Variable Structure Control of Nonlinear Multivariable Systems: A Tutorial

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Invited Paper

This paper presents, in a tutorial manner, the design of variable structure control (VSC) systems for a class of multivariable nonlinear time varying systems. By the use of the Utkin-Draženović "method of equivalent control" and generalized Lyapunov stability concepts, VSC design is described in a unified manner. Complications that arise due to multiple inputs are then described and several approaches useful in overcoming these complications are then developed.

After this, the paper investigates recent developments and the kinship of VSC and the deterministic approach to the control of uncertain systems. All points are illustrated by numerical examples. In addition, the recent VSC applications literature is surveyed.

I. INTRODUCTION

Variable Structure Control (VSC) is a viable high-speed switching feedback control (for example, the gains in each feedback path switch between two values according to some rule). This variable structure control law provides an effective and robust means of controlling nonlinear plants. It has its roots in relay and bang-bang control theory. The advances in computer technology and high-speed switching circuitry, have made the practical implementation of VSC a reality and of increasing interest to control engineers (see References).

Essentially, VSC utilizes a high-speed switching control law to drive the nonlinear plant's state trajectory onto a specified and user-chosen surface in the state space (called the sliding or switching surface), and to maintain the plant's state trajectory on this surface for all subsequent time. This surface is called the switching surface because if the state trajectory of the plant is "above" the surface a control path has one gain and a different gain if the trajectory drops "below" the surface. The plant dynamics restricted to this surface represent the controlled system's behavior. By proper design of the sliding surface, VSC attains the conventional goals of control such as stabilization, tracking, regulation, etc.

The purpose of this paper is to furnish quick readable access to key design techniques in VSC (scattered throughout the literature) for a class of nonlinear time-varying systems. Because of the paper's tutorial nature, the presentation includes only several of the basic forms of the many VSC design methods for multivariable, nonlinear, time-varying systems. These basic forms often need tweaking before application.

To minimize confusion and to maintain a unified exposition, our discussion concentrates on systems linear in the control input. Such systems are amenable to Utkin's methods [1], [2], [4]. Also for simplicity, the paper deals most of the time with ideal VSC — i.e., switching in the control law can occur infinitely fast. The ideal case is much easier to analyze and provides a baseline against which one can measure more realistic designs. Comments on the nonideal case are included for completeness at the end of the paper.

Section II introduces the reader to the flexibility offered by the variable structure control strategy via the medium of a simple example. Section III sets forth the basic definitions such as the system model, the switching surface, the associated notion of a sliding mode, and an overview of the two-phase VSC design process.

Section V examines phase one of the VSC design process, that of designing a sliding surface so that the plant restricted to the sliding surface has a desired system response. This means that the state variables of the plant dynamics are constrained to satisfy another set of equations which define the so-called switching surface. An example illustrates the ideas and the relevant literature is cited.

Section VI discusses the construction of the switched feedback gains necessary to drive the plant's state trajectory to the sliding surface. These constructions build on the generalized Lyapunov stability theory.

The remainder of the paper deals with applications, the problem of nonideal switching, and the use of the boundary layer concept to alleviate the problem of chattering induced by the high-speed switching. Relationships to the theory of uncertain systems are also pointed out and dis-
cussed along with a brief review of the recent VSC applications literature.

II. BACKGROUND

The term “variable structure control” arises because the “controller structure” around the plant is intentionally changed by some external influence to obtain a desired plant behavior or response. For example, consider a plant with two accessible states and one control input as described by the following state equations

\[
\begin{bmatrix}
  x_1 \\ x_2
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\ 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1 \\ x_2
\end{bmatrix} +
\begin{bmatrix}
  0 \\ 1
\end{bmatrix} u, \quad |u| \leq 1.
\]  

(2.1)

A block diagram representation of (2.1) appears in Fig. 1. Let the so-called switching surface be

\[
s_1 x_1 + x_2 = 0
\]

and the control law be given by

\[u = \text{sgn}(s(x_1, x_2))
\]

(2.2)

where

\[\text{sgn}(a) = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0. \end{cases}\]

A block diagram of the closed-loop system is depicted in Fig. 2. Let us now investigate the behavior of the system for different values of the parameter \(s_1\), i.e., for different switching surfaces \(s\). The phase-plane plots of the system (2.1) with control law (2.2) are given in Fig. 3. Fig. 3(a) shows phase-plane trajectories for small \(s_1 > 0\) while Fig. 3(b) illustrates those for large \(s_1 > 0\). Upward motion in the trajectories is associated with \(u = +1\) and downward motion for \(u = -1\). Suppose the relay element in the block diagram of Fig. 2 has a small delay when switching between the gains “1” and “−1.” Consider the resulting system behavior as this delay tends to zero and \(s_1\) is small. The behavior of this second-order system on the switching line \(s = s_1 x_1 + x_2 = 0\) (Fig. 3(a)) is described by the first-order differential equation

\[
x_1' + s_1 x_1 = 0.
\]

It is important to note, that the behavior of our system on \(s = 0\) is dependent only on the slope \(s_1\) of the switching line. This means the system is insensitive to any variation or perturbation of the plant parameters contained in the bottom row of the \(A\) matrix of (2.1), i.e., perturbations in the image of the “\(B\) matrix” \([0 \ 1]\). This is one dominate motivation for investigating variable structure systems. The motion of Fig. 3(b) is more complex. Here the state trajectory switches to a new parabolic motion every time it intercepts the switching line \(s = 0\). Nevertheless, the parabolic motions “spiral” into the origin.

As a second example consider a plant with two accessible states and one control input of the form

\[u = k_1(x_1, x_2)x_1 + k_2(x_1, x_2)x_2\]

where the gains \(k_1(x_1, x_2)\) take on two possible values, say \(\alpha\) or \(\beta\). To specifically illustrate the idea, consider the state model

\[
\begin{bmatrix}
  x_1(t) \\ x_2(t)
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\ 1 & 2
\end{bmatrix}
\begin{bmatrix}
  x_1(t) \\ x_2(t)
\end{bmatrix} +
\begin{bmatrix}
  0 \\ 1
\end{bmatrix} u(t)
\]

under the variable structure control law

\[u(t) = k(x_1) x_1(t)\]

where \(k(x_1)\) can be “−2” or “3”.

This system illustrated in Fig. 4 has two linear structures, one each for \(k(x_1) = 2\) and \(k(x_1) = -3\). With \(k(x_1) = -3\), the system has complex eigenvalues and with \(k(x_1) = 2\) the system has real eigenvalues.

With the switch in the upper position, the feedback pro-
duces an unstable free motion satisfying
\[
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = 
\begin{bmatrix}
  0 & 1 \\
  -2 & 2
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
\]
as shown in Fig. 5(a). With the switch in the lower position, the feedback becomes positive and the system’s free motion satisfies
\[
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = 
\begin{bmatrix}
  0 & 1 \\
  3 & 2
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}.
\]
The “unstable” equilibrium point (0, 0) is now a saddle point with asymptotes \(x_2 = 3x_1\) and \(x_2 = -x_1\), as shown in Fig. 5(b).

Switching of course is not random. It occurs with respect to a sliding or switching surface, generically denoted as \(\sigma = 0\). To illustrate this notion, consider the surface defined as \(\sigma(x_1, x_2) = s_1x_1 + x_2 = 0\) with \(s_1 > 1\). If the feedback is switched according to
\[
k(x_1) = \begin{cases} 
-3, & \text{if } \sigma(x_1, x_2)x_1 > 0 \\
2, & \text{if } \sigma(x_1, x_2)x_1 < 0
\end{cases}
\]
a behavior illustrated in the phase plane plot of Fig. 6 results.

