The ‘Concert Queueing Game’ with Feedback Routing

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Abstract

The objective of this paper is to dilate the interplay between feedback routing and strategic arrival behavior in single class queueing networks. We study a variation of the ‘Network Concert Queueing Game,’ wherein a fixed but large number of strategic users arrive at a network of queues where they can be routed to other nodes in the network following a fixed routing matrix, or potentially fedback to the end of the queue they arrive at. Working in a non-atomic setting, we prove the existence of Nash equilibrium arrival and routing profiles in three simple, but non-trivial, network topologies/architectures. In two of them, we also prove uniqueness of the equilibrium. Our results prove that Nash equilibrium decisions on when to arrive and which queue to join in a network are substantially impacted by routing, inducing ‘herding’ behavior under certain conditions on the network architecture. Our theory raises important design implications for capacity-sharing in systems with strategic users, such as ride-sharing and crowdsourcing platforms.

Keywords: Queueing, Game Theory, Strategic arrivals, Non-atomic equilibrium, routing

1. Introduction

We study the behavior of a finite, but large, number of strategic users in open queueing networks. Self-interested users choose a time to arrive and a route to take through the network. This type of behavior is evident in theme parks where users can choose which rides to visit and in which order, in ride-sharing networks where drivers can choose to visit different geographical locations at different times of the day and in some desired order, and even crowd-sourcing platforms where

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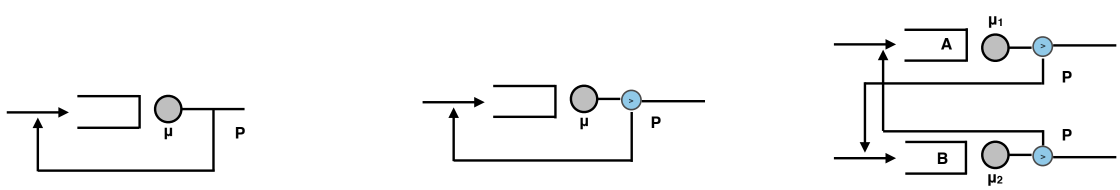
users can choose which platform to participate in.

To model this strategic behavior, we study a variation of the 'Network Concert Queueing Game' studied in [6], where the Nash equilibrium arrival timing and routing behavior of non-atomic users was studied in *feedforward* queueing networks. In the current paper, we consider network topologies with feedback. We elucidate the impact of feedback routing on strategic behavior by considering the following three network topologies/architectures [1]:

(i) a single server queue where jobs can be repeatedly returned (without bound) to the tail of the buffer after service,

(ii) a single server queue where a job can only be returned to the tail of the buffer a finite number of times after service, and

(iii) a two queue network with routing between queue 'A' and queue 'B', such that a fraction p of jobs that complete service at queue 'A' will be routed to queue 'B' and a fraction $1 - p$ will be directed out of the network (and vice versa from 'B' to 'A').



(a) Single Queue, Infinite Returns

(b) Single Queue, Finite Returns

(c) Parallel Network with Cross-Feedback

Figure 1: Network Architectures. Blue circle with arrow head indicates that jobs that have been fedback a fixed finite number of times are routed out of the network.

See Figure 1 for illustrations. Note that these are the simplest, non-trivial, architectures for which the equilibrium traffic profile can be computed. We should also note that our focus on the non-atomic setting helps to avoid modeling issues that arise in the atomic game, that detract from the broader question we wish to focus on here: *what is the impact of feedback and routing on the equilibrium arrival profile?* We note that the three network architectures are in increasing order of analytical complexity of computing the equilibrium profiles.

[1] We will use topology and architecture interchangeably henceforth.

Following [6], we consider a disutility function that trades off the cost of arriving early against the cost of waiting in the buffer. However, we modify it to explicitly account for the cost of being returned to the tail of the queue after service completion. Each user must consider the impact of these (potential) returns in making a decision of when to arrive and which route to take. Since every user must make the same calculation, an individuals' cost is determined through the interaction of the users. Now, assuming that the non-atomic users strategically choose a time to arrive and a queue to join (in architecture (iii), specifically) such that their disutility is minimized, we identify the Nash equilibrium arrival and routing profile. With feedback, users who arrive at time t must also contend with the fact that they potentially might interact with users who arrive after them. This makes the equilibrium computation substantially more complicated than the 'no feedback' case studied previously whereby users arriving after t would not directly affect the disutility of the user arriving at t.

While our primary motivation for this paper is mathematical, as noted in the first paragraph of this introduction, our results are can find application in a number of disparate service systems. As an example, consider how our results also provide modeling insights into strategic behavior in ride-sharing platforms such as Uber and Lyft. In particular, we provide a game theoretic explanation of how drivers might strategically choose a time and location to offer rides. We note, however, that our interest is in understanding the implications of feedback routing on the computation of equilibrium strategies, rather than in identifying an accurate model for ride-sharing platforms. To fix the ideas, consider the two queue first-in-first-out (FIFO) network in Figure 1c. Each queue can be viewed as a geographical location where drivers offer rides to passengers; drivers are the 'users' entering the queue and passengers are modeled as the single server. Note that the FIFO assumption implies that users obtain rides in order of their arrival, which is not an unreasonable approximation. On the other hand, the single server implies that only one passenger will be available to hail a ride at each time instant. For simplicity, we assume that the service rate is fixed; in effect, this implies that the number of passengers available to pick up per unit time is fixed. A fixed fraction p of passengers at queue 'A' wish to visit queue 'B' (and vice versa), and the remainder $1-p$ will exit this network (implying they will traverse to a different location). We assume that drivers are strategic about the amount of time they spend awaiting rides, and that this naturally translates into lost revenue. For the purposes of this discussion, we assume that the lost revenue is linear in the waiting time, and so we focus on a disutility/cost function that is a function of the sojourn time. We assume that users must make a one-shot decision of when and where to join the system; i.e., we are *not* considering a dynamic decision making process. The Nash equilibrium arrival and routing profile we compute in this paper provides insight into how drivers will react to routing and feedback, and can be used in capacity analysis and incentive design for ride-sharing networks.

1.1. Results

We start our analysis with the single queue case where a user can be returned to the back of the queue (potentially) an infinite number of times, as depicted in Figure 1a. This 'toy' problem is mathematically much simpler than the other network architectures we consider. However, it highlights the impact feedback can have on equilibrium strategies. Since there is a single queue, the only strategic decision to make is in when to arrive. After establishing the existence of a piecewise-linear Nash equilibrium arrival profile in Theorem 1, we prove that, modulo a technical condition on the parameters of the problem, this is the only Nash equilibrium in Theorem 2. Note that this result is with loss of generality. We find that, as the fraction of users being fedback increases, at equilibrium users arrive later (though potentially before service starts at time 0) and that the last user to arrive does so much later. Furthermore, the arrival epoch of the last user increases super-exponentially as the feedback fraction p approaches 1. Furthermore, the equilibrium arrival profile is piecewise linear, and we find that the slope of the arrival profile before service starts to be larger than after. One could interpret this as implying that the arrival 'rate' is greater before service starts than after. Users following the equilibrium strategy introduce additional costs on others in the form of negative externalities. It is thus useful to compare the social disutility (computed as an aggregate over the entire population) of following the equilibrium strategy against the social disutility of following the Pareto optimal strategy suggested by a central planner. We do this by computing the price of anarchy (PoA), or the ratio of the equilibrium social disutility to the optimal social disutility, and show that it is precisely equal to 2. This implies that the social disutility if doubled by following the equilibrium strategy. The PoA computation indicates that some coordination between users can result in lowered social disutility, though we do not pursue this line of research here.

Next, we consider a network architecture where users return a finite number of times. As it turns out, the analysis of the single return case is quite challenging and we focus on that exclusively.

We expect the analysis of the single return model to extend to an arbitrary (but finite) number of returns in a straightforward manner (though we do not pursue it here). Figure 1b depicts the network architecture we consider; the blue circle represents a decision 'element' that only returns users (with probability p) who have not received a second service. At each time instant $t > 0$, the service effort expended up to that point in time includes service rendered to users receiving their first service and those receiving their second service. Thus, the 'feedback rate' is dependent on the state of the queue, in contrast to the fixed 'rate' in the infinite return case. Users fedback for a second service will not affect the waiting time of users who arrive after the former depart (clearly). Contrast this with the infinite return case where every user who arrives can potentially delay all other users, due to the feedback effect. This is the crux of the equilibrium computation making it substantially more difficult than the infinite return model. In Theorem 4 we prove the existence of a piecewise linear equilibrium arrival profile by carefully constructing a sequence of time intervals where fedback users are served and showing that the size of these intervals, as measured by the Lebesgue measure, satisfy a recursive relationship. Next, in Theorem 5 we prove that this equilibrium is unique, modulo a technical condition on the parameters of the problem. Once again, we compute the price of anarchy (PoA) and show that is it equal to 2. In contrast to the infinite return case, we find that while the equilibrium queue length is linear, the arrival profile is (only) piecewise linear. Interestingly enough, we observe that the equilibrium arrival profile displays 'herding' behavior whereby users arrive at a 'high rate' in a time interval (whose length is determined by equilibrium considerations), followed by a time interval with a lower rate (again determined at equilibrium).

As it turns out the the equilibrium computation in the single return case can be extended to the network topology in Figure 1c. Here an arriving user must not only decide when to arrive, but also which queue to join. On completion of the first service, a fraction of the served users are routed to the other queue for a second service, where they join the back of the line. Once the second service is completed they are routed out of the network. Leveraging the proof of existence for the equilibrium in the single return case, we prove the existence of a Nash equilibrium arrival profile in this case as well. Once again, we observe a similar herding behavior at equilibrium. Note that, this implies that a user arriving at station 'A' (say) cannot arbitrarily improve her disutility by arriving at station 'B' instead. However, proving uniqueness of the Nash equilibrium profile turns out to be much more complicated. In Theorem 7 we prove that the Nash equilibrium is unique provided that the queueing delay faced by a user arriving at time t is equal in either queue. This, of course, need not be the case in general. The complication in the proof arises from the fact that the fraction of time server i $(= 1,\ 2)$ spends serving users routed from server $j = \mod (i, 2) + 1$ is dependent on the fraction of time server j spends serving users routed from server i. This results in a rather complicated dependency structure that prevents a straightforward resolution to uniqueness computation.

Here's a brief overview of the paper. In Section 2 we present a short literature review of the relevant prior art. We present our mathematical model of the fluid queueing network in Section 3, with special attention paid to the workload process. We commence our analysis of the game in Section 4 with the single queue and infinite return case. In Section 5 we study the single queue and single return case and leverage these results for the parallel queue/cross-feedback case in Section 6. We end with a brief summary and discussion of open problems in Section 7.

2. Literature Review

There is a long history of modeling strategic behavior in queueing networks. The book [5] is an excellent compendium of the existing literature up to 2003. Much of the research effort in this direction has been focused on strategic behavior in single queues, largely ignoring routing effects. There is an earlier stream of research initiated by Braess [2] where given fixed routes between two locations, users must strategically choose a route to take from the first location to the next that minimizes a delay cost. Another stream of research corresponds to the question of when to arrive at a queue and how this decision can be regulated. This stream of research was pioneered by Naor [11]. This paper is closely related to the latter research sequence.

Our investigations are directly motivated by the analysis in [6], where strategic users choose a time epoch to arrive and a route to take through the network *a priori* to arrival; this makes the associated game one-shot. The authors identify the non-atomic Nash equilibrium arrival and routing profile of the game by showing that, at equilibrium, the network "collapses" to a parallel queue network. It was noted in that paper that feedback routes could significantly impact the equilibrium computation. In this paper we do not allow selfish routing (it is probabilistic and fixed), since our goal is to understand how feedback routing will affect Nash equilibrium arrival and routing decisions. It should be noted that, to the best of our knowledge, besides [6] all of the remaining literature on arrival timing decisions have concerned single server queues; see [4, 8, 10]. The current paper will be first to study the impact of routing/network topology on equilibrium strategies.

