INTRODUCTION TO DYNAMICS OF STRUCTURES

A PROJECT DEVELOPED FOR THE UNIVERSITY CONSORTIUM ON INSTRUCTIONAL SHAKE TABLES

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Required Equipment:
- Instructional Shake Table
- Two Story Building
- Three Accelerometers
- MultiQ Board
- Power Supply
- Computer
- Software: Wincon and Matlab

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Objective: The objective of this experiment is to introduce you to principles in structural dynamics through the use of an instructional shake table. Natural frequencies, mode shapes and damping ratios for a scaled structure will be obtained experimentally.

1.0 Introduction

The dynamic behavior of structures is an important topic in many fields. Aerospace engineers must understand dynamics to simulate space vehicles and airplanes, while mechanical engineers must understand dynamics to isolate or control the vibration of machinery. In civil engineering, an understanding of structural dynamics is important in the design and retrofit of structures to withstand severe dynamic loading from earthquakes, hurricanes, and strong winds, or to identify the occurrence and location of damage within an existing structure.

In this experiment, you will test a small test building of two floors to observe typical dynamic behavior and obtain its dynamic properties. To perform the experiment you will use a bench-scale shake table to reproduce a random excitation similar to that of an earthquake. Time records of the measured absolute acceleration responses of the building will be acquired.

2.0 Theory: Dynamics of Structures

To understand the experiment it is necessary to understand concepts in dynamics of structures. This section will provide these concepts, including the development of the differential equation of motion and its solution for the damped and undamped case. First, the behavior of a single degree of freedom (SDOF) structure will be discussed, and then this will be extended to a multi degree of freedom (MDOF) structure.

The number of degrees of freedom is defined as the minimum number of variables that are required for a full description of the movement of a structure. For example, for the single story building shown in figure 1 we assume the floor is rigid compared to the two columns. Thus, the displacement of the structure is going to be completely described by the displacement, \( x \), of the floor. Similarly, the building shown in figure 2 has two degrees of freedom because we need to describe the movement of each floor separately in order to describe the movement of the whole structure.
2.1 One degree of freedom

We can model the building shown in figure 1 as the simple dynamically equivalent model shown in figure 3a. In this model, the lateral stiffness of the columns is modeled by the spring \( k \), the damping is modeled by the shock absorber \( c \) and the mass of the floor is modeled by the mass \( m \). Figure 3b shows the free body diagram of the structure. The forces include the spring force \( f_s(t) \), the damping force \( f_d(t) \), the external dynamic load on the structure, \( p(t) \), and the inertial force \( f_i(t) \). These forces are defined as:

\[
\begin{align*}
  f_s &= k \cdot x \\
  f_d &= c \cdot \dot{x} \\
  f_i &= m \cdot \ddot{x}
\end{align*}
\]

where the \( \dot{x} \) is the first derivative of the displacement with respect to time (velocity) and \( \ddot{x} \) is the second derivative of the displacement with respect to time (acceleration).

Summing the forces shown in figure 3b we obtain

\[
\sum F = m \cdot \ddot{x} = p(t) - c \dot{x} - kx
\]

\[
(4)
\]

\[
m \ddot{x} + c \dot{x} + kx = p(t)
\]

where the mass \( m \) and the stiffness \( k \) are greater than zero for a physical system.
2.1.1 Undamped system

Consider the behavior of the undamped system \((c=0)\). From differential equations we know that the solution of a constant coefficient ordinary differential equation is of the form

\[ x(t) = e^{\alpha t} \] (6)

and the acceleration is given by

\[ \ddot{x}(t) = \alpha^2 e^{\alpha t}. \] (7)

Using equations (6) and (7) in equation (5) and making \(p(t)\) equal to zero we obtain

\[ m\alpha^2 e^{\alpha t} + ke^{\alpha t} = 0 \] (8)

\[ e^{\alpha t}[m\alpha^2 + k] = 0. \] (9)

Equation (9) is satisfied when

\[ \alpha^2 = \frac{-k}{m} \] (10)

\[ \alpha = \pm i \sqrt{\frac{k}{m}}. \] (11)

The solution of equation (5) for the undamped case is

\[ x(t) = Ae^{\omega_n it} + Be^{-\omega_n it} \] (12)

where \(A\) and \(B\) are constants based on the initial conditions, and the natural frequency \(\omega_n\) is defined as

\[ \omega_n = \sqrt{\frac{k}{m}}. \] (13)