Observe from the dotted line trajectory that if the state vector is perturbed below the surface, \(\sigma(x_1, x_2) = s_1x_1 + x_2 = 0\) at time \(t_0\), it circles to the point \(t_1\) before intercepting the surface again. On the other hand, if the switching surface is \(\sigma_2(x_1, x_2) = s_1x_1 + x_2 = 0\) with \(s_1 < 1\), then a perturbation off the surface is always immediately forced back to the surface since the phase-plane velocity vectors always point towards the surface. Fig. 7 illustrates the phenomena.

As suggested by Figs. 3, 6, and 7, different choices of switching surfaces produce radically different system responses. The richness of variable structure control comes from this ability to choose various controller structures at different points in time.

The above example also illustrates an important notion
in VSC. For the switching surface, \( a(x_1, x_2) = s_1x_1 + x_2 = 0 \) of Fig. 7, once the state trajectory intercepts the surface it remains on the surface for all subsequent time. This property of remaining on the switching surface once intercepted is called a sliding mode.

A sliding mode will exist for a system if in the vicinity of the switching surface, the state velocity vector (the derivative of the state vector) is directed towards the surface.

The lack of a sliding mode for the \( \sigma = \sigma_1 \) scenario described in the second example disappears when using a full state feedback control law: \( u(x) = k_1(x_1, x_2)x_1 + k_2(x_1, x_2)x_2 \). With appropriate gain choice, the original system can always be forced to have a sliding mode on any surface \( \sigma = s_1x_1 + s_2 = 0 \). It was the choice of partial state feedback \( u = k_1(x_1, x_2)x_1 \) which prevented the existence of a sliding mode on the surface \( \sigma_1 = s_1x_1 + x_2 = 0 \) with \( s_1 > 1 \).

Insuring the existence of a sliding mode on the switching surface is a key necessity in VSC design. Designing the proper surface is the complementary key problem. Thus VSC design breaks down into two major phases. The first is the construction of the switching surface so that the original system or plant restricted to the surface responds in a desired manner. The second phase entails the development of a switching control law (i.e., appropriate switched feedback gains) which satisfies a set of "sufficient conditions" for the existence and reachability of a sliding mode [1], [2], [8], [11, 14], [15, 19-21], [24, 29], [30, 77].

III. DEFINITIONS AND PRELIMINARIES

System Model

This paper considers a class of systems having a state model nonlinear in the state vector \( x(t) \) and linear in the control vector \( u(t) \) of the form

\[
\dot{x}(t) = f(t, x, u) + B(t, x) u(t)
\]

(3.1)

where the state vector \( x(t) \in \mathbb{R}^n \), the control vector \( u(t) \in \mathbb{R}^m \), \( f(t, x) \in \mathbb{R}^n \), and \( B(t, x) \in \mathbb{R}^{n \times m} \); further, each entry in \( f(t, x) \) and \( B(t, x) \) is assumed to be continuous with continuous bounded derivative with respect to \( x \).

Each entry \( u_i(t) \) of the switched control \( u(t) \in \mathbb{R}^m \) has the form

\[
u_i(t, x) = \begin{cases} u_i^+(t, x) & \text{with } \sigma_i(x) > 0 \\ u_i^-(t, x) & \text{with } \sigma_i(x) < 0 \end{cases} \quad i = 1, \ldots, m
\]

(3.2)

where \( \sigma_i(x) \) is the \( i \)-th switching (also called discontinuity) surface associated with the \((n - m)\)-dimensional switching surface

\[
\sigma(x) = [\sigma_1(x), \ldots, \sigma_m(x)]^T = 0.
\]

(3.3)

The Switching Surface

The switching surface \( \sigma(x) = 0 \) is a \((n - m)\)-dimensional manifold in \( \mathbb{R}^n \) determined by the intersection of \( m \) \((n - 1)\)-dimensional switching surfaces \( \sigma_i(x) = 0 \). These switching surfaces are designed such that the system response restricted to \( \sigma(x) = 0 \) has a desired behavior such as stability or tracking. Switching surface design is taken up in a later section.

Although general nonlinear switching surfaces (3.3) are possible, linear ones are more prevalent in design [1], [6], [14], [82]. Moreover, for a large class of systems, design of linear switching surfaces proves amenable to classical linear controller techniques. Thus for clarity, convenience, and simplicity of exposition, this tutorial will focus on linear switching surfaces of the form

\[
\sigma(x) = S x(t) = 0
\]

(3.4)

where \( S \) is an \( m \times n \) matrix.

Sliding Modes

After switching surface design, the next important aspect of VSC is guaranteeing the existence of a sliding mode. A sliding mode exists, if in the vicinity of the switching surface, \( \sigma(x) = 0 \), the tangent or velocity vectors of the state trajectory always point toward the switching surface. Consequently, if the state trajectory intersects the sliding surface, the value of the state trajectory or "representative point" remains within an \( \epsilon \) neighborhood of \( \{x | \sigma(x) = 0\} \). As a point of information, if a sliding mode exists on \( \sigma(x) = 0 \), then \( \sigma(x) \) is termed a sliding surface. As seen in Fig. 8, a sliding mode may not exist on \( \sigma(x) = 0 \) separately, but only on the intersection.

An ideal sliding mode exists only when the state trajectory \( x(t) \) of the controlled plant satisfies \( \sigma(x(t)) = 0 \) at every \( t \geq t_0 \) for some \( t_0 \). This requires infinitely fast switching. In actual systems, all facilities responsible for the switching control function have imperfections such as delay, hysteresis, etc., which force switchings to occur at a finite frequency. The representative point then oscillates within a neighborhood of the switching surface. This oscillation is called chattering. If the frequency of the switching is very high compared with the dynamic response of the system, the imperfections and the finite switching frequencies are often but not always negligible. Hence our subsequent development considers primarily ideal sliding modes. The
problem of chattering and techniques for circumventing the problem are discussed in a later section of the paper.

Conditions for the Existence of a Sliding Mode

Existence of a sliding mode [1-3, 5] requires stability of the state trajectory to the sliding surface \( a(x) = 0 \) at least in a neighborhood of \( \{ x | a(x) = 0 \} \) i.e., the representative point must approach the surface at least asymptotically. The largest such neighborhood is called the region of attraction. Geometrically, the tangent vector or time derivative of the state vector must point toward the sliding surface in the region of attraction [1], [8]. For a rigorous mathematical discussion of the existence of sliding modes for such systems see [1], [2], [8], [11], [45], [46]. These types of systems are referred to as discontinuous systems in the literature.

The existence problem resembles a generalized stability problem, hence the second method of Lyapunov provides a natural setting for analysis. Specifically, stability to the switching surface requires selecting a generalized Lyapunov function \( V(t, x) \) which is positive definite and has a negative time derivative in the region of attraction. Formally stated:

**Definition 1** [2]: A domain \( D \) in the manifold \( a = 0 \) is a sliding mode domain if for each \( \epsilon > 0 \), there is \( \delta > 0 \), such that any motion starting within an \( n \)-dimensional \( \delta \)-vicinity of \( D \) may leave the \( n \)-dimensional \( \epsilon \)-vicinity of \( D \) only through the \( n \)-dimensional \( \epsilon \)-vicinity of the boundary of \( D \). See Fig. 9.

![Fig. 9. Two-dimensional illustration of domain of sliding mode.](image)

Since the region \( D \) lies on the surface \( a(x) = 0 \), dimension \([D]\) = \( n - m \). Hence:

**Theorem 1**: For the \( n - m \)-dimensional domain \( D \) to be the domain of a sliding mode, it is sufficient that in some neighborhood of \( \{ x | a(x) = 0 \} \) i.e., the representative point must approach the surface at least asymptotically. The largest such neighborhood is called the region of attraction. Geometrically, the tangent vector or time derivative of the state vector must point toward the sliding surface in the region of attraction [1], [8]. For a rigorous mathematical discussion of the existence of sliding modes for such systems see [1], [2], [8], [11], [45], [46]. These types of systems are referred to as discontinuous systems in the literature.