It is important to contrast the setting of the current paper with another sequence of research that appears similar. [12] consider a two queue multi-class network, where users must visit both queues for service. Users strategically choose which sequence of queues to visit, and the authors prove that the network is unstable when users follow Nash equilibrium strategies. On the other hand, [1] consider a two queue single class network where, again, the users must visit both queues in either sequence and strategically route themselves. Under overload conditions, the authors consider three games: one where a finite number of users are present when service starts and users sequentially route themselves. Thus each agent responds to the observed decisions made by the others. In the second game, a finite number of arrivals occur over a short time horizon (thus, there is order in the arrivals) and finally a game where users arrive according to a Poisson process. Most interestingly, the authors prove that the price of anarchy in the first game is *lower* than alternative routing schemes. In both of these papers, the network topology does not affect the arrival decisions of the users; they must necessarily visit both stations in sequence. The only strategic decision the users must make is which sequence of queues to visit. In this paper, users must make that decision in consort with arrival timing decisions.

3. Mathematical Model

The network architectures we consider are depicted in Figure 1. The queues we consider are single-class, and are assumed to follow a FIFO service schedule and a non-idling service policy. We model each queue in the network as a single server fluid queue that offers service rate μ. We assume that the total volume of non-atomic users arriving at the system is $\Lambda = 1$. A pure strategy for a user $c \in [0, 1]$ is simply the arrival epoch $t_c \in (-\infty, +\infty)$, and a mixed strategy is a probability distribution F_c with support in $(-\infty, +\infty)$. The cumulative strategy profile is defined as $F := \int_0^1 F_c m(dc)$, where $m(\cdot)$ is the Lebesgue measure. This is, in essence, an average of the mixed strategies followed by all the users. $F(t)$ is the cumulative number of users who will arrive to the system by t on average.

Consider the feedback queue in Figure 1a. The workload, or virtual waiting time, process, when each user follows a Nash equilibrium strategy, can be straightforwardly written down as

$$w(t) = \begin{cases} \mu^{-1}F(t) & t \leq 0, \\ \mu^{-1}F(t) + pt - t & t > 0. \end{cases} \tag{1}$$

This equation follows as a consequence of Lemma 2 in [6] that shows that the server does not idle when it follows a non-preemptive service policy and the traffic follows an equilibrium strategy. Note that $p\mu t$ is the fraction of served users up to time t who are fedback for a repeat service.

Next, in the single return case depicted in Figure 1b, the workload in the system is determined by the amount of time the server has spent serving users fedback for a second service. Let $\Omega \subset [0, \infty)$ be the set of all time instants where feedback users are served and let $A_t := \{s \in \Omega : s \leq t\}$. Then, the workload process is

$$w(t) = \begin{cases} \mu^{-1}F(t) & t \leq 0, \\ \mu^{-1}F(t) + p(t - m(A_t)) - t & t > 0, \end{cases} \tag{2}$$

where $m(\cdot)$ is the lebesgue measure.

Finally, consider the two queue cross-feedback network. It is beneficial to introduce the 2×2 routing matrix $\mathbf{P}$, where the entry $p_{i,j}$ is the fraction of fluid users who are routed from node i to node j. The workload process at time t is the column vector $w(t) = (w_1(t), w_2(t))$, where w_i is the workload in node i. Using the fluid model in [7], it follows that

$$w(t) = \begin{cases} \mathbf{M}^{-1}\left(F_1(t), F_2(t)\right) & t \leq 0, \\ \mathbf{M}^{-1}\left(F_1(t), F_2(t)\right) - (1,1)t + \mathbf{P}\mathbf{A}_t(1,1), & t > 0, \end{cases} \tag{3}$$

where $\mathbf{M} = \mathrm{diag}(\mu_1^{-1}, \mu_2^{-1})$ is the diagonal matrix of the inverse service rates, and $\mathbf{A}_t = \mathrm{diag}(t - A_{1,t}, t - A_{2,t})$, where $A_{1,t}$ follows the definition of A_t in (2) - albeit, $A_{1,t}$ is amount of time spent serving users routed from queue 2 (and vice-versa).

The cost function/disutility we consider generalizes the cost function introduced in [9] by incorporating the expected cost of returning or (potentially) being fedback (in the Figures 1a and 1b) or routed between queues (in Figure 1c). To be precise, the disutility of arriving at time t is defined as:

$$C(t) = \alpha w(t) + \beta(t + w(t)) + \gamma G(t), \tag{4}$$

where $\gamma > 0$ and $G(t)$ is the additional waiting time cost incurred by feedback/routing of users. Note that we choose to separate the extra waiting cost so that its impact can be explicitly quantified. The precise form of the additional waiting cost depends on the specific architecture under consideration. Thus, we delay a full description of the additional cost. Finally, following the description in [6] we define the equilibrium condition:

Definition 1 (Equilibrium Condition)**.** *The cost of arriving at the queue must be a constant in the support of the equilibrium arrival profile $F^\star$.*

4. Single Queue, Infinite Return

We first consider the case where each user can re-enter the queue after service with probability p and potentially an unbounded number of times. This case is mathematically simpler to study, but also provides insight into how feedback influences the equilibrium.

4.1. Individual Cost

Every user faces the "risk" of returning after completing service, which increases her overall waiting time through the queue. We denote the waiting time after the ith service for a (potential) user arriving at t by $R_i(t)$ and the overall additional waiting time by $R(t) = \sum_{i=1}^{N_i} R_i(t)$, where N_i is the number of times user i returns. We set $R(t) = 0$ if a user does not return. Now, let $T_i(t)$ be the time when the user arriving at time t returns for the ith potential return. We have

$$T_1(t) = \begin{cases} \frac{F(t)}{\mu}, & \text{if } t \leq 0 \\ t + \frac{Q(t)}{\mu}, & \text{if } t > 0, \end{cases} \tag{5}$$

and $T_{i+1}(t) = T_i(t) + \frac{Q(T_i(t))}{\mu}$. Consequently, $R_i(t) = \frac{Q(T_i(t))}{\mu}$. Then, the total cost of potential returns, $G(t)$, is defined as

$$G(t) := \sum_{i=1}^{\infty} p^i R_i(t) = \sum_{i=1}^{\infty} p^i \frac{Q(T_i(t))}{\mu}. \tag{6}$$

The total cost a user arriving at time t faces is

$$C(t) = \begin{cases} \frac{(\alpha+\beta)Q(t)}{\mu} - \alpha t + \gamma G(t), & \text{if } t \le 0 \\ \frac{(\alpha+\beta)Q(t)}{\mu} + \beta t + \gamma G(t), & \text{if } t > 0. \end{cases} \tag{7}$$

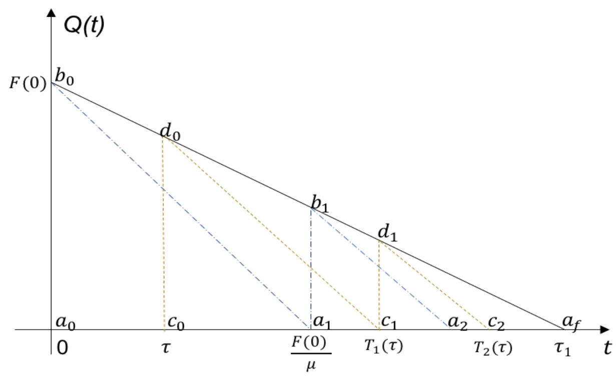


Figure 2: Return times of users arriving at time 0 and τ.

Figure 2 shows the (potential) return times of a user arriving at time 0 and a user arriving at time τ. Notice that triangles $a_0b_0a_1$ and $c_0d_0c_1$, triangles $a_1b_1a_2$ and $c_1d_1c_2$, and triangles $a_0b_0a_f$ and $a_1b_1a_f$ are similar to each other. From Thales' Theorem it follows that

$$\frac{|c_1d_1|}{|c_0d_0|} = \frac{|a_1b_1|}{|a_0b_0|} = \frac{|a_fa_1|}{|a_fa_0|}, \tag{8}$$

where $|\cdot|$ represents the Euclidean length of a line segment. τ_1 is the time at which the last user leaves the system. In [6] it is shown that the server does not idle in the arrival horizon at equilibrium. This result continues to hold even with user feedback. If p fraction of the total arriving population is fedback after service, it is straightforward to argue that $\mu\tau_1 = 1 + p\mu\tau_1$ implying that $\tau_1 = \frac{1}{\mu(1-p)}$. In Theorem 2 we show that the equilibrium queue length, $Q^\star(t)$, is necessarily a linear function of time. On the other hand, suppose the queue length $Q(t)$ is indeed linear. Figure 2 shows that $|c_0d_0| = Q(\tau)$, $|c_1d_1| = Q(T_1(\tau))$. Next, $|a_fa_0| = \tau_1$ and consequently $|a_fa_1| = |a_fa_0| - |a_1a_0| = \tau_1 - \mu^{-1}F(0)$. From (8) it follows that $|c_1d_1||a_fa_0| = |a_fa_1||c_0d_0|$, implying that $Q(T_1(\tau)) = (1-(1-p)F(0))Q(\tau)$. By recursion, it can be shown that

$$Q(T_{i+1}(\tau)) = (1-(1-p)F(0))Q(T_i(\tau)). \tag{9}$$

Now, let $q := 1 - (1-p)F(0)$, then for a user (potentially) arriving at time t,

$$G(t) = p\mu^{-1} Q(T_1(t)) \sum_{n=0}^{\infty} (pq)^n.$$

Since $pq < 1$ it follows that

$$G(t) = \frac{pQ(T_1(t))}{\mu(1-pq)}. \tag{10}$$

Finally, from equation (5), (6), (7) and (1), it follows that the cost of arriving at time t is

$$C(t) = \begin{cases} \frac{(\alpha+\beta)F(t)}{\mu} - \alpha t + \frac{\gamma p Q(\frac{F(t)}{\mu})}{\mu(1-pq)}, & \text{if } t \leq 0 \\ \frac{(\alpha+\beta)(F(t)-\mu t + p\mu t)}{\mu} + \beta t + \frac{\gamma p Q(t + \frac{Q(t)}{\mu})}{\mu(1-pq)}, & \text{if } t > 0. \end{cases} \tag{11}$$

4.2. Equilibrium Arrival Profile

Let τ_0 be the time at which the first arrival occurs and τ_f the departure time of the last user to arrive. As noted before, in [6] it is shown that at equilibrium the server does not idle and the last arrival will not face any waiting time, implying that $\tau_f = \tau_1 = (\mu(1-p))^{-1}$. Using the equilibrium condition in Definition 1, the equilibrium cost of arriving at time epoch t in the support of the support of the equilibrium arrival profile satisfies $C(t) = C(\tau_f) = \beta(\mu(1-p))^{-1}$. Note too, that the support cannot have gaps, else a strategic user could arbitrarily reduce her disutility by arriving at such a time instant. It follows that the equilibrium support must be $[\tau_0, \tau_f]$, and $C(\tau_0) = \frac{\beta}{\mu(1-p)}$. Then, (11) and the fact that $F(\tau_0) = 0$ imply $\tau_0 = \frac{\gamma p F(0)}{\mu\alpha(1-pq)} - \frac{\beta}{\mu\alpha(1-p)}$. This leads us to our first result.

