ECE 661 Homework 1

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Question 1 A set of all points in the representational space $\mathbb{R}^3$ that are homogeneous coordinates of the origin in the physical space $\mathbb{R}^2$ is

$$\mathcal{S} = \left\{ k \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| k \in \mathbb{R}, k \neq 0 \right\}.$$

Question 2 No, all points at infinity in the physical plane $\mathbb{R}^2$ are not the same.
An ideal point with HC representation as $\begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$ approaches infinity in a specific direction. For example, a point with $u = 1, v = 0$ approaches infinity along the x-axis in $\mathbb{R}^2$, or a point with $u = 1, v = 1$ approaches infinity along the $45°$ line in $\mathbb{R}^2$. However, all the points at infinity form a straight line in $\mathbb{R}^2$. The HC representation of the line formed by all ideal points is given as $\mathbf{l}_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Question 3 The HC representation of a degenerate conic $\mathbf{C}$ formed by lines $\mathbf{l}$ and $\mathbf{m}$ is given as

$$\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T. \tag{1}$$

Let $\mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$ and $\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$. Analysing the first matrix in the sum in (1), i.e., $\mathbf{l}\mathbf{m}^T$.

$$\mathbf{l}\mathbf{m}^T = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix} = \begin{bmatrix} l_1m_1 & l_1m_2 & l_1m_3 \\ l_2m_1 & l_2m_2 & l_2m_3 \\ l_3m_1 & l_3m_2 & l_3m_3 \end{bmatrix} = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \end{bmatrix},$$

where $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ represent 1st, 2nd, 3rd columns of the product matrix $\mathbf{l}\mathbf{m}^T$, respectively.
Here, $\mathbf{a_2} = \frac{m_2}{m_1}\mathbf{a_1}$ and $\mathbf{a_3} = \frac{m_3}{m_1}\mathbf{a_1}$.
Hence, the matrix $\mathbf{l}\mathbf{m}^T$, has only one independent column and its rank is 1.

Similarly, for $\mathbf{m}\mathbf{l}^T$,

$$\mathbf{m}\mathbf{l}^T = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} = \begin{bmatrix} m_1l_1 & m_1l_2 & m_1l_3 \\ m_2l_1 & m_2l_2 & m_2l_3 \\ m_3l_1 & m_3l_2 & m_3l_3 \end{bmatrix} = \begin{bmatrix} \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \end{bmatrix},$$

where $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$ represent 1^{st}, 2^{nd}, 3^{rd} columns of the product $\mathbf{ml}^T$, respectively.
Here, $\mathbf{b_2} = \frac{l_2}{l_1}\mathbf{b_1}$ and $\mathbf{b_3} = \frac{l_3}{l_1}\mathbf{b_1}$.
Hence, the matrix $\mathbf{ml}^T$, has only one independent column and its rank is 1.

Now, considering $\mathbf{C}$ with $\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}$ as its 1^{st}, 2^{nd}, 3^{rd} columns, respectively,

$$\begin{aligned}\mathbf{C} &= \begin{bmatrix}\mathbf{c_1} & \mathbf{c_2} & \mathbf{c_3}\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3}\end{bmatrix} + \begin{bmatrix}\mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3}\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{a_1}+\mathbf{b_1} & \mathbf{a_2}+\mathbf{b_2} & \mathbf{a_3}+\mathbf{b_3}\end{bmatrix}\end{aligned}$$

The rank of the matrix $\mathbf{C}$ is the number of independent columns in it. Columns $\mathbf{c_1}, \ldots, \mathbf{c_n}$ are linearly independent if $k_1\mathbf{c_1} + \cdots + k_n\mathbf{c_n} = 0$ happens only when $k_1 = \cdots = k_n = 0$ [1].
Solving this equation for $\mathbf{C}$ to find k_1, k_2, k_3,

$$\begin{aligned}k_1\mathbf{c_1} + k_2\mathbf{c_2} + k_3\mathbf{c_3} &= 0\\ k_1\mathbf{a_1} + k_1\mathbf{b_1} + k_2\mathbf{a_2} + k_2\mathbf{b_2} + k_3\mathbf{a_3} + k_3\mathbf{b_3} &= 0\\ k_1\mathbf{a_1} + k_1\mathbf{b_1} + \frac{k_2m_2}{m_1}\mathbf{a_1} + \frac{k_2l_2}{l_1}\mathbf{b_1} + \frac{k_3m_3}{m_1}\mathbf{a_1} + \frac{k_3l_3}{l_1}\mathbf{b_1} &= 0\\ \left(k_1 + \frac{k_2m_2}{m_1} + \frac{k_3m_3}{m_1}\right)\mathbf{a_1} + \left(k_1 + \frac{k_2l_2}{l_1} + \frac{k_3l_3}{l_1}\right)\mathbf{b_1} &= 0 \qquad (2)\end{aligned}$$

<u>**Case 1:**</u> $\mathbf{a_1}$ and $\mathbf{b_1}$ are independent. Then only solution to (2) is

$$\begin{aligned}k_1 + \frac{k_2m_2}{m_1} + \frac{k_3m_3}{m_1} &= 0\\ k_1 + \frac{k_2l_2}{l_1} + \frac{k_3l_3}{l_1} &= 0\end{aligned}$$

Here, we have three unknowns and two equations. This system of equations will have infinite number of solutions and there is no unique solution. Hence, non-zero values of k_1, k_2, k_3 can satisfy (2). Hence, the matrix $\mathbf{C}$ cannot have 3 independent columns and its rank can not exceed 2.
<u>**Case 2:**</u> $\mathbf{a_1}$ and $\mathbf{b_1}$ are not independent.
If $\mathbf{a_1}$ and $\mathbf{b_1}$ are dependent then we will have only equation and only one independent column in $\mathbf{C}$. Hence, the rank of the matrix $\mathbf{C}$ would be 1, which is less than 2.