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Since the region \( D \) lies on the surface \( a(x) = 0 \), dimension \([D]\) = \( n - m \). Hence:

**Theorem 1**: For the \( n - m \)-dimensional domain \( D \) to be the domain of a sliding mode, it is sufficient that in some \( n \)-dimensional \( U \supset D \), there exists a function \( V(t, x, a) \) continuously differentiable with respect to all of its arguments, satisfying the following conditions:

1. \( V(t, x, a) \) is positive definite with respect to \( a \), i.e., \( V(t, x, a) > 0 \) for \( a \neq 0 \) and arbitrary \( t, x \), and \( V(t, x, 0) = 0 \); and on the sphere \( |a| = \rho \) for all \( x \in \Omega \) and any \( t \) the relations
   \[ \inf_{|a| = \rho} V(t, x, a) = h_\rho, \quad h_\rho > 0 \tag{3.5} \]

2. The total time derivative of \( V(t, x, a) \) for the system (3.1) has a negative supremum for all \( x \in \Omega \) except for \( x \) on the switching surface where the control inputs are undefined, and hence the derivative of \( V(t, x, a) \) does not exist.

**Proof**: The proof is given in [1].

A sliding mode is globally reachable if the domain of attraction is the entire state space. Otherwise the domain of attraction is a subset of the state space.

The structure of the function \( V(t, x, a) \) determines the ease with which one computes the actual feedback gains implementing a VSC design. For poorly chosen Lyapunov functions, the feedback gain computations can be untenable.

For all single input systems a suitable Lyapunov function is \( V(t, x) = a^2(x) \) which clearly is globally positive definite. In VSC, \( a \) will depend on the control and hence if switched feedback gains can be chosen so that

\[ \dot{a}^2 = a^2 \frac{da}{dt} < 0 \tag{3.7} \]

in the domain of attraction, then the state trajectory converges to the surface and is restricted to the surface for all subsequent time. The feedback gains which would implement an associated VSC design are straightforward to compute in this case [1], [2], [8], [11], [41], [29].

**Illustrative Design Examples**

To illustrate the single input VSC design procedure consider the single pendulum system of [22], [23] having nonlinear state model \( x(t) = A(x(t)) + Bu(t) \) where \( x = [x_1, x_2]^T \)

\[
A(x) = \begin{bmatrix} 0 & 1 \\ -\sin(x_1) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

with control \( u(t) = k_1(x_1) x_1 + k_2(x_2) x_2 \) where

\[
k_1(x) = \begin{cases} \alpha_1(x), & \text{if } a(x)x_1 > 0 \\ \beta_1(x), & \text{if } a(x)x_1 < 0 \end{cases}
\]

and \( a(x) = [s_1, s_2]^T x \). The feedbacks \( \alpha_1(x) \) and \( \beta_1(x) \) are chosen so that \( a(x)x_1 < 0 \). Some simple substitutions show that

\[
a(x)x_1 = a(x)x_1 \left[ s_2 \left( \frac{\sin(x_1)}{x_1} \right) \right] + a(x)x_2 \left[ s_1 + s_2 k_2(x) \right] < 0
\]

if

\[
\alpha_1(x) = \alpha_1 < \min_{x_1} \left[ \frac{\sin(x_1)}{x_1} \right] = -1
\]

\[
\beta_1(x) = \beta_1 > \max_{x_1} \left[ \frac{\sin(x_1)}{x_1} \right] = 1
\]

and \( \alpha_2 < -(s_2/s_2) \) and \( \beta_2 > -(s_2/s_2) \). Hence, computation of the feedback gains is straightforward.

For multivariable systems, useful Lyapunov functions prove difficult to find, except in special cases [1], [2]. Specifically:

1. Suppose there exists a positive definite, symmetric transformation \( W(t, x) \) such that \( R(t, x) = -W(t, x) SB(t, x) \)

\[ h_\rho > 0 \tag{3.6} \]

hold, where \( h_\rho \) and \( H_\rho \) depend on \( \rho (h_\rho \neq 0 \text{ if } \rho \neq 0) \).

2. The total time derivative of \( V(t, x, a) \) for the system (3.1) has a negative supremum for all \( x \in \Omega \) except for \( x \) on the switching surface where the control inputs are undefined, and hence the derivative of \( V(t, x, a) \) does not exist.

**Proof**: The proof is given in [1].
where $R = [r_{ij}]$ has the property $r_{ii} > \sum_{j \neq i}^{m}|r_{ij}|$ for $i = 1, \ldots, m$, i.e., $R(t, x)$ is diagonally dominant. If so, the recommended form of $V(t, x, a)$ is a simple quadratic $V(t, x, a) = a^T R a$ where $R$ is symmetric and diagonalizes $SB$.

2) Suppose $SB(t, x)$ is symmetric. The recommended form here is $V(t, x, a) = a^T R a$ where $R$ is symmetric and diagonalizes $SB$.

3) Suppose $SB(t, x)$ is diagonally dominant. Here the recommended form of $V(t, x, a)$ is a simple quadratic $V(t, x, a) = a^T R a$ where $W(t, x)$ is a nonsingular diagonal matrix.

4) Finally suppose $SB(t, x)$ is diagonal. The recommended form of $V(t, x, a)$ is $V(t, x, a) = a^T a$.

For these cases, the necessary feedback gains needed to implement a VSC design are straightforward to compute. For other cases, one usually executes some type of transformation to obtain such a form. Section VI provides a detailed discussion of case 4) above.

To illustrate that a poor choice of a Lyapunov function may lead to extreme difficulties in solving for the necessary control gains, consider the hypothetical multivariable system

\[
\dot{x}(t) = A x(t) + B u(t) \tag{3.8a}
\]

where

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.8b}
\]

with the sliding surface

\[
\sigma(x) = S x = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \tag{3.9}
\]

and the control law $u(t) = \Psi x(t)$ where $\Psi = [\Psi_i]$ and

\[
\Psi_i = \begin{cases} \Psi^+_i, & \text{if } x_i \sigma_i > 0 \\ \Psi^-_i, & \text{if } x_i \sigma_i < 0 \end{cases} \tag{3.10}
\]

for $i = 1, 2$ and $j = 1, 2, 3$. Observe that

\[
\sigma^T \sigma = (1 - \Psi_{11} + 2 \Psi_{21}) x_1 \sigma_1 + (-1 + \Psi_{11} - \Psi_{22}) x_1 \sigma_2 \\
+ (3 - \Psi_{12} + 2 \Psi_{22}) x_2 \sigma_1 + (\Psi_{12} - \Psi_{22}) x_2 \sigma_2 \\
+ (5 - \Psi_{13} + 2 \Psi_{23}) x_3 \sigma_1 + (-4 + \Psi_{13} - \Psi_{23}) x_3 \sigma_2. \tag{3.11}
\]

A sliding mode exists if (3.11) is negative in the domain of attraction. In the single-input case this is usually accomplished by making each term in the sum negative. By considering just the first two terms in (3.11), it is necessary to simultaneously satisfy

\[
(1 - \Psi_{11} + 2 \Psi_{21}) x_1 \sigma_1 < 0 \\
(-1 + \Psi_{11} - \Psi_{22}) x_1 \sigma_2 < 0. \tag{3.12}
\]

There are four cases to consider.

Case 1: $x_1 \sigma_1 > 0$ and $x_1 \sigma_2 > 0$

Case 2: $x_1 \sigma_1 > 0$ and $x_1 \sigma_2 < 0$

Case 3: $x_1 \sigma_1 < 0$ and $x_1 \sigma_2 > 0$

Case 4: $x_1 \sigma_1 < 0$ and $x_1 \sigma_2 < 0$.

After some calculations, one concludes that

\[
\begin{align*}
&i) \quad \Psi^+_1 < 0 \\
&ii) \quad \Psi^+_2 > 0 \\
&iii) \quad 1 + \Psi^+_1 > \Psi^+_1 > 1 + 2 \Psi^+_1.
\end{align*}
\]

Given (i) and (ii), it is not possible to satisfy (iii).