Theorem 1. *Suppose $Q^\star$ is piecewise-linear on $(\tau_0, 0]$ and $(0, \tau_1)$ (respectively), then there exists a piecewise-linear Nash equilibrium arrival profile $F^\star$.*

Proof. First, (1) implies that it suffices to find the coefficients of the equilibrium linear queue length $Q^\star$ to establish the existence of $F^\star$. Recall, from (7) and (10), that

$$C(t) = \begin{cases} \frac{(\alpha+\beta)Q^\star(t)}{\mu} - \alpha t + \frac{\gamma p Q^\star\left(\frac{Q^\star(t)}{\mu}\right)}{\mu(1-pq)}, & \text{if } t \leq 0 \\ \frac{(\alpha+\beta)Q^\star(t)}{\mu} + \beta t + \frac{\gamma p Q^\star\left(t + \frac{Q^\star(t)}{\mu}\right)}{\mu(1-pq)}, & \text{if } t > 0. \end{cases}$$

Suppose $t > 0$, then at any Nash equilibrium it follows that

$$\frac{(\alpha+\beta)Q^\star(t)}{\mu} + \beta t + \frac{\gamma p Q^\star\left(t + \frac{Q^\star(t)}{\mu}\right)}{\mu(1-pq)} - \frac{\beta}{\mu\alpha(1-p)} = 0.$$

Suppose $Q^\star(t) = a_1 t + a_2$ for $t > 0$, where $(a_1, a_2) \in \mathbb{R}^2$, then $0 =$

$$t\left(\frac{(\alpha+\beta)a_1}{\mu} + \beta + \frac{\gamma p a_1}{\mu(1-pq)} + \frac{\gamma p a_1^2}{\mu^2(1-pq)}\right) + \left(\frac{(\alpha+\beta)a_2}{\mu} + \frac{\gamma p a_2}{\mu(1-pq)} + \frac{\gamma p a_1 a_2}{\mu^2(1-pq)} - \frac{\beta}{\alpha\mu(1-p)}\right). \tag{12}$$

It follows that $a_1^2 \frac{\gamma p}{\mu^2(1-pq)} + a_1\left(\frac{(\alpha+\beta)}{\mu} + \frac{\gamma p}{\mu(1-pq)}\right) + \beta = 0$ and $a_2\left(\frac{\alpha+\beta}{\mu} + \frac{\gamma p}{\mu(1-pq)}\right) + a_1 a_2 \frac{\gamma p}{\mu^2(1-pq)} - \frac{\beta}{\alpha\mu(1-p)} = 0$. Note that the former (quadratic) equation is a function of a_1 alone. A real-valued solution for a_1 exists if and only if the discriminant satisfies

$$\left(\frac{\alpha+\beta}{\mu} + \frac{\gamma p}{\mu(1-pq)}\right)^2 - 4\beta\frac{\gamma p}{\mu^2(1-pq)} \geq 0. \tag{13}$$

Let a_1^* be the solution of the quadratic equation, then

$$a_2^* = \frac{\beta}{\alpha\mu(1-p)}\left(\frac{\alpha+\beta}{\mu} - \frac{\gamma p}{\mu(1-pq)} + a_1^*\frac{\gamma p}{\mu^2(1-pq)}\right)^{-1}.$$

Thus, the equilibrium queue length when $t \geq 0$ is $Q^\star(t) = a_1^* t + a_2^*$, and denote it as $Q^\star(t)|_{t\geq 0}$.

Likewise, at any Nash equilibrium, if $t \leq 0$,

$$\frac{(\alpha+\beta)Q^\star(t)}{\mu} - \alpha t + \frac{\gamma p Q^\star\left(\frac{Q^\star(t)}{\mu}\right)}{\mu(1-pq)} - \frac{\beta}{\mu\alpha(1-p)} = 0.$$

Since $\frac{Q^\star(t)}{\mu} > 0$, we use $Q^*(t)|_{t>0}$ to expand the term $\frac{\gamma p Q^\star(\cdot)}{\mu(1-pq)}$. Suppose $Q^\star(t) = b_1 t + b_2$ for $t \leq 0$, then it follows that

$$\frac{(\alpha+\beta)(b_1 t + b_2)}{\mu} - \alpha t + \frac{\gamma p(a_1^* b_1 t + a_1^* b_2)}{\mu^2(1-pq)} + \frac{\gamma p a_2^*}{\mu(1-pq)} - \frac{\beta}{\mu\alpha(1-p)} = 0. \tag{14}$$

Clearly, $b_1 a_1^* \frac{\gamma p}{\mu^2(1-pq)} + b_1\frac{\alpha+\beta}{\mu} - \alpha = 0$ and $b_2\left(\frac{\alpha+\beta}{\mu} + \frac{\gamma p a_1^*}{\mu^2(1-pq)} + \frac{\gamma p}{\mu(1-pq)}\right) - \frac{\beta}{\mu\alpha(1-p)} = 0$. Since we know that a_1^* and a_2^* exist provided condition (4.2) holds, there exist unique solutions for b_1^* and b_2^*. Thus $Q^\star(t) = b_1^* t + b_2^*$ is the linear equilibrium queue length when $t \leq 0$ and we denote it as $Q^\star(t)|_{t\leq 0}$. $\square$

Note that the linear equilibrium arrival profile in Theorem 1 can be expressed as

$$F^\star(t) = \begin{cases} (1 - F^\star(0))(1-p)\mu t + F^\star(0), & \text{if } t > 0 \\ F^\star(0)\left(\frac{\beta}{\mu\alpha(1-p)} - \frac{\gamma p F^\star(0)}{\mu\alpha(1-pq)}\right)^{-1} t + F^\star(0), & \text{if } t \leq 0. \end{cases} \tag{15}$$

Thus, it suffices to know the exact expression for $F^\star(0)$ to solve for the equilibrium arrival profile. From (11) it follows that

$$C(0) = \frac{\beta}{\mu(1-p)} = \frac{(\alpha+\beta)F^\star(0)}{\mu} + \frac{\gamma p((1-F^\star(0))(1-p)F^\star(0) + pF^\star(0))}{\mu(1-pq)}.$$

Solving for $F^\star(0)$, we obtain

$$F^\star(0) = \frac{-\sqrt{(\beta p - \gamma p + (\alpha+\beta)(p-1))^2 - \beta(4p(\alpha+\beta)(p-1) - 4\gamma p(p-1))} + \gamma p - \beta p - (\alpha+\beta)(p-1)}{2p(\alpha+\beta)(1-p) + 2\gamma p(p-1)}.$$

Our next result proves the (partial) necessity of the linearity of an equilibrium arrival profile by proving that $F^\star$ is unique modulo a condition on the parameters of the cost function.

Theorem 2. *$F^\star(t)$ is the only Nash equilibrium if $\frac{(1-p)(\alpha+\beta)}{p\gamma} > 1$.*

First, though, consider the following lemmas.

Lemma 1. *The queue length function is continuous at any Nash equilibrium.*

Proof. Note that the only discontinuities possible are upwards jumps, on account of (possible) singularities in the equilibrium arrival profile. Suppose $Q^\star(t)$ is discontinuous at t_x such that $Q^\star(t_x^+) \le Q^\star(t_x^-)$. Then we claim that the user arriving at $Q^\star(t_x^-)$ has an incentive to arrive at $Q^\star(t_x^+)$ since by delaying her arrival by an infinitesimal amount of time she can reduce her cost. Thus $Q^\star$ must necessarily be continuous. □

Lemma 2. *In any Nash equilibrium, $Q^\star(T_n(t)) \to 0$ as $n \to \infty$.*

Proof. Suppose $Q^\star(t_0) = 0$ for some $t_0 \in [0, \tau_1)$, then it is trivial that customers arriving after t_0 will have incentive to switch to t_0, which contradicts with the equilibrium condition. Hence, $Q^\star(t) > 0$ for $\forall t \in [0, \tau_1)$. By definition of $T_i(t)$ introduced in equation (5), $T_i(t) \to \tau_1$. Since $Q^\star(t)$ is a continuous function by lemma 1, $Q^\star(T_n(t)) \to 0$ as $n \to \infty$. □

Lemma 3. *Let $c_0 = (\alpha+\beta)$, $c_i = p^i\gamma$ and $a_0 = \delta \neq 0$. For sequence $\{a_n\}$, if*

$$a_n = \begin{cases} -\frac{1}{c_0}\sum_{j=1}^{\infty} c_j a_j & if\ n = 0 \\ -\frac{1}{c_0}\sum_{j=n+1}^{\infty} c_{j-n} a_j - \frac{1}{c_0}\beta \sum_{j=0}^{n-1} a_j, & if\ n \ge 1, \end{cases} \tag{16}$$

and

$$0 = \sum_{j=0}^{\infty} a_j, \tag{17}$$

then a_n does not converge to 0 if $\frac{(1-p)(\alpha+\beta)}{p\gamma} > 1$.

Proof. Suppose $a_n \to 0$ as $n \to \infty$. Then $\forall \epsilon \to 0$, $\exists N_1$ such that $|a_n| \le \epsilon$ for $n > N_1$. By equation (16), for $n > N_1$,

$$-\epsilon < -\frac{1}{c_0}\sum_{j=n+1}^{\infty} c_{j-n}a_j - \frac{\beta}{c_0}\sum_{j=0}^{n-1} < -\frac{\epsilon}{c_0}\sum_{j=n+1}^{\infty} c_{j-n} - \frac{\beta}{c_0}\sum_{j=0}^{n-1} a_j.$$

Furthermore, $\frac{\beta}{c_0}\sum_{j=0}^{n-1} a_j < \epsilon(1 - \frac{\gamma}{c_0}\frac{p}{1-p})$. By equation (17), $\frac{\beta}{c_0}\sum_{j=0}^{n-1} a_j \to 0$ as $n \to \infty$. We claim that $\epsilon(1 - \frac{\gamma}{c_0}\frac{p}{1-p}) \ge 0$. Since otherwise, we can find $N_2 \ge N_1$ such that $\frac{\beta}{c_0}\sum_{j=0}^{n-1} a_j > \epsilon(1 - \frac{\gamma}{c_0}\frac{p}{1-p})$, which leads to a contradiction. Therefore, a_n does not converge to 0 if $\epsilon(1 - \frac{\gamma}{c_0}\frac{p}{1-p}) < 0$. Since $\epsilon > 0$ is arbitrary, it follows that $\frac{(1-p)(\alpha+\beta)}{p\gamma} > 1$. □

Proof of Theorem 2. Suppose there is another equilibrium $F^{**}(t)$ with a corresponding queue length $Q^{**}(t)$, such that $Q^{**}(t_x) \neq Q^{\star}(t_x)$ for some t_x. Let $\delta = Q^{**}(t_x) - Q^{\star}(t_x)$. Also, let $T_i(t)$ and $T_i'(t)$ be the time sequences as defined in equation (5) corresponding to $Q^{*}(t)$ and $Q^{**}(t)$ respectively. By equation (7) and definition of the equilibrium it follows that

$$\frac{(\alpha+\beta)Q^{**}(t)}{\mu} - \alpha t + \sum_{i=1}^{\infty} p^i\gamma\frac{Q^{**}(T_i'(t))}{\mu} = \frac{(\alpha+\beta)Q^{\star}(t)}{\mu} - \alpha t + \sum_{i=1}^{\infty} p^i\gamma\frac{Q^{\star}(T_i(t))}{\mu}, \text{ if } t \le 0, \quad (18)$$

$$\frac{(\alpha+\beta)Q^{**}(t)}{\mu} + \beta t + \sum_{i=1}^{\infty} p^i\gamma\frac{Q^{**}(T_i'(t))}{\mu} = \frac{(\alpha+\beta)Q^{\star}(t)}{\mu} + \beta t + \sum_{i=1}^{\infty} p^i\gamma\frac{Q^{\star}(T_i(t))}{\mu}, \text{ if } t > 0. \quad (19)$$