Hence proved that the matrix rank of a degenerate conic can never exceed 2.

Question 4 Using HC representations,
Part 1

$$\mathbf{l}_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}3\\5\\1\end{bmatrix} = \begin{bmatrix}-5\\3\\0\end{bmatrix}$$

$$\mathbf{l}_2 = \begin{bmatrix}-3\\4\\1\end{bmatrix} \times \begin{bmatrix}-7\\5\\1\end{bmatrix} = \begin{bmatrix}-1\\-4\\13\end{bmatrix}$$

Intersection of $\mathbf{l}_1$ and $\mathbf{l}_2$:

$$\mathbf{l}_1 \times \mathbf{l}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ -4 \\ 13 \end{bmatrix} = \begin{bmatrix} 39 \\ 65 \\ 23 \end{bmatrix}$$

The physical point of intersection of $\mathbf{l}_1$ and $\mathbf{l}_2$ in $\mathbb{R}^2$ is $\left(\frac{39}{23}, \frac{65}{23}\right)$

Part 2

$$\mathbf{l}_2 = \begin{bmatrix} -7 \\ -5 \\ 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -14 \\ 0 \end{bmatrix}$$

The constant term c of this line 0, i.e. it passes through origin. Also, the constant term for $\mathbf{l}_1$ is 0 and it passes through origin. Hence, the two lines intersect at the origin. **In this case, two steps are needed to find the point of intersection.**

Question 5 Using HC representations, **Part 1**

$$\mathbf{l}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$$

$$\mathbf{l}_2 = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 15 \end{bmatrix}$$

Intersection of $\mathbf{l}_1$ and $\mathbf{l}_2$:

$$\mathbf{l}_1 \times \mathbf{l}_2 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ 5 \\ 15 \end{bmatrix} = \begin{bmatrix} 75 \\ -45 \\ 0 \end{bmatrix}$$

The physical point of intersection of $\mathbf{l}_1$ and $\mathbf{l}_2$ is an ideal point which is at infinity. The two lines $\mathbf{l}_1$ and $\mathbf{l}_2$ are parallel.

Question 6 The equation of an ellipse with center at (h, k) and major axis parallel to the x-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where a and b are lengths of semi-major and semi-minor axis, respectively. Forming the equation for the ellipse in the given question,

$$\begin{aligned} \frac{(x-3)^2}{1^2} + \frac{(y-2)^2}{(\frac{1}{2})^2} &= 1 \\ (x-3)^2 + 4(y-2)^2 &= 1 \\ x^2 - 6x + 9 + 4(y^2 - 4y + 4) &= 1 \\ x^2 + 4y^2 - 6x - 16y + 24 &= 0 \end{aligned} \tag{3}$$

HC representation of the ellipse in (3) is

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & -8 \\ -3 & -8 & 24 \end{bmatrix}$$

HC representation of the given point $\mathbf{p}$, which is origin of the $\mathbb{R}^2$ physical plane, is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then the HC representation of the polar line $\mathbf{l}_{polar}$ is

$$\begin{aligned} \mathbf{l}_{polar} &= \mathbf{Cp} \\ &= \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & -8 \\ -3 & -8 & 24 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \mathbf{l}_{polar} &= \begin{bmatrix} -3 \\ -8 \\ 24 \end{bmatrix} \end{aligned} \tag{4}$$

HC representation of x-axis is $\mathbf{l}_{x-axis} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and of y-axis is $\mathbf{l}_{y-axis} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Intersection of $\mathbf{l}_{polar}$ with x-axis:

$$\mathbf{l}_{polar} \times \mathbf{l}_{x-axis} = \begin{bmatrix} -3 \\ -8 \\ 24 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -24 \\ 0 \\ -3 \end{bmatrix}$$

Hence, the polar line $\mathbf{l}_{polar}$ intersects x-axis at the point $(8, 0)$ in the physical $\mathbb{R}^2$ plane.

Intersection of $\mathbf{l}_{polar}$ with y-axis:

$$\mathbf{l}_{polar} \times \mathbf{l}_{y-axis} = \begin{bmatrix} -3 \\ -8 \\ 24 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 24 \\ 8 \end{bmatrix}$$

Hence, the polar line $\mathbf{l}_{polar}$ intersects y-axis at thr point $(0, 3)$ in the physical $\mathbb{R}^2$ plane.

Question 7 HC representation of a line $ax + by + c = 0$ is the physical $\mathbb{R}^2$ plane is $\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

HC representation of for the line with equation $x = 1/2$ is $\mathbf{l}_1 = \begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix}$.

HC representation of for the line with equation $y = -1/3$ is $\mathbf{l}_2 = \begin{bmatrix} 0 \\ 1 \\ 1/3 \end{bmatrix}$.

Intersection of $\mathbf{l}_1$ and $\mathbf{l}_2$ is

$$\mathbf{l}_1 \times \mathbf{l}_2 = \begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/3 \\ 1 \end{bmatrix}$$

Hence, lines $x = 1/2$ and $y = -1/3$ intersect at the point $(\frac{1}{2}, -\frac{1}{3})$ in the physical $\mathbb{R}^2$ plane.

References

[1] Gilbert Strang. *Linear algebra and its applications.* Cengage learning, fourth edition, 2006.