(\vec{k})=\langle\hat{u}_i'^*(\vec{k})\hat{u}_j'(\vec{k})\rangle$ is first constructed. The covariance is then transferred to the spherical coordinate through mapping $\hat{R}(\vec{k})=A\hat{R}'(\vec{k})$, where A is the transformation matrix computed as $A=[(\cos\theta\cos\phi,-\sin\theta,\cos\theta\sin\phi),(\sin\theta\cos\phi,-\cos\theta,\sin\theta\sin\phi),(-\sin\phi,0,\cos\phi)]$. Test calculations were performed for three sets of ensembles with 4000, 6000, and 12000 wave vectors. All three ensembles yielded identical results. All the results presented below are obtained from an ensemble of 6000 wave vectors.

We first investigate the RDT behavior at the incompressible limit (low Mach number) for *homogeneous shear* (HS) distortion, $\partial_{x_j}\bar{U}_i=[(0,S,0),(0,0,0),(0,0,0)]$. This mean velocity field automatically satisfies the homogeneity requirement and no body force is needed. Figure 1 shows the evolutions of (a) normalized turbulent kinetic energy and (b) Reynolds-stress anisotropies at low Mach-number limit

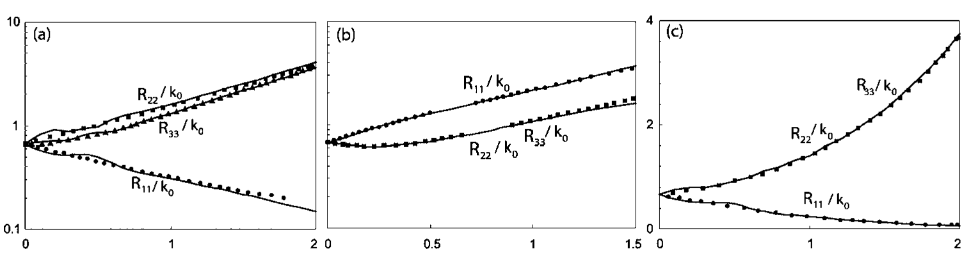

FIG. 4. The evolution of the Reynolds stresses at incompressible limit ($M_{a0}=0.01$) for different rapid distortions. (a) PS, (b) AE, and (c) AC. Symbols are incompressible RDT data from Pope (Ref. 14).

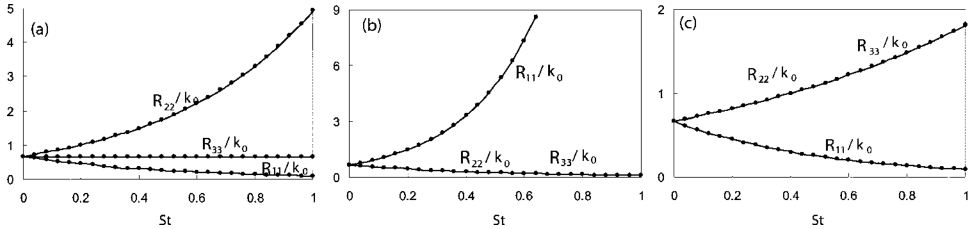

FIG. 5. The evolution of the Reynolds stresses at high Mach-number limit ($M_{a0}=100$) for different rapid distortions. (a) PS, (b) AE, and (c) AC. Symbols are analytical solutions.

($M_{a0}=0.1$). Symbols are from incompressible RDT calculations of Pope.[14] It is reasonable to conclude that incompressible flow results are recovered at the low Mach-number limit.

At the Burgers limit ($M_{a0}\rightarrow\infty$), Eq. (8a) can be solved analytically (omitting the pressure terms) for the homogeneous shear case to obtain

$$k(t)/k_0 = 1 + (St)^2/3. \tag{9}$$

At this limit, the other covariances can also be calculated analytically from their respective equations after neglecting the pressure-related terms. Figure 2(a) shows the evolution of normalized turbulent kinetic energy at different Mach numbers from low to high as indicated. The dashed line is computed from Eq. (9). As can be seen, the solution converges to the Burgers limit as Mach number increases. The growth of density-temperature correlation for a broad range of Mach number range is plotted in Fig. 2(b). It is seen that the correlation attains a value of unity very rapidly in all cases indicating perfect correlation between the fluctuating density and temperature. However, the density-velocity correlation (not shown) is nearly zero for all levels of shear. Thus, velocity fluctuations are out of phase with density and temperature fluctuations.