From equations (18) and (19), we obtain

$$(\alpha+\beta)\Big(Q^{**}(t_x) - Q^{*}(t_x)\Big) = \sum_{i=1}^{\infty} p^i\gamma\Big(Q^{\star}(T_i(t_x)) - Q^{**}(T_i'(t_x))\Big).$$

Let $a_0 = \delta$, $a_i = Q^{\star}(T_i(t_x)) - Q^{**}(T_i'(t_x))$ for $i > 0$, $c_0 = (\alpha+\beta)$ and $c_i = p^i\gamma$ for $i > 0$. Substituting in the expression above, we obtain $-c_0a_0 = \sum_{i=1}^{\infty} c_ia_i$. Similarly, if we notice that $T_{i+1}(t) - T_{i+1}'(t) = \mu^{-1}\sum_{j=0}^{i}\Big(Q^{*}(T_i(t)) - Q^{**}(T_i'(t))\Big) = \mu^{-1}\sum_{j=0}^{i} a_j$, then for the users arriving at time $T_i(t_x)$, we have

$$-c_0a_i - \beta\sum_{j=1}^{i-1} a_j = \sum_{j=i+1}^{\infty} c_{j-i}a_j.$$

Also, by Lemma 2, $T_n(t) \to \tau_1$ and $T_n'(t) \to \tau_1$ as $n \to \infty$. We, thus, obtain $T_{i+1}(t) - T_{i+1}'(t) = \mu^{-1}\sum_{j=0}^{i} a_j \to 0$ as $i \to \infty$, which implies that $\sum_{j=0}^{i} a_j \to 0$ as $i \to \infty$. Applying Lemma 3 it follows that a_n does not converge to 0 if $\frac{(1-p)(\alpha+\beta)}{p\gamma} > 1$. This, however, contradicts Lemma 2 where $Q^{\star}(T_n(t)) \to 0$ as $n \to \infty$, and further $a_n \to 0$ as $n \to \infty$. Hence, it cannot be the case that

$Q^{**}(t_x) - Q^\star(t_x) = \delta \neq 0$. Therefore, there does not exist a point t_x such that $Q^{**}(t_x) \neq Q^\star(t_x)$. i.e. $Q^{**}(t)$ and $Q^\star(t)$ must be equal pointwise. Consequently, $F^\star(t)$ is the unique Nash equilibrium arrival profile, provided $\frac{(1-p)\alpha}{p\gamma} > 1$. $\square$

4.3. Price of Anarchy

We define the social cost of following strategy profile F as

$$J(F) := \int C_F(t)dF(t).$$

Let J_{opt} denote the social cost of following the optimal arrival profile and J_{eq} as the social cost of following the equilibrium arrival strategy profile. It is expected that J_{eq} will be greater than J_{opt}. To characterize the inefficiency of the equilibrium arrival profile, consider the price of anarchy (PoA) η, defined as:

$$\eta = \sup_{F \in \mathcal{N}} \frac{J_{eq}}{J_{opt}(F)},$$

where $\mathcal{N}$ is the set of Nash equilibrium profiles. For brevity, let J_{eq} represent the equilibrium social cost when the Nash equilibrium strategy is unique.

Theorem 3. *The price of anarchy of the single queue, infinite return, arrival game is 2, when* $\frac{(1-p)(\alpha+\beta)}{p\gamma} > 1$.

Proof. Let $C_{F^\star}$ be the equilibrium disutility of following the unique Nash equilibrium strategy profile $F^\star$. Recall that $C_{F^\star}(t) = \beta(\mu(1-p))^{-1}$, which implies

$$J_{eq} = \frac{\beta}{\mu(1-p)}.$$

Note that we have assumed the total about of arrivals to be 1.

Next, the socially optimal arrival strategy would be to arrange jobs such that they do not face any delay. In other words, the *effective* arrival rate, taking the feedback effect into account, should be equal to the service rate. Let the optimal arrival profile be

$$F'_{opt}(t) = \mu(1-p)$$

for $t > 0$. In this case, the effective arrival rate is $\sum_{i=0}^{\infty} \mu(1-p)p^i = \mu$. It follows that $C_{opt}(t) = \beta t$, implying

$$J_{opt} = \int_0^{\tau_1} \mu\beta t(1-p)dt = \frac{\beta}{2\mu(1-p)}.$$

Consequently, the price of anarchy for the infinite return case is

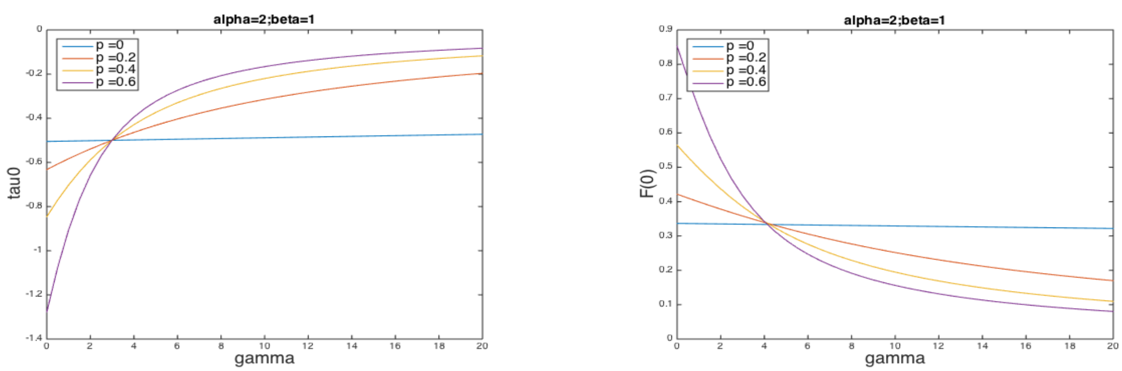
$$\eta = \frac{J_{eq}}{J_{opt}} = 2. \tag{20}$$

$\square$

4.4. Equilibrium Behavior

In this section we take a closer look at how p and γ influence the equilibrium profile. Recall from (15) that $F^{\star}(t)$ is piecewise linear on $(\tau_0, 0]$ and $(0, \tau_1)$ and can be expressed in terms of τ_0 and $F^{\star}(0)$. It suffices to consider how τ_0 and $F^{\star}(0)$ depend on p and γ.

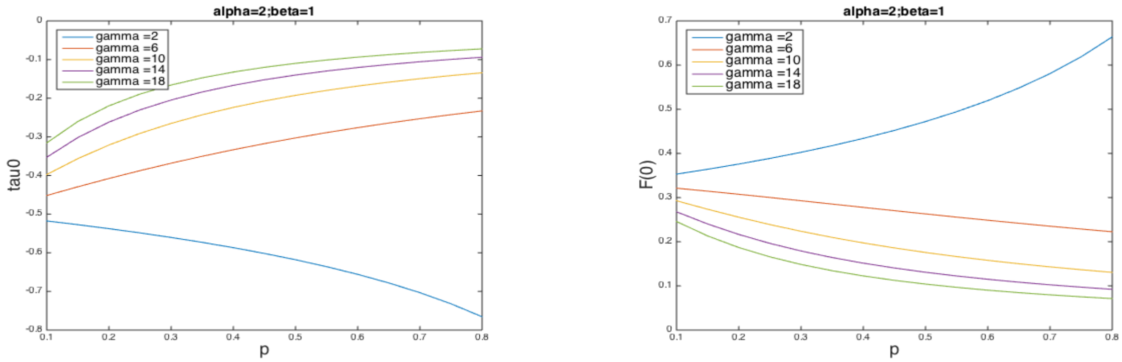
Figure 3 shows how γ influences τ_0 and F_0. Essentially, $F^{\star}(0)$ is a monotone decreasing convex function of γ and τ_0 is a monotone increasing concave function of γ. A larger p (i.e. return rate) magnifies this trend. Thus, as γ grows, the cumulative number of arrivals before time 0 (when service commences) decreases while the first arrival epoch increases. In essence, this implies that users are sensitive to the risk of being fedback to the end of the line, and are thus unwilling to arrive earlier than necessary.



(a) τ_0-γ (b) $F^{\star}(0)$-γ

Figure 3: Influence of gamma on τ_0 and $F^{\star}(0)$

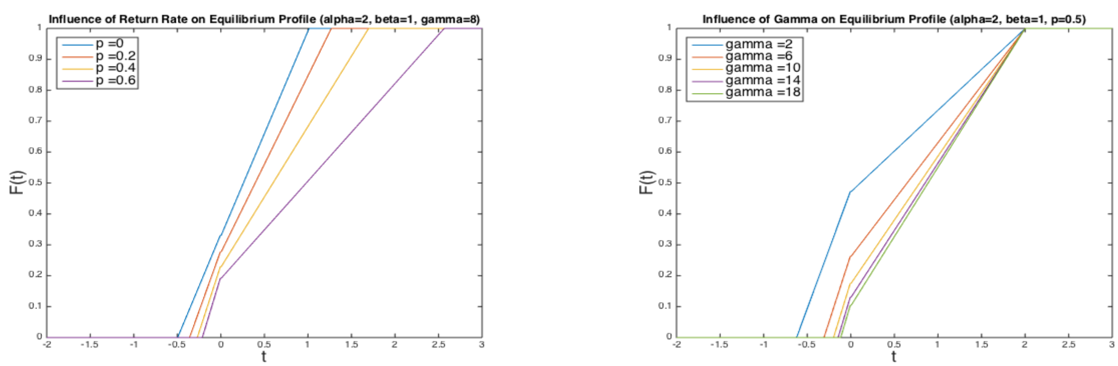
This is further confirmed by Figure 4 which shows the influence of p. In this case, we notice that for small γ's, both τ_0 increases with p while $F^{\star}(0)$ decreases with p. This implies that when p increases, the earliest arrival occurs earlier and the number of arrivals before service starts increases as well. As γ increases, both τ_0 is increasing with p and $F^{\star}(0)$ is decreasing with p. In this case, the first arrival occurs later while the number of arrivals before time 0 is smaller as p increases. Furthermore, as p increases both τ_0 and $F^{\star}(0)$ changes much faster as γ increases.



(a) τ_0-p (b) $F^{\star}(0)$-p

Figure 4: Influence of p on τ_0 and $F^{\star}(0)$

Figure 5 shows the equilibrium arrival profiles under different p and γ. It is useful to contrast these curves with the case with no feedback (or $p = 0$). In that case, [9] proved that the Nash equilibrium arrival profile is uniform (which can also be observed in Figure 5a). As noted before, with feedback, strategic users choose to arrive closer to the start of service, due to the increased risk of being sent back to the end of the line.



(a) Equilibrium-p (b) Equilibrium-γ

Figure 5: Influence of γ and p on the equilibrium

5. Single Queue, Single Return

We now consider the case where a user is fed back only a finite number of times. The analysis of the single return case, itself, turns out to be quite complicated, so we only focus on that. We believe that the generalization to the multiple, but finite, return case should be relatively straightforward using the mathematical analysis we develop here, but we do not pursue this.

5.1. Individual Cost

The expected "risk" of being returned to the end of line after completing service for a user who arrived at time t is $G(t) = pR(t)$, where

$$R(t) = \begin{cases} \frac{1}{\mu} Q\left(\frac{F(t)}{\mu}\right), & \text{if } t \leq 0 \\ \frac{1}{\mu} Q\left(p(t - m(A_t)) + \frac{F(t)}{\mu}\right), & \text{if } t > 0, \end{cases} \tag{21}$$

where $m(\cdot)$ is the Lebesgue measure and A_t is the set of time instances before t at which returning users are served, as defined in (2). Thus the cost of arriving at time t is $C(t) = (\alpha+\beta)w(t) + \beta t + \gamma p R(t)$, and $w(t)$ is defined in (2).