Using Euler’s formula and rewriting equation (12) yields

\[ e^{i\alpha t} = \cos \alpha t + i \sin \alpha t \] (14)

\[ x(t) = A(\cos (\omega_n t) + i \sin (\omega_n t)) + B(\cos (-\omega_n t) + i \sin (-\omega_n t)) \] (15)

\[ x(t) = A\cos (\omega_n t) + A\sin (\omega_n t) + B\cos (-\omega_n t) + B\sin (-\omega_n t). \] (16)

Using \(\cos (-\alpha) = \cos (\alpha)\) and \(\sin (-\alpha) = -\sin (\alpha)\) we have

\[ x(t) = A\cos (\omega_n t) + A\sin (\omega_n t) + B\cos (\omega_n t) - B\sin (\omega_n t) \] (17)

or,
\[ x(t) = (A + B) \cos(\omega_n t) + (A - B)i \sin(\omega_n t). \]  
(18)

Letting \( A + B = C \) and \( A - B = D \) we obtain
\[ x(t) = C(\cos \omega_n t) + Di(\sin \omega_n t) \]  
(19)

where \( C \) and \( D \) are constants that are dependent on the initial conditions of \( x(t) \).

From equation (19) it is clear that the response of the system is harmonic. This solution is called the \textit{free vibration response} because it is obtained by setting the forcing function, \( p(t) \), to zero. The value of \( \omega_n \) describes the frequency at which the structure vibrates and is called the \textit{natural frequency}. Its units are \textit{radians/sec}. From equation (13) the natural frequency, \( \omega_n \), is determined by the stiffness and mass of the structure.

The vibration of the structure can also be described by the natural period, \( T_n \). The period of the structure is the time that is required to complete one cycle given by
\[ T_n = \frac{2\pi}{\omega_n}. \]  
(20)

2.1.2 Damped system

Consider the response with a nonzero damping coefficient \( c \neq 0 \). The homogenous solution of the differential equation is of the form
\[ x(t) = e^{\alpha t} \]  
(21)

and
\[ \dot{x}(t) = \alpha e^{\alpha t} \]  
(22)
\[ \ddot{x}(t) = \alpha^2 e^{\alpha t}. \]  
(23)

Using equations (21), (22) and (23) in equation (5) and making the forcing function \( p(t) \) equal to zero we have
\[ e^{\alpha t}[m\alpha^2 + c\alpha + k] = 0. \]  
(24)

Solving for \( \alpha \) we have
\[ \alpha_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} \]  
(25)

Defining the critical damping coefficient as
\[ c_{cr} = \sqrt{4km} \]  
(26)

and the damping ratio as
we can rewrite equation (25)

\[ \alpha_{1,2} = -\zeta \omega_n \pm i \omega_n \sqrt{1 - \zeta^2}. \]  

Defining the damped natural frequency as

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]  

equation (28) can be rewritten as

\[ \alpha_{1,2} = -\zeta \omega_n \pm i \omega_d. \]  

Thus, the solution for the differential equation of motion for a damped unforced system is

\[ x(t) = A e^{-\zeta \omega_n t} e^{-i \omega_d t} + B e^{-\zeta \omega_n t} e^{i \omega_d t}, \text{ or} \]

\[ x(t) = e^{-\zeta \omega_n t} (A e^{-i \omega_d t} + Be^{i \omega_d t}) \]  

Using equation (14) (Euler’s formula)

\[ x(t) = e^{-\zeta \omega_n t} (C \cos(\omega_d t) + D \sin(\omega_d t)) \]  

where \( C \) and \( D \) are constants to be determined by the initial conditions.

Civil structures typically have low damping ratios of less than 0.05 (5%). Thus, the damped natural frequency, \( \omega_d \), is typically close to the natural frequency, \( \omega_n \).

Comparing the solutions of the damped structure in equation (19) and the undamped structure in equation (33), we notice that the difference is in the presence of the term \( e^{-\zeta \omega_n t} \). This term forces the response to be shaped with an exponential envelope as shown in figure 4.

Figure 4. Response of damped structures.
Summary: In this section you learned basic concepts for describing a single degree of freedom system (SDOF). In the followings section you will extend these concepts to the case of multiple degree of freedom systems.