Because of this and the obvious difficulty of solving (3.12) directly, the use of the Lyapunov function $V = .5a^T a$ and the control law (3.10) make this problem more difficult than necessary. By using a different Lyapunov function or another control law it is relatively straightforward to compute the control gains. For example, let

\[
V = .5a^T a
\]

and

\[
u(x) = -(SB)^{-1} S A x - (SB)^{-1} \sigma.
\]

This control forces $a^T \sigma = -1 < 0$; thus a sliding mode exists on the sliding surface (3.9) and is reachable for all $x \in \mathbb{R}^3$.

Also, a different sliding surface will work, for example if

\[
S = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 1 & -1 \end{bmatrix}
\]

where $\epsilon$ is a small positive constant, it is possible to solve for the gains $\Psi_i$ since $SB$ is diagonally dominant. Lastly, we point out that using a multi-input diagonalization method, described in a later part of the paper, controller design for this problem is easily accomplished.

Design Procedure Overview

From the above discussion it becomes clear that VSC design breaks down into two phases. Phase 1 entails constructing switching surfaces so that the system restricted to the switching surfaces produces a desired behavior. Phase 2 entails constructing switched feedback gains which drive the plant state trajectory to the sliding surface and maintain it there.

The actual details of this procedure are developed in Sections IV through VII.

IV. EXISTENCE AND UNIQUENESS OF SOLUTIONS TO VSC SYSTEMS

VSC produces system dynamics with discontinuous right-hand sides due to the switching action of the controller. Thus they fail to satisfy conventional existence and uniqueness results of differential equation theory. Nevertheless an important aspect of VSC design is the presumption that the plant behaves in a unique way when restricted to $\sigma(x) = 0$. Therefore the problem of existence and uniqueness of differential equations with discontinuous right-hand sides is of fundamental importance.

Various types of existence and uniqueness theorems can be found in [1], [8], [35]. Supplementary material also exists in [52], [53]. However, one of the earliest and conceptually straightforward approaches is the method of Filippov [46]. We will briefly review this method as background to the
the previous section. In particular, computation of

\section*{V. SLIDING SURFACE DESIGN}

above referenced results and as an aid in understanding variable structure system behavior on the switching surface.

Consider the following nth order, single input system

\[ \dot{x}(t) = f(t, x, u) \quad (4.1) \]

with the following general control strategy

\[ u = \begin{cases} u^+(t, x), & \text{if } \sigma(x) > 0 \\ u^-(t, x), & \text{if } \sigma(x) < 0 \end{cases} \quad (4.2) \]

It can be shown from Filippov's work in [46] that the state trajectories of (4.1) with control (4.2) on \( \sigma(x) = 0 \) are the solutions of the equation (see Fig. 10)

\[ \dot{x}(t) = \alpha f^+ + (1 - \alpha) f^- = f^0, \quad 0 \leq \alpha \leq 1 \quad (4.3) \]

where \( f^+ = f(t, x, u^+), f^- = f(t, x, u^-) \), and \( f^0 \) is the resulting velocity vector of the state trajectory in a sliding mode. Solving the equation \( \langle da, f^0 \rangle = 0 \) for \( \alpha \) yields

\[ \alpha = \frac{\langle da, f^- \rangle}{\langle da, f^- - f^+ \rangle} \]

provided 1) \( \langle da, f^- - f^+ \rangle > 0 \), and 2) \( \langle da, f^+ \rangle \leq 0 \) and \( \langle da, f^- \rangle \geq 0 \) where the notation \( \langle a, b \rangle \) denotes the inner product of \( a \) and \( b \) also written as \( "a \cdot b" \), and \( da = \text{grad } \sigma(x) \).

Therefore one may conclude that on the average, the solution to (4.1) with control (4.2) exists and is uniquely defined on \( \sigma(x) = 0 \). Notice also that this technique can be used to determine the behavior of the plant in a sliding mode.

\section*{V. SLIDING SURFACE DESIGN}

Filippov's method is one possible technique for determining the system motion in a sliding mode as outlined in the previous section. In particular, computation of \( \hat{s} \) represented the "average" velocity \( \dot{s} \) of the state trajectory restricted to the switching surface. A more straightforward technique easily applicable to multi-input systems is the method of equivalent control, as proposed by Utkin in [1], [4], and Draženović in [77].

\textbf{The Method of Equivalent Control}

The method of equivalent control is a means for determining the system motion restricted to the switching surface \( \sigma(x) = 0 \). Suppose that \( t_0 \) the state trajectory of the plant intercepts the switching surface and a sliding mode exists for \( t \geq t_0 \). The existence of a sliding mode implies 1) \( \sigma(x(t_0)) = 0 \), and 2) \( \sigma(x(t)) = 0 \) for all \( t \geq t_0 \). From the chain rule \( \langle \sigma(\dot{x}), \dot{x} \rangle = 0 \). Substituting for \( \dot{x} \) yields

\[ \left[ \frac{\partial \sigma}{\partial x} \right] \dot{x} = \left[ \frac{\partial \sigma}{\partial x} \right] (f(t, x) + B(t, x) u_{eq}) = 0 \]

where \( u_{eq} \) is the so-called equivalent control which solves this equation. After substituting this \( u_{eq} \) into (3.1), the motion of (3.1) describes the behavior of the system restricted to the switching surface provided the initial condition \( x(t_0) \) satisfies \( \sigma(x(t_0)) = 0 \).

To compute \( u_{eq} \), let us assume that the matrix product \( \partial \sigma(\dot{x}) B(t, x) \) is nonsingular for all \( t \) and \( x \). Then

\[ u_{eq} = -\left[ \frac{\partial \sigma}{\partial x} B(t, x) \right]^{-1} \frac{\partial \sigma}{\partial x} f(t, x). \quad (5.1) \]

Therefore, given \( \sigma(x(t_0)) = 0 \), the dynamics of the system on the switching surface for \( t \geq t_0 \) is given by

\[ \dot{x} = \left[ I - B(t, x) \left[ \frac{\partial \sigma}{\partial x} B(t, x) \right]^{-1} \frac{\partial \sigma}{\partial x} \right] f(t, x). \quad (5.2) \]

In the special case of a linear switching surface \( \sigma(x) = Sx \) \( = 0 \), \( \dot{x}, \partial \sigma/\partial x = S \), (5.2) reduces to

\[ \dot{x} = \left[ I - B(t, x) [SB(t, x)]^{-1} S \right] f(t, x). \quad (5.3) \]

This structure can be advantageously exploited in switching surface design.

Observe that (5.2) in conjunction with the constraint \( \sigma(x) = 0 \) determines the system motion on the switching surface. As such, the motion on the switching surface will be governed by a reduced order set of equations. This order reduction comes about because of the set of state variable constraints, \( \sigma(x) = 0 \).

The remaining parts of the section will describe 1) how one determines a reduced order set of dynamical equations governing the system motion on the switching surface, and 2) how to choose surface parameters \( S \) for a linear switching surface \( \sigma(x) = Sx = 0 \), so that the system in a sliding mode exhibits the desired behavior.