The additional waiting time effected by the previous returns is much trickier to compute now since the effective 'return rate' is time dependent. Essentially, a user who has returned once will not return again, and consequently the effective return rate is less than or equal to $p\mu$, in contrast to the infinite return case. Intuitively, this phenomenon starts as soon as the first user to be fedback receives her second service. Our first result establishes the existence of a Nash equilibrium arrival profile when the queue length is piecewise linear.

Theorem 4. *There exists a Nash Equilibrium such that the corresponding queue length function Q is piecewise linear.*

Proof. Suppose $Q(t),\ t > 0$ is linear, we conclude that $\frac{R(t)}{w(t)} = \frac{1+p-F(0)}{1+p}$ for $t > 0$ using the geometric argument aluded to in Figure 2. Then, the cost of arriving at time t is

$$\begin{aligned} C(t) &= (\alpha+\beta)\ w(t) + \beta t + \gamma p R(t) \\ &= \left(\gamma p \frac{1+p-F(0)}{1+p} + \alpha + \beta\right) w(t) + \beta t, \end{aligned}$$

where $w(t) = \mu^{-1}Q(t) - \min(0, t)$. Now, suppose $t > 0,\ t + \delta t \leq 1 + p$, and $\delta t > 0$. Then,

$$\begin{aligned} C(t) &= \left(\gamma p \frac{1+p-F(0)}{1+p} + \alpha + \beta\right) \frac{Q(t)}{\mu} + \beta t, \text{ and} \\ C(t+\delta t) &= \left(\gamma p \frac{1+p-F(0)}{1+p} + \alpha + \beta\right) \frac{Q(t+\delta t)}{\mu} + \beta (t+\delta t). \end{aligned}$$

Since the arrival cost at equilibrium is constant in the arrival horizon, by setting $C(t) = C(t+\delta t)$ we obtain

$$\frac{Q(t+\delta t)-Q(t)}{\delta t} = \frac{-\beta\mu}{\gamma p\frac{1+p-F(0)}{1+p}+\alpha+\beta} =: k_1.$$

Letting $\delta t \to 0$, it follows that the first order derivative of Q is constant for $t > 0$. Thus, it follows that $Q(t) = k_1 t + b_1$ for $t > 0$, where b_1 can be determined by solving the corresponding differential equation and using the boundary condition $Q_1(\tau_1) = 0$.

Next, suppose $t < 0,\ t+\delta t \leq 0$, and $\delta t > 0$. Then,

$$\begin{aligned}
C(t) &= (\alpha+\beta)w(t)+\beta t+\gamma pR(t)\\
&= \frac{(\alpha+\beta)Q(t)}{\mu}+\beta t+\gamma p\mu^{-1}Q\left(\frac{Q(t)}{\mu}\right)\\
&= \frac{k_1\gamma p\mu+\alpha+\beta}{\mu}Q(t)+\beta t+\gamma pb_1, \text{ and}\\
C(t+\delta t) &= \frac{k_1\gamma p\mu+\alpha+\beta}{\mu}Q(t+\delta t)+\beta(t+\delta t)+\gamma pb_1.
\end{aligned}$$

Hence,

$$\frac{Q(t+\delta t)-Q(t)}{\delta t} = \frac{\mu\beta}{k_1\gamma p\mu+\alpha+\beta} =: k_2,$$

implying that the first order derivative of $Q(t)$ $t \leq 0$ is a constant, and we have $Q(t) = k_2 t + b_2$ where b_2 can be determined by solving the differential equation with boundary condition $Q(\tau_0) = 0$. Therefore, with such a piecewise linear $Q(t)$, the cost is invariant over the arrival horizon $[\tau_0, \tau_1]$. The corresponding $F(t)$ is the Nash equilibrium arrival profile. □

Note that this result does not provide an expression for the equilibrium arrival profile $F^\star$. Based on the linearity of the queue length, one might conclude that $F^\star(t)$ is linear. However, this intuition is, in fact, incorrect since the return 'rate' (defined as the fraction of all users who might be fedback to the queue) is now time dependent. On the contrary, the return rate is affected by the users who have already returned once - and they will never return again. We next establish a description of the workload that formalizes this intuition. Recall that $\Omega :=$ $\{$Set of all time instants where returning users are served$\}$ and $m : \mathbb{R}_+ \to \mathbb{R}_+$ the standard Lebesgue measure.

Proposition 1. *Let $t_0 = 0$, and $t_k = t_{k-1} + \frac{Q(t_{k-1})}{\mu}$ $k \geq 1$ be the time point when users in the* *queue at time epoch t_{k-1} are served. Denote $A_t = \{x \in \Omega \ : \ x \leq t\}$. Then, at equilibrium,*

$$\begin{aligned} m(A_{t_0}) &= 0 \\ m(A_{t_{k+1}}) &= m(A_{t_k}) + p(t_k - t_{k-1} - (m(A_{t_k}) - m(A_{t_{k-1}}))) \ \forall k \geq 1. \end{aligned}$$

Proof. Since the server operates without idling at equilibrium the total time spent serving feedback users is $m(\Omega) = \mu^{-1}p$. Clearly, $m(A_t) = 0$ for $t \leq 0$ and $m(A_{t_0}) = 0$. Since $t_1 = \mu^{-1}Q(0)$ is the first time epoch at which a feedback user receives her second service, we have $m(A_{t_1}) = 0$.

From t_1 to t_2, the set of users receiving their second service is precisely those fedback between t_0 and t_1. Since all the users arriving for the first time between t_0 and t_1 will have a chance to return, it follows that $m(A_{t_2}) = m(A_{t_1}) + p(t_1 - t_0)$.

From t_2 to t_3, the set of users being served for the second time is precisely those fedback between t_1 and t_2. On the other hand the workload from users who will not return between t_1 and t_2 is $m(A_{t_2}) - m(A_{t_1})$, implying that the total workload fedback is $p(t_2 - t_1 - (m(A_{t_2}) - m(A_{t_1})))$. Hence $m(A_{t_3}) = m(A_{t_2}) + p(t_2 - t_1 - (m(A_{t_2}) - m(A_{t_1})))$. Continuing in this manner it follows that $m(A_{t_{k+1}}) = m(A_{t_k}) + p(t_k - t_{k-1} - (m(A_{t_k}) - m(A_{t_{k-1}}))) \ \forall k \geq 1$. $\square$

Now, the expected disutility of arriving at time t is

$$C(t) = \begin{cases} \frac{(\alpha+\beta)Q(t)}{\mu} - \alpha t + \gamma p \frac{Q(\frac{F(t)}{\mu})}{\mu}, & \text{if } t \leq 0 \\ \frac{(\alpha+\beta)Q(t)}{\mu} + \beta t + \gamma p \frac{Q(t+\frac{Q(t)}{\mu})}{\mu}, & \text{if } t > 0 \end{cases} \tag{22}$$

where

$$Q(t) = \begin{cases} F(t), & \text{if } t \leq 0 \\ F(t) - \mu t + p\mu(t - m(A_t)), & \text{if } t > 0. \end{cases} \tag{23}$$

Our next result establishes the uniqueness of the equilibrium under a sufficient condition on the parameters of the disutility.

Theorem 5. *The Nash equilibrium is unique if $\frac{\alpha+\beta}{\gamma p} > 1$.*

First, consider the following lemma.

Lemma 4. *Let a_n and b_n be two sequences such that*

$$c_1 a_n + b_n + c_2 a_{n+1} = 0, \tag{24}$$

where $b_0 = 0$, $a_0 = \delta \neq 0$, $b_{n+1} = \sum_{i=0}^{i=n} a_i$. Then a_n does not converge to 0 if $\frac{c_1}{c_2} > 1$.

Proof. Recursively expanding a_n, we obtain:

$$\begin{aligned}
a_{n+1} &= -\frac{c_1}{c_2}a_n - \frac{1}{c_2}b_n \\
&= (-\frac{c_1}{c_2})^{n+1}\delta - \sum_{i=0}^{n}\frac{b_i}{c_2}(-\frac{c_1}{c_2})^{n-i} \\
&= (-\frac{c_1}{c_2})^{n+1}\delta - \sum_{i=0}^{n}(-\frac{c_1}{c_2})^{n-i}\sum_{j=0}^{i-1}a_j \\
&= (-\frac{c_1}{c_2})^{n+1}\delta - \frac{\delta}{c_2}\sum_{j=0}^{n-1}a_j\sum_{i=j+1}^{n}(-\frac{c_1}{c_2})^{n-i} \\
&= (-\frac{c_1}{c_2})^{n+1}\delta - \delta\sum_{j=0}^{n-1}\frac{a_j}{c_1+c_2}(-\frac{c_1}{c_2})^{n}\Bigg((-\frac{c_2}{c_1})^{j+1} - (-\frac{c_2}{c_1})^{n+1}\Bigg).
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_{n+1} - a_n &= \delta(-\frac{c_1}{c_2})^{n}\Bigg(-\frac{c_1}{c_2} - 1 - \frac{1}{(c_1+c_2)}\Bigg(a_{n-1}(-\frac{c_2}{c_1})^{n} - a_{n-1}(-\frac{c_2}{c_1})^{n+1})\Bigg)\Bigg) \\
&= -\frac{\delta}{c_2}\Bigg((c_1+c_2)(-\frac{c_1}{c_2})^{n} + \frac{c_2}{c_1}a_{n-1}\Bigg).
\end{aligned}$$

Suppose $a_n \to 0$ as $n \to \infty$. i.e. $\forall\epsilon > 0$, $\exists N$ such that $|a_n| < \epsilon$ for $\forall n > N$, which implies

$$a_{n+1} - a_n \geq -\frac{\delta}{c_2}\Bigg((c_1+c_2)(-\frac{c_1}{c_2})^{n} + \frac{c_2}{c_1}\epsilon\Bigg).$$

It follows that

$$a_{2n} - a_{2n-1} \geq -\frac{\delta}{c_2}\Bigg(-(c_1+c_2)(\frac{c_1}{c_2})^{2n-1} + \frac{c_2}{c_1}\epsilon\Bigg).$$

Next, let

$$N' = \left\lceil 2^{-1}log_{\frac{c_1}{c_2}}\frac{\epsilon c_2(2+\frac{\delta}{c_1})}{\delta(c_1+c_2)}\right\rceil,$$

so that

$$a_{2n} - a_{2n-1} \geq -\frac{\delta}{c_2}\Bigg(-(c_1+c_2)(\frac{c_1}{c_2})^{2n-1} + \frac{c_2}{c_1}\epsilon\Bigg) > 2\epsilon \tag{25}$$

for all $n > \max\{N, N'\}$. However, this implies either $|a_{2n}| > \epsilon$ or $|a_{2n-1}| > \epsilon$. But, this contradicts our assumption that $|a_n| < \epsilon$, and by *reductio* the proof is complete. □

Proof of Theorem 5. Suppose there are two different Nash equilibria $F^\star(t)$ and $F^{**}(t)$ with corresponding queue length functions $Q^\star(t)$ and $Q^{**}(t)$. We describe the difference of the two equilibriums by $Q^\star(t_x) - Q^{**}(t_x) = \delta > 0$ and then argue by contradiction.