2.2 Multiple degree of freedom systems

A multiple degree of freedom structure and its equivalent dynamic model are shown in figure 5. The differential equations of motion of a multiple degree of freedom system is

$$M\ddot{x} + C\dot{x} + Kx = p(t)$$

(34)

where $M$, $C$ and $K$ are matrices that describe the mass, damping and stiffness of the structure, $p(t)$ is a vector of external forces, and $x$ is a vector of displacements. A system with $n$ degrees of freedom has mass, damping, and stiffness matrices of size $n \times n$, and $n$ natural frequencies. The solution to this differential equation has $2n$ terms.

The structure described by Eq. (34) will have $n$ natural frequencies. Each natural frequency, $\omega_n$, has an associated mode shape vector, $\phi_n$, which describes the deformation of the structure when the system is vibrating at each associated natural frequency. For example, the mode shapes for the four degree of freedom structure in figure 5 are shown in figure 6. A node is a point that remains still when the structure is vibrating at a natural frequency. The number of nodes is related with the natural frequency number by

$$\#\text{nodes} = n - 1$$

(35)

where $n$ is the frequency number associated with the mode shape.

![Figure 5. Multiple degree of freedom system and dynamic model.](image)

![Figure 6. Diagram of mode shapes for a four degree of freedom structure.](image)
2.3 Frequency Domain Analysis

The characteristics of the structural system can also be described in the frequency domain. The Fourier transform of a signal $x(t)$ is defined by

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} \, dt \quad (36)$$

and is related to the Fourier transform of the derivatives of this function by

$$[\dot{x}(t)] = j2\pi fX(f) \quad (37)$$

$$[\ddot{x}(t)] = -(2\pi f)^2 X(f) \quad (38)$$

Plugging this into the equation of motion (equation (5)) for the SDOF system, we obtain

$$[-(2\pi f)^2 m + j2\pi fc + k]X(f) = P(f) \quad (39)$$

and the ratio of the frequency domain representation of the output to the frequency domain representation of the input is determined

$$H(f) = \frac{X(f)}{P(f)} = \frac{1}{k - (2\pi f)^2 m + j2\pi fc} \quad (40)$$

which is called the complex frequency response function, or transfer function. Note that this is a function of the frequency, $f$, and provides the ratio of the structural response to the input loading at each frequency.

Figure 7 shows an example of a transfer function for a two degree of freedom structure. Here the magnitude of the complex function in Eq. (40) is graphed. The X axis represents frequency (in either radians per second or Hz) and the Y axis is provided in decibels. One decibel is defined as

$$\text{dB} = 20\log(\text{Amplitude}) \quad (41)$$

Peak(s) in the transfer function correspond to the natural frequencies of the structure, as shown in Figure 7.

2.4 Experimental determination of the damping in a structure

A structure is characterized by its mass, stiffness and damping. The first two may be obtained from the geometry and material properties of the structure. However, damping should be determined through experiments. For purposes of this experiment you will assume that the only damping present in the structure is due to viscous damping. Two commonly used methods to determine the damping in structures are the logarithmic decrement method and the half power bandwidth method. The logarithmic decrement technique obtains the damping properties of a structure from a free vibration test using time domain data. The half power bandwidth method uses the transfer function of the structure to determine the amount of damping for each mode.
2.4.1 Exponential decay

Using free vibration data of the acceleration of the structure one may obtain the damping ratio. Figure 8 shows a free vibration record of a structure. The logarithmic decrement, $\delta$, between two peaks is defined as

$$
\delta = \ln \frac{y_1}{y_2}
$$

(42)

where $y_1$ and $y_2$ are the amplitudes of the peaks.

From the solution of the damped system (equation (33)) we can say that $y_1$ and $y_2$ can be written as

$$
y_1 = Ce^{-\zeta \omega_n t}
$$

(43)

$$
y_2 = Ce^{-\zeta \omega_n (t + T)}
$$

(44)

where the constant C includes the terms of the sine and cosines in equation (33), and $T$ is the period of the system. Using equations (43) and (44) in equation (42)

$$
\delta = \ln \frac{y_1}{y_2} = \ln \frac{C e^{-\zeta \omega_n t}}{C e^{-\zeta \omega_n (t + T)}} = \zeta \omega_n T
$$

(45)

and when the damping ratio is small, can be approximated as

$$
\delta \equiv 2\pi \zeta.
$$

(46)

Solving for $\zeta$

$$
\zeta = \frac{\delta}{2\pi} = \frac{\ln y_1}{2\pi y_2}
$$

(47)
Using equation (47) we can obtain the damping ratio $\zeta$ of the structure using the amplitude of the signal at two consecutive peaks in a free vibration record of displacement or acceleration.