Before closing this subsection, the reader should note that some control applications require a time-varying switching surface \( \sigma(x, t) = 0 \). In this case, \( \dot{x}(t, x) = (\partial \sigma/\partial t) + (\partial \sigma/\partial x) \dot{x} \) and the equivalent control takes the form

\[ u_{eq} = -\left[ \frac{\partial \sigma}{\partial x} B(t, x) \right]^{-1} \frac{\partial \sigma}{\partial x} \left( \frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x} \right) f(t, x). \quad (5.4) \]

For simplicity of exposition, we have avoided the added complexity of the time-varying surface throughout most of the paper. Generalizations incorporating a time-varying component of \( \sigma(x, t) \) are straightforward to construct. The time-varying surface structure will appear briefly when discussing diagonalization methods in Section VI and more commonly when discussing uncertain systems and VSC in Section VII.
Reduction of Order

For sake of clarity, we concentrate on the case of linear switching surface, \( \sigma(x) = Sx = 0 \). As mentioned above, in a sliding mode, the equivalent system must satisfy not only the \( n \)-dimensional state dynamics (5.2), but also the "\( m \)" algebraic equations, \( \sigma(x) = 0 \). The use of both constraints reduces the system dynamics from an \( n \)-th-order model to an \((n - m)\)-th-order model.

Specifically, suppose the nonlinear system of (3.1) is restricted to the switching surface of (3.4), i.e., \( \sigma(x) = Sx = 0 \), then it is possible to solve for \( m \) of the state variables, in terms of the remaining \( n - m \) state variables, if the rank \( [S] = m \).

The condition that rank \( [S] = m \) holds under the earlier assumption that \( [\partial \sigma/\partial x] B(t, x) \) is nonsingular for all \( t \) and \( x \). To obtain the solution, solve for \( m \) of the state variables (e.g., \( x_{n-m+1}, \ldots, x_n \)) in terms of the \( n - m \) remaining state variables. Substitute these relations into the remaining \( n - m \) equations of (5.3) and the equations corresponding to the \( m \) state variables. The resultant \((n - m)\)-th-order system fully describes the equivalent system given an initial condition satisfying \( \sigma(x) = 0 \).

Example 5.4: To clarify the above procedure and to pave the way for later examples consider the system \( x(t) = A(t, x) x(t) + Bu(t) \), where

\[
A(t, x) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Assume the third and fifth rows of \( A(t, x) \) have nonlinear time-varying entries which are bounded: \( a_{3i}^{\text{max}} \leq a_i(t, x) \leq a_{3i}^{\text{min}} \) for all \( t \in R^n \) and \( x \in \{0, \infty\} \).

The method of equivalent control produces the following equivalent system (as per (5.3))

\[
\dot{x}(t) = [I - B(SB)^{-1}S] A(t, x) x(t)
\]

provided \( \sigma(x(t_0)) = 0 \) for some \( t_0 \).

If the linear switching surface parameters are

\[
S = \begin{bmatrix}
s_{11} & s_{12} & s_{13} & s_{14} & s_{15} \\
s_{12} & s_{22} & s_{23} & s_{24} & s_{25} \\
\end{bmatrix}
\]

then

\[
SB = \begin{bmatrix}
s_{11} & s_{15} \\
s_{21} & s_{25} \\
\end{bmatrix}
\]

To simplify the example let us choose \( s_{13} s_{25} - s_{15} s_{23} = 1 \). Specifically, choose \( s_{13} = 2, s_{15} = s_{23} = s_{25} = 1 \). Then

\[
(SB)^{-1} = \begin{bmatrix}
s_{25} & -s_{15} \\
-s_{23} & s_{13} \\
-s_{13} s_{25} - s_{15} s_{23} & 1 \\
1 & -1 \\
\end{bmatrix}
\]

This produces

\[
\dot{x}(t) = [I - B(SB)^{-1}S] A(t, x) x(t)
\]

subject to \( \sigma(x) = 0 \), in which case

\[
\begin{bmatrix}
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & s_{11} - s_{11} & s_{22} - s_{12} & s_{24} - s_{14} & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & s_{11} - 2s_{21} & s_{12} - 2s_{22} & 0 & s_{14} - 2s_{24} \\
\end{bmatrix}
\]

(5.10)

Solving for \( x_3 \) and \( x_4 \) yields

\[
\begin{bmatrix}
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(5.11)

The reduced order equivalent linear time-invariant system is

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
\]

(5.12)

where \( \dot{x}_1 = x_1, \dot{x}_2 = x_2, \dot{x}_3 = x_3 \).

To see how control design might be accomplished, suppose a design constraint requires the spectrum of the equivalent system be \( \{ -1, -2, -3 \} \); the desired characteristic polynomial is

\[
\pi_4(\lambda) = \lambda^3 + 6\lambda^2 + 11\lambda + 6.
\]

The characteristic polynomial of the equivalent system given in (5.13) is

\[
\pi_4(\lambda) = \lambda^3 + (s_{12} - s_{22} + 2s_{24} - s_{14})\lambda^2
\]

\[
+ (s_{12} s_{24} - s_{14} s_{22} + s_{11} - s_{21})\lambda + (s_{11} s_{24} - s_{14} s_{21})
\]

Equating coefficients of like powers of \( \lambda \) produces the set of equations

\[
\begin{bmatrix}
0 & 1 & -1 & 0 & -1 & 2 \\
1 & s_{24} & -s_{22} & -1 & 0 & 0 \\
0 & 0 & -s_{14} & 0 & 0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
s_{11} \\
s_{12} \\
s_{14} \\
s_{21} \\
2 \\
\end{bmatrix}
\]

(5.13)

subject to \( \sigma(x) = 0 \), in which case

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & s_{11} - s_{11} & s_{22} - s_{12} & s_{24} - s_{14} & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & s_{11} - 2s_{21} & s_{12} - 2s_{22} & 0 & s_{14} - 2s_{24} \\
\end{bmatrix}
\]

(5.10)

subject to \( \sigma(x) = 0 \), in which case

\[
\begin{bmatrix}
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(5.11)

The reduced order equivalent linear time-invariant system is

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
\]

(5.12)

where \( \dot{x}_1 = x_1, \dot{x}_2 = x_2, \dot{x}_3 = x_3 \).

To see how control design might be accomplished, suppose a design constraint requires the spectrum of the equivalent system be \( \{ -1, -2, -3 \} \); the desired characteristic polynomial is

\[
\pi_4(\lambda) = \lambda^3 + 6\lambda^2 + 11\lambda + 6.
\]

The characteristic polynomial of the equivalent system given in (5.13) is

\[
\pi_4(\lambda) = \lambda^3 + (s_{12} - s_{22} + 2s_{24} - s_{14})\lambda^2
\]

\[
+ (s_{12} s_{24} - s_{14} s_{22} + s_{11} - s_{21})\lambda + (s_{11} s_{24} - s_{14} s_{21})
\]

Equating coefficients of like powers of \( \lambda \) produces the set of equations

\[
\begin{bmatrix}
0 & 1 & -1 & 0 & -1 & 2 \\
1 & s_{24} & -s_{22} & -1 & 0 & 0 \\
0 & 0 & -s_{14} & 0 & 0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
s_{11} \\
s_{12} \\
s_{14} \\
s_{21} \\
2 \\
\end{bmatrix}
\]

(5.13)
One solution accomplishing the control design objective is

\[
S = \begin{bmatrix} 1 & 1.8333 & 2 & -6 & 1 \\ 1 & 1.8333 & 1 & 0 & 1 \end{bmatrix}.
\]

In conclusion, the reduced order equivalent system with the desired eigenvalues is \( \dot{\lambda} = A_1 \lambda \), where

\[
\lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 6 \\ -1 & -1.8333 & -6 \end{bmatrix}.
\]

This example worked out so cleanly because the original system dynamics were given in the Luenberger canonical form. Systems not in this form often require a transformation to a more general form called the regular form [28].

**Regular Form and the Reduced Order Dynamics**

The regular form of the plant dynamics (3.1), is

\[
x_1 = f_1(t, x)
\]

\[
x_2 = f_2(t, x) + B(t, x)u
\]

where \( x_1 \in \mathbb{R}^{n-m} \) and \( x_2 \in \mathbb{R}^m \). A system in this form has simply computed reduced order equivalent dynamics, also referred to as the system equations of slow motion. The computation of this form assumes \( B(t, x) \) is an \( m \times m \) nonsingular mapping. This assumption is necessary for the existence of the equivalent control.