It is argued in theorem 2 that cost of the two equilibrium profiles are equal (i.e. $C(\tau^\star)$) in infinite return case. It is trivial that the argument also applies for single return case. We define two time sequences as follows:

$$T_i(t_x) = \begin{cases} t_x, & \text{if } i = 0 \\ T_{i-1}(t_x) + \frac{Q^\star(T_{i-1}(t_x))}{\mu}, & \text{if } i > 0. \end{cases} \tag{26}$$

$$T'_i(t_x) = \begin{cases} t_x, & \text{if } i = 0 \\ T'_{i-1}(t_x) + \frac{Q^{\star\star}(T'_{i-1}(t_x))}{\mu}, & \text{if } i > 0. \end{cases} \tag{27}$$

Let $C^\star(t)$ and $C^{\star\star}(t)$ be cost functions of $F^\star$ and $F^{\star\star}$ respectively. We have $C^\star(T_n(t)) = C^{\star\star}(T'_n(t))$. i.e.

$$\frac{\alpha+\beta}{\mu}\Big(Q^\star(T_n(t_x)) - Q^{\star\star}(T'_n(t_x))\Big) + \beta\Big(T_n(t) - T'_n(t)\Big) + \frac{\gamma p}{\mu}\Big(Q^\star(T_{n+1}(t_x)) - Q^{\star\star}(T'_{n+1}(t_x))\Big) = 0. \tag{28}$$

WLOG, set $\mu = 1$ and let $\frac{\alpha+\beta}{\beta} = c^\star$, $\frac{\gamma p}{\beta} = c^{\star\star}$, $Q^\star(T_n(t_x)) - Q^{\star\star}(T'_n(t_x)) = a_n$ and $T_n(t) - T'_n(t) = b_n$. With this setting, it can be argued from equation (26) and (27) that $b_{n+1} = \sum_{i=0}^{i=n} a_i$. Using notations as defined above, equation (28) can be written as $c^\star_n + b_n + c^{\star\star} a_{n+1} = 0$.

Lemma 4 implies that a_n diverges if $a_0 = \delta > 0$ and $\frac{c^\star}{c^{\star\star}} > 1$, which contradicts with the claim of Lemma 2 [2]. Hence, we obtain that $a_0 = 0$. i.e. $Q^\star(t) - Q^{\star\star}(t) \equiv 0$ for $\forall t \in [\tau_0, \tau_1]$. □

5.2. Equilibrium Arrival Profile

As noted before, the system does not idle in the support of the equilibrium arrival profile. Recall that τ_1 is the time of service termination. It is straightforward to argue that $\tau_1 = \mu^{-1}(1+p)$, since $\mu^{-1}p$ is the total feedback workload. The equilibrium cost of arriving at τ_1 is $C(\tau_1) = \beta\tau_1 = \beta\mu^{-1}(1+p)$. Since the Nash equilibrium condition in Definition 1 implies that the arrival cost is constant in the support of the equilibrium arrival profile, it follows that

$$C(\tau_0) = C(\tau_1) = \frac{(1+p)\beta}{\mu}. \tag{29}$$

From equation (22), (29) and the fact that $F^\star(\tau_0) = 0$, we have $\tau_0 = (\mu\alpha)^{-1}\left(\gamma p F^\star(0) - \beta(1+p)\right)$.

[2] Note that while Lemma 2 is proved for the infinite-return case the same arguments hold for the single-return case as well.

Now, assume that the equilibrium queue length $Q^\star$ is piecewise linear. We know that $Q^\star(0) = F^\star(0)$ and $Q^\star(\tau_1) = 0$. It follows that the slope of $Q^\star(t)$ for $t \geq 0$ equals

$$\frac{0 - F^\star(0)}{\tau_1 - 0} = -\frac{\mu F^\star(0)}{1+p},$$

and the abscissa/intercept is $F^\star(0)$. On the other hand, if $t < 0$, we know that $Q^\star(\tau_0) = 0$ and $Q^\star(0-) = F^\star(0)$, implying the slope is

$$\frac{F^\star(0) - 0}{0 - \tau_0} = \frac{\mu\alpha F^\star(0)}{\beta(1+p) - \gamma p F^\star(0)}$$

and the abscissa is $F^\star(0)$. Thus, it follows that

$$Q^\star(t) = \begin{cases} -\frac{\mu F^\star(0)}{1+p} t + F^\star(0), & \text{if } t > 0 \\ \frac{\mu\alpha F^\star(0)}{\beta(1+p) - \gamma p F^\star(0)} t + F^\star(0), & \text{if } t \leq 0. \end{cases} \tag{30}$$

Recall, from the proof of Theorem 3, that $\lim_{\delta t \to 0} \frac{Q^\star(t+\delta t) - Q^\star(t)}{\delta t} = -\beta\mu \left(\gamma p \frac{1+p-F^\star(0)}{1+p} + \alpha + \beta\right)^{-1}$, implying that

$$F^\star(0) = (2p\gamma)^{-1} \left(\frac{\alpha+\beta+\gamma p}{p+1} - \sqrt{(\frac{\alpha+\beta+\gamma p}{p+1})^2 - \frac{4\gamma\beta p}{(p+1)^2}} \right) (p+1)^2. \tag{31}$$

Now, recall that $t_0 = 0$ and $t_k = t_{k-1} + \frac{Q^\star(t_{k-1})}{\mu}$ for $k \geq 1$. The (piecewise) linearity of Q implies that

$$\frac{t_{i+2} - t_{i+1}}{t_{i+1} - t_i} = \frac{Q^\star(t_{i+1})}{Q^\star(t_i)} = \frac{1 + p - F^\star(0)}{1+p}.$$

Let $q = (1+p)^{-1}(1 + p - F^\star(0))$, then $Q^\star(t_i) = F^\star(0) q^i$ and $t_i = \mu^{-1} \sum_{j=1}^{j=i} F^\star(0) q^{j-1}$ for $i > 0$. Furthermore, Proposition 1 showed that $m(A_{t_{i+2}}) = m(A_{t_{i+1}}) + p(t_{i+1} - t_i - (m(A_{t_{i+1}} - m(A_{t_i})))$, implying that

$$\frac{q\left(m(A_{t_{i+2}}) - m(A_{t_{i+1}})\right)}{t_{i+2} - t_{i+1}} = p\left(1 - \frac{(m(A_{t_{i+1}}) - m(A_{t_i}))}{t_{i+1} - t_i}\right).$$

Denoting $k_i = \frac{(m(A_{t_{i+1}}) - m(A_{t_i}))}{t_{i+1} - t_i}$ (note that $k_0 = 0$) we have, $k_{i+1} = q^{-1} p(1 - k_i)$ for $i \geq 1$, so that $k_i = \sum_{j=1}^{i} (-1)^{j-1} (\frac{p}{q})^j$ for $i \geq 1$, and therefore

$$m(A_{t_{i+1}}) = m(A_{t_i}) + k_i(t_{i+1} - t_i) = m(A_{t_i}) + q^i t_1 k_i.$$

Since $m(A_{t_0}) = 0$, it follows that $m(A_{t_{i+1}}) = \sum_{j=1}^{i} q^j t_1 k_j$.

By definition $F^\star(t) = Q^\star(t) + \mu t - p\mu(t - m(A_t))$, implying that

$$F^\star(t) = \begin{cases} Q^\star(t) + \mu t - p\mu t, & \text{if } t \in [t_0, t_1] \\ Q^\star(t) + \mu t - p\mu(t - \sum_{j=1}^{i} q^j t_1 k_j - k_i(t - t_i)), & \text{if } t \in (t_{i+1}, t_{i+2}] \\ Q^\star(t), & \text{if } t < 0. \end{cases} \tag{32}$$

Since

$$Q^\star(t) = \begin{cases} -\frac{\mu F^\star(0)}{1+p} t + F^\star(0), & \text{if } t > 0 \\ \frac{\mu\alpha F^\star(0)}{\beta(1+p) - \gamma p F^\star(0)} t + F^\star(0), & \text{if } t \le 0, \end{cases} \tag{33}$$

we conclude that

$$F^\star(t) = \begin{cases} -\frac{\mu F^\star(0)}{1+p} t + F^\star(0) + \mu t - p\mu t, & \text{if } t \in (t_0, t_1] \\ -\dfrac{\mu F^\star(0)}{1+p} t + F^\star(0) + \mu t - p\mu \Bigg[t - \displaystyle\sum_{j=1}^{i} \Bigg(\left(\dfrac{1+p-F^\star(0)}{1+p} \right)^j & \text{if } t \in (t_{i+1}, t_{i+2}] \\ t_1 \displaystyle\sum_{k=1}^{j} (-1)^{k-1} \left(\dfrac{p(1+p)}{1+p-F^\star(0)} \right)^k \Bigg) - \displaystyle\sum_{j=1}^{i} (-1)^{i-1} \left(\dfrac{p(1+p)}{1+p-F^\star(0)} \right)^i (t - t_i) \Bigg] & \\ \mu\alpha F^\star(0) \left(\beta(1+p) - \gamma p F^\star(0)\right)^{-1} t + F^\star(0), & \text{if } t \le 0. \end{cases} \tag{34}$$

5.3. Price of Anarchy

Using the same notations and definition in Section 4.3, we show that the price of anarchy (PoA), under the uniqueness condition is precisely 2. Thus, the social cost induced by strategic behavior is twice that imposed by a central planner.

Theorem 6. *The price of anarchy of the single queue, single return arrival game is* 2*, when* $(\alpha + \beta)(\gamma p)^{-1} > 1$*.*

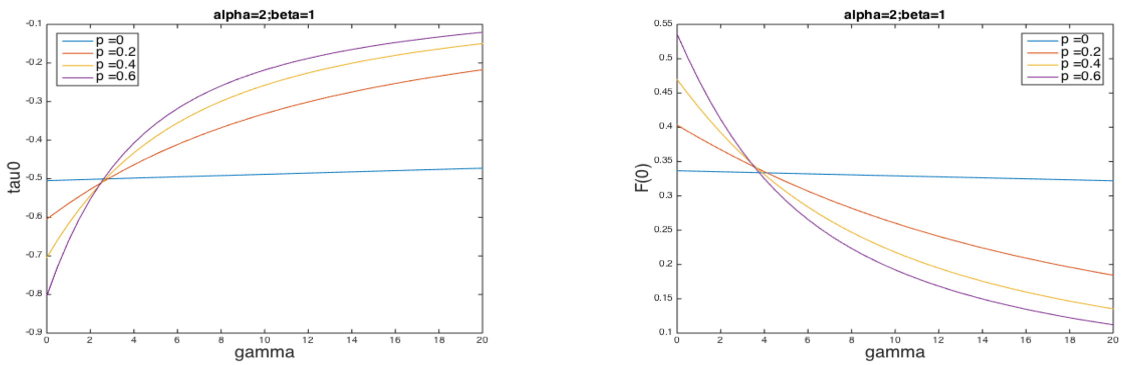
Proof. Following the analysis in Theorem 3, it can be shown (straightforwardly) that

$$\begin{aligned} J_{eq} &= \frac{\beta(1+p)}{\mu}, \\ C_{opt}(t) &= \beta t, \\ F'_{opt}(t) &= \frac{\mu}{1+p}, \\ J_{opt} &= \int_0^{\tau_1} \frac{\mu}{(1+p)} \beta t dt = \frac{\beta(1+p)}{2\mu}. \end{aligned}$$

It follows that the price of anarchy is 2. ∎

5.4. Equilibrium Behavior

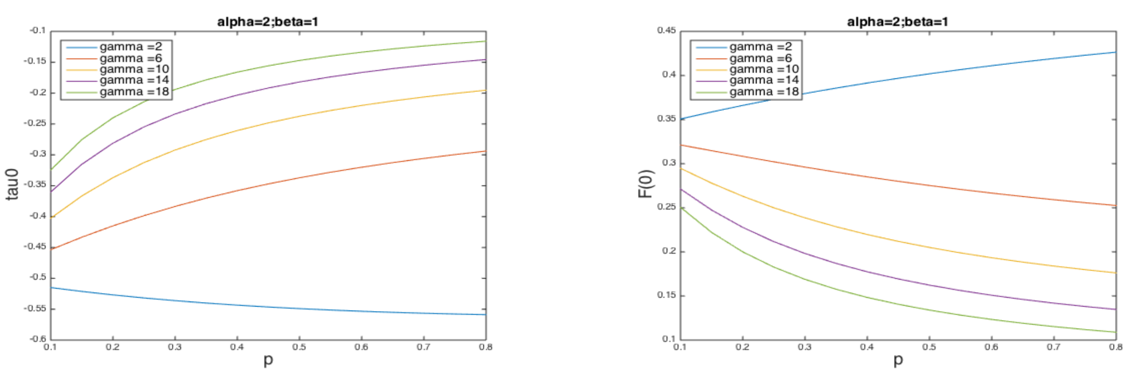
Figure 6 graphs the influence of γ on τ_0 and $F^\star(0)$. Essentially, τ_0 increases while $F^\star(0)$ decreases with increasing γ, implying that, as γ increases, the earliest arrival occurs later and number of arrivals before time 0 decreases. Once again, notice that as the risk of being returned and γ increases, the fraction of users who arrive before time drops dramatically.