### 2.4.2 Half power bandwidth method

The second method to obtain an estimation of the damping of a structure is the half power bandwidth method. In contrast to the previous method, the half power bandwidth method uses the transfer function plot to obtain the damping. The method consists of determining the frequencies at which the amplitude of the transfer function is $A_2$ where

$$A_2 = \frac{A_1}{\sqrt{2}} \quad (48)$$

and $A_1$ is the amplitude at the peak. The frequencies $f_a$ and $f_b$ associated with the half power points on either side of the peak are obtained, as shown in figure 9. Then the damping ratio $\zeta$ is obtained using the formula

$$\zeta = \frac{f_b - f_a}{f_b + f_a} \quad (49)$$

The damping ratio associated with each natural frequency can be obtained using the half power bandwidth method.

![Figure 9. Half Power Bandwidth method.](image-url)
3.0 Experimental Setup: Equipment

3.1 Required Equipment

- Data acquisition system (MultiQ board and computer)
- Instructional shake table
- Standard test structure
- Three accelerometers (one on the shake table)
- Power unit for sensors and shake table
- Relevant cables
- Software: Wincon and Matlab (signal processing toolbox required)
- Passive control device (optional, instructions provided)

3.2 Data Acquisition System

A data acquisition system is used to obtain measurements of physical quantities using sensors. These measurements may be temperature, pressure, wind, distance, acceleration, etc. In civil engineering applications the most common types of sensors measure displacement, acceleration, force and strain. In this experiment, we are going to use acceleration sensors to obtain records of acceleration over time for a simple model of a building.

Photos of the experimental components are shown in figure 10. The data acquisition system consists of a computer and a MultiQ board. Accelerometers are attached to each floor of the test structure to measure accelerations. A power supply is used to provide current to the accelerometers and to the shake table. The accelerometers are connected to the power supply. The ground accelerometer should be connected at “S1”, the first floor accelerometer should be connected at “S2”, and the second floor accelerometer should be connected at “S3”. The MultiQ board should be connected to the power supply as well. To make this connection use the appropriate cable to connect the “From MultiQ” on the power supply to the “Shaker X” on the MultiQ board. (Hint: Also, be sure that all the accelerometers are facing in the same direction as the ground accelerometer).

3.3 Shake Table

Figure 11 shows the bench-scale shake table used to excite the structure. This small shake table is a uniaxial shake table with a design capacity of 25 pounds. It is controlled by a computer which has the capability to excite the building with different types of signals including sine wave, random or step signals. With this instrument, it is also possible to reproduce an earthquake and study the characteristics of structures under specific earthquakes. A safety circuit is provided which stops the shake table in case it travels beyond the range of operation. To enable the shake table, one must depress the deadman button. The shake table stops when the deadman button is released. For a more detailed guide on how to operate the shake table see the “Bench-Top Shake Table User’s Guide” available in the University Consortium of Instructional Shake Tables web page (http://ucist.cive.wustl.edu/)
3.4 Test Structure

The test structure is a simple model of a two story building. The building’s height is 50 centimeters (19.68 inches) per floor and has a weight of 2 kilograms per floor. A small shock absorber can be attached to the structure as a passive control device and an active mass driver can be adapted to the structure as an active control device.
4.0 Experimental Procedure

<table>
<thead>
<tr>
<th>Important Notes: Safe Operation of the Shake Table</th>
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<tbody>
<tr>
<td>• The “safety override” button on the power supply should <strong>ALWAYS</strong> remain in the off position.</td>
</tr>
<tr>
<td>• Turn the power supply <strong>off</strong> if you turn off or reboot the computer.</td>
</tr>
<tr>
<td>• The deadman switch must be depressed to excite the shake table. Press this button and hold it before you begin each segment of the experiment (before you hit the “Start” button on the Wincon server).</td>
</tr>
<tr>
<td>• It may be necessary to reboot the computer if it locks up during the test.</td>
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4.1 Transfer function calculation