To investigate the evolution of other correlations of interest, we define the following nondimensional parameters $\Lambda=d_t k/(Sk)$, $\Phi=d_t\langle\rho^2\rangle/\langle\rho^2\rangle$, $\Gamma=d_t\langle\rho T\rangle/(S\langle\rho T\rangle)$, $\Pi=d_t\langle T^2\rangle/(S\langle T^2\rangle)$ characterizing the growth rates of kinetic energy, density, density-temperature correlation, and temperature, respectively. The time evolutions of these growth rates are plotted in Fig. 3. Again, the dashed lines correspond to high Mach-number analytical results. It is shown that beyond $\mathrm{Ma}_0=100$, all growth rates converge to their corresponding high-Mach limits. Thus, we demonstrate for homogeneous shear flow that kinetic energy and other key statistics have the correct behavior at the incompressible and pressure-release limits.

Next we turn our attention to other mean velocity gradients: PS, *plain strain* $[(S,0,0),\ (0,-S,0),\ (0,0,0)]$; AC, *axisymmetric contraction* $[(S,0,0),\ (0,-S/2,0),\ (0,0,-S/2)]$; and AE, *axisymmetric expansion* $[(-2S,0,0),\ (0,S,0),\ (0,S,0)]$. It must be pointed out that these mean velocity fields require body forces to balance the mean-flow acceleration.

We compare our compressible RDT computation with $M_{a0}=0.01$ against incompressible RDT data from Pope[14] in Fig. 4. High Mach-number comparisons are shown in Fig. 5. The dashed lines again represent analytical solutions obtained from the governing equations after suppressing the pressure terms corresponding to the pressure-release limit. Again, it is clearly seen that our RDT results recover the incompressible and high Mach-number limits very well. Clearly, the current RDT formulation is not restricted to homogeneous shear flow, unlike previous ones.

In conclusion, we present a general RDT tool for investigating homogeneous ideal-gas compressible turbulence. A body force is introduced to decouple the mean velocity field from the mean thermodynamic properties, thus extending the range of compressible RDT to a broader class of mean velocity gradients. A closed set of spectral covariance equations is derived. Validation in homogeneous mean shear flow as well as strained mean flows demonstrates that the spectral covariance equations recover incompressible and pressure-release limits very accurately. In the future, we propose to use this method to investigate flow-thermodynamic variable interactions in a variety of compressible flows over a wide range of Mach numbers. We also plan to extend compressible RDT to the analysis of density-weighted variables.

[1] G. K. Batchelor and I. Proudman, "The effect of rapid distortion of a fluid in turbulent motion," Q. J. Mech. Appl. Math. **7**, 121 (1954).
[2] A. M. Savill, "Recent developments in rapid distortion theory," Annu. Rev. Fluid Mech. **19**, 531 (1987).
[3] J. C. R. Hunt and D. J. Carruthers, "Rapid distortion theory and the 'problems' of turbulence," J. Fluid Mech. **212**, 497 (1990).
[4] S. C. Kassinos and W. C. Reynolds, "A particle representation model for the deformation of homogeneous turbulence," *Annual Research Briefs* (Center for Turbulence Research, NASA Ames/Stanford University, Stanford, CA, 1996), p. 31.
[5] P. A. Durbin and O. Zeman, "Rapid distortion theory for homogenous compressed turbulence with application to modelling," J. Fluid Mech. **242**, 349 (1992).
[6] G. A. Blaisdell, G. N. Coleman, and D. N. N. Mansour, "Rapid distortion theory for compressible homogeneous turbulence under isotropic mean strain," Phys. Fluids **8**, 2692 (1996).
[7] G. N. Coleman and D. N. N. Mansour, "Modeling the rapid spherical compression of isotropic turbulence," Phys. Fluids A **3**, 2255 (1991).
[8] G. N. Coleman and D. N. N. Mansour, "Simulation and modeling of homogeneous compressible turbulence under isotropic mean compression," in *Turbulent Shear Flows 8* (Springer, Berlin, 1992).
[9] C. Cambon, G. N. Coleman, and D. N. N. Mansour, "Rapid distortion analysis and direct simulation of compressible homogeneous turbulence at finite Mach number," J. Fluid Mech. **257**, 641 (1993).
[10] L. Jacquin, C. Cambon, and E. Blin, "Turbulence amplification by a shock wave and rapid distortion theory," Phys. Fluids A **5**, 2539 (1993).
[11] A. Simone, G. N. Coleman, and C. Cambon, "The effect of compressibility on turbulent shear flow: A rapid-distortion-theory and direct-numerical-simulation study," J. Fluid Mech. **330**, 307 (1997).
[12] D. Livescu and C. K. Madnia, "Small scale structure of homogenous turbulent shear flow," Phys. Fluids **16**, 2864 (2004).
[13] S. S. Girimaji, J. R. O'Neil, and D. Yu, "Rapid distortion analysis of homogenous turbulence subjected to rotating shear," Phys. Fluids **18**, 085102 (2006).
[14] S. B. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, 2000).
[15] S. S. Girimaji, E. Jeong, and S. V. Poroseva, "Pressure-strain correlation in homogeneous anisotropic turbulence subject to rapid strain-dominated distortion," Phys. Fluids **15**, 3209 (2003).