(a) τ_0-γ (b) $F^\star(0)$-γ

Figure 6: Influence of gamma on τ_0 and $F^\star(0)$

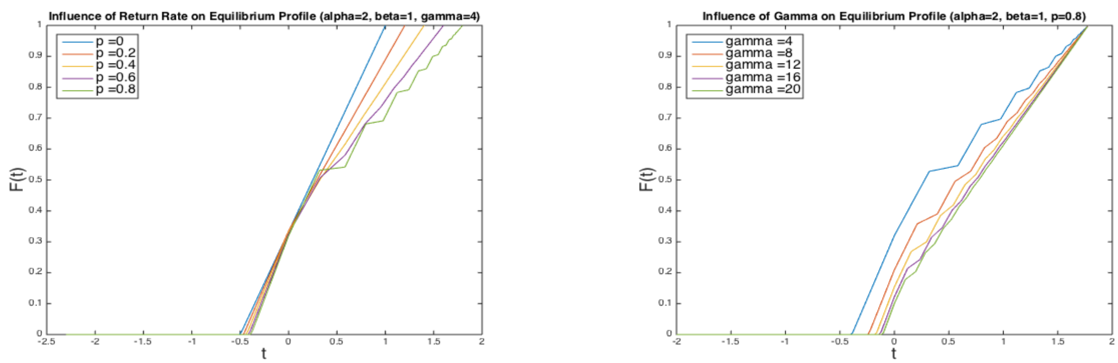
Figure 7 depicts the dependence of τ_0 and $F^\star(0)$ on p. Clearly, the curves are monotone as p approaches 1. Notice that τ_0 increases and $F^\star(0)$ decreases monotonically at a fixed p, as $\gamma > 2$ increases. This indicates that if users weight the risk of returning lower, fewer of them tend to arrive before time 0. Similarly, if they weight the waiting time after returns higher, more users will arrive before service starts.



(a) τ_0-p (b) $F^\star(0)$-p

Figure 7: Influence of p on τ_0 and $F^\star(0)$

The equilibrium profile as shown in Figure 8 is more than a combination of Figure 6 and Figure 7. The difference is caused by piecewise linearity of the return rate on $(t_i, t_{i+1}]$ as defined in Section 5.1. Notice that the piecewise linearity of the arrival profile becomes more apparent as p approaches 1. This stands in contrast with Figure 5, where the fact that each user can return an infinite number of times smooths out the equilibrium arrival profile. It also implies that there is a 'herding' type behavior here, and users tend to arrive in clumps - at a higher rate followed by a lower rate. Figure 8 (b), on the other hand shows that for $p = 0.8$ and small values of γ more users tend to arrive after service starts at time 0, and thus the "feedback effects" are exaggerated. On the other hand, for larger values of γ (at the same fixed p), more users arrive before time 0, and consequently feedback effects are more obvious near 0 than later, since there are fewer "new" users who can turn up after 0.



(a) Equilibrium-p

(b) Equilibrium-γ

Figure 8: Influence of γ and p on the equilibrium

6. Two Queues with Cross-Feedback

We extend the analysis in the previous sections to a two queue network with "cross-feedback," as depicted in Figure 1c. The service rates of the queues are μ_1 and μ_2 (respectively) and both of them commence service at time 0. Each user strategically picks a queue to join and a time epoch to join the queue. Users may be routed to the other queue (after service) with probability p. A user so routed will leave the network after service at the second queue, making this model analogous to the single return case in Section 5. Our goal, of course, is to identify the Nash equilibrium arrival profile (which now includes both routing and timing choices) when the cost of arriving at time t follows (7).

6.1. Queueing Dynamics

Before discussing the Nash equilibrium computation, we briefly review the queue length dynamics in the two queue network. Following the construction in Proposition 1, let $t_0 = \frac{Q_1(0)}{\mu_1}$, $t_0' = \frac{Q_2(0)}{\mu_2}$ and

$$\begin{aligned} t_{k+1} &= t_k' + \frac{Q_1(t_k')}{\mu_1}, \\ t_{k+1}' &= t_k + \frac{Q_2(t_k)}{\mu_2}. \end{aligned}$$

Using the same induction as in the proof of Proposition 1, we obtain:

$$\begin{aligned} m(A_{t_0}) &= 0, \\ m(A_{t_{k+1}}) &= m(A_{t_k}) + p(t_k - t_{k-1} - (m(A_{t_k}) - m(A_{t_{k-1}}))) \;\; \forall k \geq 1, \end{aligned}$$

and

$$\begin{aligned} m(A'_{t_0'}) &= 0, \\ m(A'_{t_{k+1}'}) &= m(A'_{t_k'}) + p(t_k' - t_{k-1}' - (m(A'_{t_k'}) - m(A'_{t_{k-1}'}))) \;\; \forall k \geq 1, \end{aligned}$$

where the set A_t corresponds to the first queue and A_t' is defined for the second queue. Note that $m(A_t)$ is linear on $(t_k, t_{k+1}]$, while $m(A_t')$ is linear on $(t_k', t_{k+1}']$ for $\forall k \geq 0,\ k \in \mathbb{N}$. Recall that $Q_1(t)$ and $Q_2(t)$ be queue lengths of the two queues. Let $F_1^\star(t)$ and $F_2^\star(t)$ be the corresponding distribution functions. Then, unpacking (3)

$$Q_1(t) = \begin{cases} F_1^\star(t) & \text{if } t \leq 0, \\ F_1^\star(t) - \mu t + p\mu(t - m(A_t)) & \text{if } t > 0, \end{cases} \tag{35}$$

and

$$Q_2(t) = \begin{cases} F_2^\star(t), & \text{if } t \leq 0 \\ F_2^\star(t) - \mu t + p\mu(t - m(A_t')), & \text{if } t > 0. \end{cases} \tag{36}$$

6.2. Equilibrium Arrival Profile

We start by considering a couple of lemma's that illuminate the equilibrium arrival and routing behavior in the network. Recall that the "makespan" of a queueing system is the departure epoch of the last user to arrive.

Lemma 5. *The makespan of the two queues are the same when traffic follows a Nash equilibrium arrival profile.*

Proof. Let $C_1(t)$ and $C_2(t)$ be the cost of arriving at time t, at queue 1 and queue 2 (respectively), as defined in (4). Let τ_1 and τ_1' be the makespan of the queues. Since the Nash equilibrium condition applies to both of the two queues, it must be the case that $C_1(\tau_1) = C_2(\tau_1')$, else a user arriving at the queue with higher arrival cost can arbitrarily improve her utility by choosing to arrive at the lower cost queue. Since the last user to arrive choses to do so (at either queue) such that she faces no delay it follows that $\tau_1 = \tau_1'$. □

Lemma 6. *The ratio of the total number of users arriving at the two queues is*

$$\frac{J_1}{J_2} = \frac{\mu_2(1+p) - \mu_1}{\mu_1(1+p) - \mu_2}. \tag{37}$$

Proof. Let J_1 and J_2 be total number of arrivals of the two queues. In[6] it is shown that the servers will not idle at equilibrium. Then, by Lemma 5 we have $\frac{(1+p)J_1+J_2}{\mu_2} = \frac{(1+p)J_2+J_1}{\mu_1}$, and hence $\frac{J_1}{J_2} = \frac{\mu_2(1+p)-\mu_1}{\mu_1(1+p)-\mu_2}$. □

Recall from Theorem 4, in the "single queue, single return" case, that there exists a piecewise linear $Q^\star(t)$ to which the corresponding $F^\star(t)$ is a Nash equilibrium arrival profile. The goal of the following analysis is to find linear $Q_1^\star(t)$ and $Q_2^\star(t)$ such that the corresponding $F_1^\star(t)$ and $F_2^\star(t)$ are Nash equilibria. To proceed, first suppose that $\frac{F_1^\star(0)}{\mu_1} = \frac{F_2^\star(0)}{\mu_2}$. Then, the waiting time of a routed user at equilibrium must be the same in either queue given that $Q_i^\star$ is piecewise linear. To see this, note that $Q_1^\star(t) = \frac{F_1^\star(0)}{\tau_1}t + F_1^\star(0)$ and $Q_2^\star(t) = \frac{F_2^\star(0)}{\tau_1}t + F_2^\star(0)$ for $t > 0$, implying that $\frac{Q_1^\star(t)}{\mu_1} = \frac{Q_2^\star(t)}{\mu_2}$. It follows that Theorem 4 will apply to both queues and the Nash equilibrium arrival profile is the tuple $(F_1^\star, F_2^\star)$ (note that this includes both arrival and routing decisions).

Consider queue 1 alone. We have $\tau_1' = \tau_1$ and $\frac{J_1}{J_2} = \frac{\mu_2(1+p)-\mu_1}{\mu_1(1+p)-\mu_2}$ from Lemma 5 and Lemma 6 respectively. Using the fact that $J_1 + J_2 = 1$, it follows that $\tau_1 = \frac{1+p}{\mu_1+\mu_2}$. Following the derivation of cost function in Section 4.2, we have

$$C_1(t) = \begin{cases} \frac{(\alpha+\beta)Q_1^\star(t)}{\mu_1} - \alpha t + \gamma p \frac{Q_1^\star\left(\frac{F_1^\star(t)}{\mu_1}\right)}{\mu_1} & \text{if } t \leq 0, \\ \frac{(\alpha+\beta)Q_1^\star(t)}{\mu_1} + \beta t + \gamma p \frac{Q_1^\star\left(t+\frac{Q_1^\star(t)}{\mu_1}\right)}{\mu_1} & \text{if } t > 0. \end{cases} \tag{38}$$

Notice that $C_1(\tau_1) = \frac{\beta(1+p)}{(\mu_1+\mu_2)}$, so by setting $C_1(0) = C_1(\tau_1)$ and solving for $F_1^\star(0)$ we obtain,

$$F_1^\star(0) = \left(2\frac{\gamma p(\mu_1+\mu_2)}{\mu_1^2(1+p)}\right)^{-1} \left(\frac{\alpha+\beta}{\mu_1} + \frac{\gamma p}{\mu_1} - \sqrt{\left(\frac{\alpha+\beta}{\mu_1} + \frac{\gamma p}{\mu_1}\right)^2 - 4\left(\frac{\gamma p \beta}{\mu_1^2}\right)}\right). \tag{39}$$