1. Check that all the connections are correct (accelerometers, shake table, MultiQ board, *etc.*)
2. Turn on the power supply and wait to see that the right and left indicator lights blink.
3. On the computer, use Windows Explorer to open UCIST directory: C:/UCIST/boot
4. Double click on the file <boot.exe>. The left and right indicator lights on the power supply should stop blinking.
5. Double click on the shortcut <sample>. This program will start matlab and a figure window with a menu will appear (see figure 14). Now you are ready to begin the experiment.
6. Calibrate and center the shake table by clicking the first button of the menu “Calibrate shake table” (see figure 14). When the Wincon server starts hit the “Start” button to perform the calibration. The program will ask you if you can download the program and you should select “yes”.
7. Click on the second button, “Obtain data”, to run a sine sweep excitation. Hit the “Start” button on the Wincon server to generate the excitation. Graphs of the acceleration responses obtained with the data acquisition system will appear. Three plots should appear, one for the acceleration on the ground (shake table) and one for each floor of the building.
8. Click the third button “Plot transfer functions”, to calculate the transfer functions. This will take a few seconds. Two transfer functions will be computed, including
   • Ground excitation to first floor
   • Ground excitation to second floor
When the plots are displayed another menu will appear. This tool is provided so that you can graphically determine the frequencies of the structure. You should identify the natural frequencies of the structure using this tool. Be sure to hit “quit” when you have identified all of the frequencies, or you will lock up the computer.

You will use the transfer function data to obtain the damping ratio using the half-power bandwidth method. Zoom in on the peaks in the transfer functions and make a printout of these plots so that you can obtain the peaks and the half-power points from the graphs. Remember that the data in the plot is in decibels.

Please answer the following questions.

• How many natural frequencies does the structure have?
• What are the values of the natural frequencies?
• Are these values the same in the two transfer functions? Why or why not?

4.2 Determination of Mode Shapes

In Section 4.1 you found the natural frequencies of the test structure. Now you are going to identify the mode shapes of the structure using a sine wave excitation. Use the next button of the menu, “Sine Wave Excitation Test” (see figure 14). A new window will appear with two dials. One dial allows you to change the frequency and the second allows you to change the amplitude. Make sure that the structure is at rest before starting the excitation. Please keep the amplitude low as you are exciting the structure at resonance. Also, change the frequency of the excitation slowly.

Excite the building with each of the natural frequencies obtained in Section 4.1. Turn the frequency indicator that appears on the screen until you reach the value you are looking for. The value of the frequency chosen appears above the control dial.

Please do the following.

• Sketch each of the mode shapes of the structure.
• Obtain the number of nodes in each mode shape.
• Does this result satisfy equation (35)? Explain.

4.3 Damping estimation

4.3.1 Exponential decay

In this test you will excite the structure with a finite duration sinusoidal excitation to examine the free response in each mode of the structure. The sinusoidal excitation lasts for 30 seconds, and
then the structure is in free vibration. Then a record of the acceleration of the two floors of the structure will appear.

The next button on the menu, “Free Vibration Test” (See figure 14), will perform this test. When you hit this button a control panel will appear and you must insert the frequency into the blue box for each test. You must then hit the “Start” button on the Wincon server to begin the excitation. Be sure that the structure is at rest before performing this test. When the test is over, a plot will appear for the free vibration portion of the response. Do this test for each mode of the structure. Using these records obtain the damping ratio using the exponential decay method described in 2.4.1.

Please do the following.
• What is the damping ratio obtained using this method?
• Compare this damping ratio with that obtained in 4.3.2.

4.3.2 Half Power Bandwidth method
Use the half power bandwidth method (section 2.4.2) and transfer function obtained in section 4.1 to determine the damping in the structure.

Please do the following.
• From the transfer functions obtained in 4.1 estimate the damping using the half power bandwidth method described in 2.4.2. What is the damping ratio associated with each natural frequency?
• Compare the damping values for each of the two modes.
• Discuss the advantages and disadvantages of these two methods?

5.0 References
PAZ, M., Structural Dynamics, Chapman & Hall, New York, 1997