To compute the reduced order dynamics, assume a linear switching surface (this will be generalized later) of the form

\[
s(x) = [S_1 \quad S_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.
\]

Without loss of generality assume \( S_1 \) is nonsingular. Thus in a sliding mode

\[
x_2 = -S_2^{-1}S_1 x_1
\]

and

\[
x_1 = f_1(t, x) = f_1(t, x, -S_2^{-1}S_1 x_1)
\]

which is the reduced order dynamics. Observe that if \( f_1 \) has the very desirable linear structure

\[
x_1 = f_1(t, x) = A_{11} x_1 + A_{12} x_2
\]

then the reduced order dynamics becomes

\[
x_1 = [A_{11} - A_{12} S_2^{-1} S_1] x_1
\]

which has the feedback structure "\( A_{11} + A_{12} F \)" with \( F = -S_2^{-1} S_1 \) and \( A_{12} \) playing the role of the input matrix. If the pair \( (A_{11}, A_{12}) \) is controllable, then it is possible to effectively use classical feedback control design techniques to compute an \( F \) such that \( A_{11} + A_{12} F \) has desired characteristics. Having found \( F \), one can compute \( [S_1 \quad S_2] \) such that \( F = -S_2^{-1} S_1 \), thus completing the switching surface design. Note that one can use pole placement techniques, linear optimal control techniques, etc., to design \( F \). For more details of the linear case see Young et al. [25] and El-Ghezawi et al. [14].

For the more general case of a nonlinear switching sur-

\[
s(x) = \sigma(x) + S_2 x_2 = 0
\]

face consider

\[
s(x) = \sigma(x) + S_2 x_2 = 0
\]

which is linear in \( x_2 \) and possibly nonlinear in \( x_1 \). For this case, the reduced order dynamics in a sliding mode will have the form

\[
x_1 = f_1(t, x, -S_2^{-1} \sigma(x), x_2)
\]

An example of designing a nonlinear switching surface will be given later.

The next important question is how one transforms the given system dynamics (3.1) to the regular form of (5.14). We first consider the case of a linear switching surface of (3.13) and a nonsingular linear time invariant transformation \( z = T x \). Taking the time derivative of \( z \) yields

\[
z = T x = T f(t, x) + T B(t, x) u.
\]

If it is true that

\[
TB = \begin{bmatrix} 0 \\ \tilde{B}_1 \end{bmatrix}
\]

then in the new coordinates the dynamics of the plant (3.1) become

\[
\dot{z}_1 = \tilde{f}_1(t, z)
\]

\[
\dot{z}_2 = \tilde{f}_2(t, z) + \tilde{B}_1(t, z) u.
\]

Hence in a sliding mode the equivalent reduced order dynamics are given by (5.17) modulo the coordinate change, i.e.,

\[
\dot{z}_1 = \tilde{f}_1(t, z, -S_2^{-1} S_1 z_1)
\]

where \( [S_1 \quad S_2] = [S_1 \quad S_2] T^{-1} \).

If there is no linear transformation such that (5.22) is satisfied, then one must resort to nonlinear transformations of the form

\[
z = \tilde{T}(t, x) = \begin{bmatrix} T_1(t, x) \\ T_2(t, x) \end{bmatrix}
\]

where \( \tilde{T}(\cdot, \cdot) \) is a diffeomorphic transformation, 2) \( \tilde{T}(\cdot, \cdot): R \times R^n \rightarrow R^{n-m} \), and \( T(\cdot, \cdot): R \times R^n \rightarrow R^m \). Diffeomorphic [66],[67] means that there exists a continuous differentiable inverse mapping \( \tilde{T}(t, z) = x \) satisfying \( \tilde{T}(t, 0) = 0 \) for all \( t \).

Differentiating \( z \) in (5.25) with respect to time produces

\[
\dot{z} = \frac{\partial \tilde{T}}{\partial t} (t, x) + \frac{\partial \tilde{T}}{\partial x} (t, x) \dot{x}.
\]

Substituting (3.1) into (5.26) yields

\[
\dot{z} = \frac{\partial \tilde{T}}{\partial x} f(t, x) + \frac{\partial \tilde{T}}{\partial x} B(t, x) u(t) + \frac{\partial \tilde{T}}{\partial t}.
\]

If the transformation has the property that

\[
\frac{\partial \tilde{T}}{\partial x} B(t, x) = \begin{bmatrix} \tilde{T}_1 \frac{\partial \tilde{T}_1}{\partial x} \\ \tilde{T}_2 \frac{\partial \tilde{T}_2}{\partial x} \end{bmatrix} B(t, x) = \begin{bmatrix} 0 \\ \tilde{B}_1(t, x) \end{bmatrix}
\]

then in the new coordinates, the equations describing our system or plant are
\[ z_1 = \frac{\partial f}{\partial x}(t, \dot{t}(t, x)) + \frac{\partial f}{\partial t}(t, \dot{t}(t, x)) \]
\[ z_2 = \frac{\partial f}{\partial x}(t, \dot{t}(t, x)) + \frac{\partial f}{\partial t}(t, \dot{t}(t, x)) + \frac{\partial B(t, \dot{t}(t, x))}{\partial x} u \]
\[ \triangle \delta f(t, z) + \delta B(t, x) u. \]  
(5.29)

The problem of converting a nonlinear system to a canonical form, in particular the regular form, was explored in [28], [66], [67] among others.

VI. CONTROLLER DESIGN

Controller design is the second phase of the VSC design procedure mentioned earlier. Here the goal is to determine switched feedback gains which will drive the plant state trajectory to the switching surface and maintain a sliding mode condition. The presumption is that the sliding surface has been designed. In general, the control is an \( m \)-vector \( u(t) \) each of whose entries have the structure of the form

\[ u_i = \begin{cases} u_i^+(t, x), & \text{for } \sigma_i(x) > 0 \\ u_i^-(t, x), & \text{for } \sigma_i(x) < 0 \end{cases} \]  
(6.1)

where \( \sigma(x) = [\sigma_1(x), \ldots, \sigma_m(x)]^T = 0 \).

Diagonalization Methods

Our purpose here is to describe two different approaches to controller design labeled in the literature as diagonalization methods. The essential feature of these methods is conversion of a multi-input design problem into \( m \) single-input design problems.

Method 1 entails constructing a new control vector \( u^* \) via a nonsingular transformation, \( Q^{-1}(t, x) \) \( \frac{\partial \sigma}{\partial x} \) \( B(t, x) \) of the original control \( u \) defined as

\[ u^*(t) = Q^{-1}(t, x) \left[ \frac{\partial \sigma}{\partial x} (x) \right] B(t, x) u(t) \]  
(6.2)

where \( Q(t, x) \) is an arbitrary \( m \times m \) diagonal matrix with elements \( q_i(t, x) (i = 1, \ldots, m) \) such that \( \inf q_i(t, x) > 0 \) for all \( t \geq 0 \) and all \( x \). The actual conversion of the \( m \)-input design problem to \( m \)-input single design problems is accomplished by the \( \frac{\partial \sigma}{\partial x} \) \( B(t, x) \)-term with the diagonal entries of \( Q^{-1}(t, x) \) merely allowing flexibility in the design, for example by weighting the various control channels of \( u^* \).