Let τ_0 be such that $F_1^\star(t) = 0$ for $t \leq \tau_0$ and set $C_1(\tau_0) = C_1(\tau_1)$, implying that $\tau_0 = \frac{\gamma p F_1^\star(0)}{\alpha \mu_1} - \frac{\beta(1+p)}{\alpha(\mu_1+\mu_2)}$. Therefore, following the computations in “Single Queue, Single Return,” we have

$$Q_1^\star(t) = \begin{cases} -\frac{F_1^\star(0)}{\tau_0} t + F_1^\star(0) & \text{if } t \leq 0, \\ -\frac{F_1^\star(0)}{\tau_1} t + F_1^\star(0) & \text{if } t > 0, \end{cases} \tag{40}$$

and

$$F_1^\star(t) = \begin{cases} -\frac{F_1^\star(0)}{\tau_1} t + F_1^\star(0) + \mu_1 t - p\mu_2 t & \text{if } t \in (t_0, t_1], \\ -\dfrac{F_1^\star(0)}{\tau_1} t + F_1^\star(0) + \mu_1 t - p\mu_2 \Bigg[t - \displaystyle\sum_{j=1}^{i} \left(\left(\dfrac{1+p-F_1^\star(0)}{1+p}\right)^j \right. & \\ \left. t_1 \displaystyle\sum_{k=1}^{j} (-1)^{k-1} \left(\dfrac{p(1+p)}{1+p-F_1^\star(0)}\right)^k \right) - \displaystyle\sum_{j=1}^{i} (-1)^{i-1} \left(\dfrac{p(1+p)}{1+p-F_1^\star(0)}\right)^i (t-t_i) \Bigg] & \text{if } t \in (t_{i+1}, t_{i+2}] \\ -\frac{F_1^\star(0)}{\tau_0} t + F_1^\star(0) & \text{if } t \leq 0. \end{cases} \tag{41}$$

Now, by the previous construction, $F_2^\star(0) = F_1^\star(0)\frac{\mu_2}{\mu_1}$. Let τ_0' be such that $F_2^\star(t) = 0$ for $t \leq \tau_0'$, then $\tau_0' = \frac{\gamma p F_2^\star(0)}{\alpha \mu_2} - \frac{\beta(1+p)}{\alpha(\mu_1+\mu_2)}$. Therefore, similarly,

$$Q_2^\star(t) = \begin{cases} -\frac{F_2^\star(0)}{\tau_0'} t + F_2^\star(0) & \text{if } t \leq 0, \\ -\frac{F_2^\star(0)}{\tau_1} t + F_2^\star(0) & \text{if } t > 0, \end{cases} \tag{42}$$

and

$$
F_2^\star(t) = \begin{cases} -\frac{F_2^\star(0)}{\tau_1}t + F_2^\star(0) + \mu_2 t - p\mu_1 t & \text{if } t \in (t_0, t_1], \\ -\dfrac{F_2^\star(0)}{\tau_1}t + F_2^\star(0) + \mu_2 t - p\mu_1 \Bigg[t - \displaystyle\sum_{j=1}^{i} \Bigg(\left(\dfrac{1+p-F_2^\star(0)}{1+p} \right)^j & \\ t_1 \displaystyle\sum_{k=1}^{j} (-1)^{k-1} \left(\dfrac{p(1+p)}{1+p-F_2^\star(0)} \right)^k \Bigg) - \displaystyle\sum_{j=1}^{i} (-1)^{i-1} \left(\dfrac{p(1+p)}{1+p-F_2^\star(0)} \right)^i (t - t_i) \Bigg] & \text{if } t \in (t_{i+1}, t_{i+2}] \\ -\frac{F_2^\star(0)}{\tau_0'}t + F_2^\star(0) & \text{if } t \leq 0. \end{cases} \tag{43}
$$

Theorem 7. *The Nash equilibrium is unique if* $\frac{\alpha+\beta}{\gamma p} > 1$ *and* $\frac{Q_1^\star(t)}{\mu_1} = \frac{Q_2^\star(t)}{\mu_2}$.

Proof. It's sufficient to show that $Q_1^\star$ is unique since proof for $Q_2^\star$ is identical. Suppose there exists another Nash equilibrium such that the corresponding queue length function $Q_1^{\star\prime}(t_x) - Q_1^\star(t_x) = \delta > 0$ for some t_x in the support of the equilibrium arrival profile. Following the proof of Theorem 5 in Section 5.1, we obtain the equilibrium equation:

$$
\frac{\alpha+\beta}{\mu_1}\big(Q_1^\star(T_i(t)) - Q_1^{**}(T_i'(t))\big) - \beta(T_i(t) - T_i'(t)) + \frac{\gamma p}{\mu_2}\big(Q_2^\star(T_{i+1}(t)) - Q_2^{**}(T_{i+1}'(t))\big) = 0. \tag{44}
$$

Since $\frac{Q_1^\star(t)}{\mu_1} = \frac{Q_2^\star(t)}{\mu_2}$, equation (44) can be written as:

$$
\frac{\alpha+\beta}{\mu_1}\big(Q_1^\star(T_i(t)) - Q_1^{**}(T_i'(t))\big) - \beta(T_i(t) - T_i'(t)) + \frac{\gamma p}{\mu_1}\big(Q_1^\star(T_{i+1}(t)) - Q_1^{**}(T_{i+1}'(t))\big) = 0. \tag{45}
$$

Let $\frac{\alpha+\beta}{\beta\mu_1} = c_1$, $\frac{\gamma p}{\beta\mu_1} = c_2$, $Q_1^\star(T_n(t_x)) - Q_1^{**}(T_n'(t_x)) = a_n$ and $T_n(t) - T_n'(t) = b_n$. Then the desired result follows directly from lemma 4. $\square$

Figure 9 demonstrates the arrival profile of the Nash equilibrium as derived in equation (41) and (43). We see that the increasing pattern of $F_i^\star(t)(i = 1, 2)$ is the similar to that in figure 8. Notice, too, the 'herding' behavior where users choose to arrive in clumps of higher and lower arrival 'rates.'

7. Conclusion

Our goal in this paper was to study the effect of feedback routing on the Nash equilibrium arrival timing and routing behavior of strategic users. As we demonstrate, the network architecture will affect the Nash equilibrium profile. The infinite return and single return risk affects when users

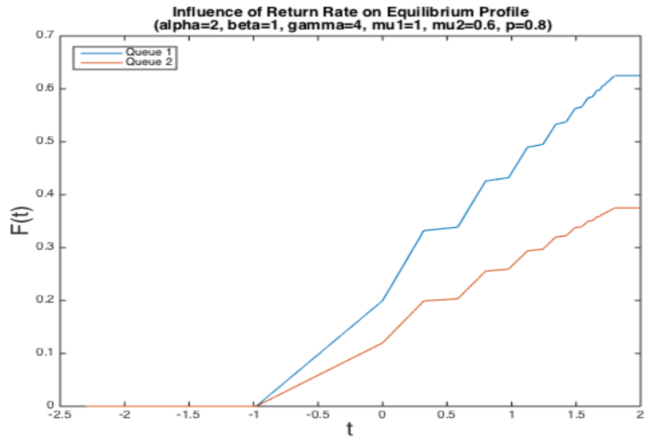


Figure 9: Parallel-Equilibrium

choose to arrive, in the single queue architectures, and the routing decisions in the cross-feedback network. We also showed that the price of anarchy (PoA) is exactly equal to 2 in the single queue architectures. The equilibrium and PoA computation are, however, substantially more complicated in the case of the cross-feedback network, as we are only able to prove uniqueness under a strong technical condition. To see why the proof of uniqueness is complicated, we first argue algebraically; as equation (44) implies,

$$\frac{\alpha+\beta}{\mu_1}\big(Q_1(T_i(t)) - Q_1'(T_i'(t))\big) - \beta(T_i(t) - T_i'(t)) + \frac{\gamma p}{\mu_2}\big(Q_2(T_{i+1}(t)) - Q_2'(T_{i+1}'(t))\big) = 0.$$

If the condition $\frac{Q_1(t)}{\mu_1} = \frac{Q_2(t)}{\mu_2}$ is removed, then it is not possible to formulate a simple induction since $Q_1(T_i(t)) - Q_1'(T_i'(t))$ and $Q_2(T_{i+1}(t)) - Q_2'(T_{i+1}'(t))$ are not related in a straightforward manner. Consequently, the proof by contradiction is no longer appropriate

To further fix the intuition about the existence of other equilibria, consider Figure 10, which shows the queue length function of both queues at an equilibrium. With the same setting as that of figure 2, t_1 and t_1' are service completion time of the customers arriving at time 0, entering queue 1 and queue 2 respectively. While t_2 and t_2' are the corresponding service completion time of potential returns. Note that, $\frac{Q_1(t)}{\mu_1} = \frac{Q_2(t)}{\mu_2}$ does not hold. The equilibrium condition tells us that

$$(\alpha+\beta)\big(t_1 - t_1'\big) + \gamma p\big(t_2 - t_2'\big) = 0.$$

This condition can easily be satisfied by carefully choosing slopes (i.e. $Q_1'(0)$ and $Q_2'(0)$) of the two queues. If we study the customers arriving at arbitrary time $t \geq 0$, it can be easily argued using

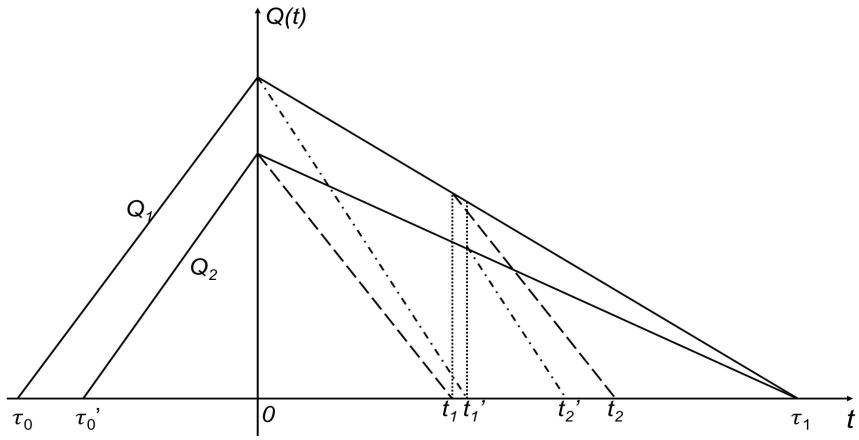


Figure 10: Graphic Interpretation

similar triangles that the equilibrium condition also holds. This construction shows the potential existence of alternative equilibria when the condition $\frac{Q_1(t)}{\mu_1} = \frac{Q_2(t)}{\mu_2}$ is removed.

In this paper, we only considered single-class networks, which also contributes to the complications in computing the equilibrium profile. An alternative model would be consider a multi-class network where by fedback/routed users are queued up in separate buffers from external arrivals. In this case, the equilibrium strategy profiles will be a function of not only the network architecture, but also the scheduling policy. To the best of our knowledge, [12] is the only work in the literature that considers strategic routing behavior in multi-class networks. As noted before in Section 2, the authors consider the question of how strategic users might strategically route themselves in a version of the Kumar-Seidman network; see [3, Chapter 8]. More broadly, in future work, we will consider the effect of service allocation policies on strategic behavior and how strategic behavior, in turn, can influence the design of such policies. For instance, from a performance analysis perspective the primary focus is on achieving some form of optimality - for instance, maximum throughput. However, the presence of strategic traffic flows implies a loss of capacity. Can this loss in capacity be quantified in a meaningful way? A second key question is how to design policies to achieve Pareto optimal outcomes. Put another way, is it possible to view scheduling policies as mechanisms? We leave these questions to forthcoming papers.

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