Often \( Q(t, x) \), is chosen as the identity. In terms of \( u^* \) the state dynamics become

\[ \dot{x}(t) = f(t, x) + B(t, x) \left[ \frac{\partial \sigma}{\partial x} (x) \right] B(t, x) u^*(t). \]  
(6.3)

Although this new control structure looks more complicated, the structure of \( \dot{\sigma}(x) = 0 \) permits one to independently choose the \( m \)-entries of \( u^* \) to satisfy the sufficient conditions for the existence and reachability of a sliding mode. Once \( u^* \) is known it can be unraveled by inverting the transformation to yield the required \( u \). To see this, recall that for existence and reachability of a sliding mode it is enough to satisfy the condition \( \dot{\sigma}(x) \sigma(x) < 0 \). In terms of \( u^* \)

\[ \dot{\sigma}(x) = \frac{\partial \sigma}{\partial x} (x) f(t, x) + Q(t, x) u^*(t). \]  
(6.4)

Thus if the entries \( u_i^+ \) and \( u_i^- \) are chosen to satisfy

\[ q_i(t, x) u_i^+ < -\sigma_i(x) f(t, x) \]
\[ = -\sum_{i=1}^{n} q_i f(t, x) \text{ when } \sigma_i(x) > 0 \]  
(6.5a)

\[ q_i(t, x) u_i^- > -\sigma_i(x) f(t, x) \]
\[ = -\sum_{i=1}^{n} q_i f(t, x) \text{ when } \sigma_i(x) < 0 \]  
(6.5b)

then sufficient conditions for the existence and reachability are satisfied where \( s_j \) equals the \( j \)-entry of \( \sigma(x) \) is which is the \( j \)-th row of \( \frac{\partial \sigma}{\partial x} \). In particular, the conditions of (6.5) force each term in the summation of \( \sigma^2 \) to be negative definite. As mentioned, the control actually implemented is

\[ u(t) = \left( \frac{\partial \sigma}{\partial x} (x) B(t, x) \right)^{-1} Q(t, x) u^*(t). \]  
(6.6)

Other sufficient conditions for the existence of a sliding mode can also be used.

The second method of diagonalization requires a nonsingular transformation of \( \sigma \) rather than the control \( u \). In particular, consider the new switching surface

\[ \sigma^*(t, x) = Q(t, x) \sigma(x) = 0 \]  
(6.7)

for an appropriate transformation \( Q(t, x) \). This method is based on the fact that the equivalent system is invariant to a nonsingular switching surface transformation as verified in the following theorem.

**Theorem 2** [1]: Suppose that the original system is given by (3.1), (3.2) with switching surfaces \( \sigma_i(t, x) = 0 \) \( (i = 1, \ldots, m) \), then the sliding motion (trajectory of the equivalent system) is invariant to the transformation of the switching surface \( \sigma^*(t, x) = Q(t, x) \sigma(x) = 0 \in R^m \) if \( \|Q\| \) and \( \|Q^{-1}\| \) are bounded for all \( t, x \in S \subseteq R \times R^m \).

**Proof:** First, by the method of equivalent control

\[ \dot{\sigma}^* = \Omega(t, x) \frac{\partial \sigma}{\partial x} (x) (f(t, x) + Bu^*) + \Omega \frac{\partial \dot{\sigma}}{\partial t} \dot{\sigma} = 0. \]  
(6.8)

Since \( \Omega(t, x) \) is a nonsingular \( m \times m \) matrix

\[ u_{eq}^* = -\left( \frac{\partial \sigma}{\partial x} (x) B \right)^{-1} \frac{\partial \sigma}{\partial x} (x) f - \left( \frac{\partial \sigma}{\partial x} (x) B \right)^{-1} \Omega^{-1} \dot{\sigma} = 0. \]  
(6.9)

which differs from \( u_{eq} \) only by the term \( \left( \frac{\partial \sigma}{\partial x} (x) B \right)^{-1} \Omega^{-1} \dot{\sigma} \). However, on the switching surface, \( \sigma = 0 \), thus

\[ u_{eq}^* = -\left( \frac{\partial \sigma}{\partial x} (x) B \right)^{-1} \frac{\partial \sigma}{\partial x} (x) f - \left( \frac{\partial \sigma}{\partial x} (x) B \right)^{-1} \frac{\partial \sigma}{\partial t} = u_{eq}. \]  
(6.10)

Hence the equivalent systems are identical and the motions in the sliding mode coincide.

Loosely stated, Theorem 2 says that the motion in the sliding mode is independent of a nonsingular possibly time-varying transformation of the switching surfaces. Observe that any nonsingular transformation \( \Omega \) with bounded derivatives will produce the same "equivalent" system.

In this second diagonalization procedure, we select
\( \Omega(t, x) \) so that \( \Omega(t, x) (d\sigma/dx)(x) B(t, x) \) is a diagonal matrix, say \( Q(t, x) = \text{diag}(q_1(t, x), \ldots, q_{n_x}(t, x)) \) whose entries are bounded away from zero. Specifically select \( \Omega(t, x) \) as

\[
\Omega(t, x) = Q(t, x) \left[ \frac{d\sigma}{dx}(x) B(t, x) \right]^{-1}
\]

(6.11)

for appropriate \( Q(t, x) \). Again \( Q(t, x) \) is often chosen as the identity matrix.

In order to determine the existence and reachability conditions it is necessary to compute \( \sigma^* \) as

\[
\dot{\sigma}^* = Q(t, x) \left[ \frac{d\sigma}{dx}(x) B(t, x) \right]^{-1} \frac{d\sigma}{dx}(x) \]

\( f(t, x) + Q(t, x) u + \dot{\sigma}^* \Omega^{-1}(t, x) \sigma^* \)

(6.12)

where again the control term \( u \) "enters" \( \dot{\sigma}^* \) via the diagonal matrix \( Q(t, x) \).

Sufficient conditions for reachability/existence of sliding mode are met if for any point in the state space and for all \( t \geq t_0 \), \( q_1^*(t, x) \) and \( q_{n_x}^*(t, x) \) are of opposite sign. Specifically, this requires that

\[
q_i(t, x) u_i^* < -\sigma_i^*(t, x) - q_{n_x}^*(t, x), \quad \text{for } \sigma_i^* > 0
\]

\[
q_i(t, x) u_i^* > -\sigma_i^*(t, x) - q_{n_x}^*(t, x), \quad \text{for } \sigma_i^* < 0
\]

(6.13)

where \( q_i(t, x) \) is the \( i \)th entry of \( q_1^*(t, x) \).

Example 6.14: To clarify the diagonalization methods, consider the system of Example 5.4 where

\[
\dot{x}(t) = A(t, x) x(t) + B u(t)
\]

(6.15a)

and

\[
A(t, x) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(6.15b)

The surface \( \sigma(x) = Sx = 0 \) was designed to have

\[
S = \begin{bmatrix}
1 & 1.8333 & 2 & -6 & 1 \\
1 & 1.8333 & 1 & 0 & 1 \\
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25}
\end{bmatrix} = \begin{bmatrix}
S_1 \\
S_2
\end{bmatrix}
\]

(6.16)

The objective of this example is to illustrate phase 2 of the VSC controller design process using the first and second diagonalization methods described above. The first design employs method 1 which transforms the control \( u \) as per (6.2)

\[
\sigma^*(t) = Q^{-1}(t, x) S B(t, x) u(t)
\]

(6.17)

where \( Q(t, x) \) is a nonsingular diagonal matrix such that \( \inf q_i(t, x) > 0 \). For simplicity choose

\[
Q = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\]

(6.18)

The choice was random. However, the diagonal entries of \( Q \) can be chosen to weight different control channels or to compensate somewhat for the "distortion" introduced by \( [S B(t, x)]^{-1} \). As per (6.3), the state dynamics driven by \( u^* \) are

\[
x(t) = A(t, x) x(t) + B(t, x) [S B(t, x)]^{-1} Q(t, x) u^*(t).
\]

(6.19)

In computing the feedback gains to meet the existence conditions, (6.4) becomes

\[
\dot{\sigma}(t) = S x(t) = S A(t, x) x(t) + Q(t, x) u^*(t, x).
\]

(6.20)

It follows that for the first switching surface \( \sigma_1(x) = [S_{11}, \ldots, S_{15}]^T \) \( S_1 x(t) \) one has

\[
S_1 A(t, x) x(t) = \{ (2a_{11} + a_{21}) x_1 + (1 + 2a_{21} + a_{22}) x_2 x_3 + (1.8333 + 2a_{13} + a_{23}) x_3 + (2a_{15} + a_{25}) x_4 + (2a_{45} + a_{55}) - 6 \}
\]

(6.21)

Recall the assumption: \( a_{i,j}^\text{min} \leq a_{i,j}(t, x) \leq a_{i,j}^\text{max}, i = 1, 2, j = 1, \ldots, 5 \). Under this assumption and control law \( u^* = K x \) where \( K = \{ k_j \} \), to satisfy the existence condition of (6.20), \( k_j \) must satisfy the following:

\[
k_{11} = \begin{cases}
- (2a_{11}^\text{max} + a_{11}^\text{min}) & \text{if } \sigma_1 x_1 > 0 \\
- (2a_{11}^\text{max} + a_{11}^\text{min}) & \text{if } \sigma_1 x_1 < 0
\end{cases}
\]

(6.22)

With regard to the second switching surface, \( \sigma_2(x) = [S_{21}, \ldots, S_{25}]^T \) \( S_2 x(t) \) one has

\[
S_2 A(t, x) x(t) = \{ (a_{11} + a_{12}) x_1 + (1 + a_{12} + a_{22}) x_2 x_3 + (1.8333 + a_{13} + a_{23}) x_3 + (a_{44} + a_{54}) x_4 + (a_{45} + a_{55}) x_5 \}
\]

(6.23)
Let $u_i^* = k_i x_i$ to satisfy the existence conditions (6.20), $k_i$ must satisfy the following:

$$
k_{21} = \begin{cases} 
-1/(a_{11}^{\text{max}} + a_{1}^{22}) & \text{if } s_2 x_1 > 0 \\
-1/(a_{11}^{\text{min}} + a_{1}^{22}) & \text{if } s_2 x_1 < 0 
\end{cases}
$$

$$
k_{22} = \begin{cases} 
-1/(1 + a_{2}^{22} + a_{22}^{\text{max}}) & \text{if } s_2 x_2 > 0 \\
-1/(1 + a_{2}^{22} + a_{22}^{\text{min}}) & \text{if } s_2 x_2 < 0 
\end{cases}
$$

$$
k_{23} = \begin{cases} 
-1/(1.8333 + a_{2}^{23} + a_{22}^{\text{max}}) & \text{if } s_2 x_3 > 0 \\
-1/(1.8333 + a_{2}^{23} + a_{22}^{\text{min}}) & \text{if } s_2 x_3 < 0 
\end{cases}
$$

$$
k_{24} = \begin{cases} 
-1/(a_{24}^{\text{max}} + a_{24}^{\text{min}}) & \text{if } s_2 x_4 > 0 \\
-1/(a_{24}^{\text{max}} + a_{24}^{\text{min}}) & \text{if } s_2 x_4 < 0 
\end{cases}
$$

$$
k_{25} = \begin{cases} 
-1/(a_{25}^{\text{max}} + a_{25}^{\text{min}}) & \text{if } s_2 x_5 > 0 \\
-1/(a_{25}^{\text{max}} + a_{25}^{\text{min}}) & \text{if } s_2 x_5 < 0 
\end{cases}
$$

Summarizing, $u^*(t) = Kx(t)$ where the entries of $K$ are specified by (6.22) and (6.24). Since $u(t) = (SB)^{-1}Qu^*(t, x)$, the actual control is $u(t) = (u_1(t), u_2(t))^T$, so that

$$
\begin{bmatrix}
  u_1 \\
u_2
\end{bmatrix} = \begin{bmatrix}
  1 & -2 \\
-1 & 4
\end{bmatrix} \begin{bmatrix}
  u_1^* \\
u_2^*
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}.
$$

(6.25)

This completes the illustration of the first method of diagonalization. Attention now turns to the second method. Again assume that the switching surface design is complete. In the second diagonalization method, the objective is to decouple the controls by making a nonsingular transformation of the switching surface. The control components, the entries of $u$, now switch on $a_i^*(x) = 0, i = 1, 2$, with $a_i^*(x)$ given by (6.7). To perform the diagonalization choose $\Omega(t, x)$ according to (6.11) where $(\partial \Omega/\partial x) = S$. If $Q$ is chosen as per (6.18), then

$$
\Omega(t, x) = Q(t, x)(SB(t, x))^{-1}
$$

(6.26)

To construct a controller meeting the existence conditions of a sliding mode consider the derivative of $a^*$ given by (6.12), noting that $\Omega = 0$

$$
\dot{a}^*(t, x) = \Omega(t, x) SA(t, x) x(t) + Q(t, x) u(t)
$$

(6.27)

where

$$
\Omega(t, x) SA(t, x) x(t) = \begin{bmatrix}
a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + a_{14} x_4 + (a_{15} - 6) x_5 \\
2 a_{21} x_1 + 2(1 + a_{22}) x_2 + 2(a_{23} + 1.8333) x_3 \\
2 a_{24} x_4 + 2(a_{25} + 6) x_5
\end{bmatrix}
$$

(6.28)

and $Qu = [u_1, 2 u_2]^T$.

The conditions for the existence of a sliding mode (6.13), for this example are

$$
q_i u_i^* < -a_i^2(t, x), \quad i = 1, 2
$$

(6.29)

$$
q_i u_i^* > -a_i^2(t, x)
$$

(6.29)

where $a_i^*(x, t)$ is the $i$th component of $a^*(t, x) = \Omega(t, x) SA(t, x) x(t)$.

Let $u = Kx$ where $K = [k_{ij}]$, $i = 1, 2, j = 1, \cdots, 5$. Since $a_{ij}^\text{min} \leq a_i(t, x) \leq a_{ij}^\text{max}$, $i = 1, 2, j = 1, \cdots, 5$, then to satisfy (6.29), it is required that

$$
k_{11} = \begin{cases} 
-\infty & \text{if } s_1 x_1 > 0 \\
\infty & \text{if } s_1 x_1 < 0 
\end{cases}
$$

$$
k_{21} = \begin{cases} 
-\infty & \text{if } s_2 x_2 > 0 \\
\infty & \text{if } s_2 x_2 < 0 
\end{cases}
$$

$$
k_{12} = \begin{cases} 
-\infty & \text{if } s_1 x_2 > 0 \\
\infty & \text{if } s_1 x_2 < 0 
\end{cases}
$$

$$
k_{22} = \begin{cases} 
-\infty & \text{if } s_2 x_2 > 0 \\
\infty & \text{if } s_2 x_2 < 0 
\end{cases}
$$

$$
k_{13} = \begin{cases} 
-\infty & \text{if } s_1 x_3 > 0 \\
\infty & \text{if } s_1 x_3 < 0 
\end{cases}
$$

$$
k_{23} = \begin{cases} 
-\infty & \text{if } s_2 x_3 > 0 \\
\infty & \text{if } s_2 x_3 < 0 
\end{cases}
$$

$$
k_{14} = \begin{cases} 
-\infty & \text{if } s_1 x_4 > 0 \\
\infty & \text{if } s_1 x_4 < 0 
\end{cases}
$$

$$
k_{24} = \begin{cases} 
-\infty & \text{if } s_2 x_4 > 0 \\
\infty & \text{if } s_2 x_4 < 0 
\end{cases}
$$

$$
k_{15} = \begin{cases} 
-\infty & \text{if } s_1 x_5 > 0 \\
\infty & \text{if } s_1 x_5 < 0 
\end{cases}
$$

$$
k_{25} = \begin{cases} 
-\infty & \text{if } s_2 x_5 > 0 \\
\infty & \text{if } s_2 x_5 < 0 
\end{cases}
$$

(6.30)

Since $\Omega(t, x)$ in (6.26) is constant, $\dot{\Omega} = 0$ and the above are also sufficient conditions for reaching the sliding surface.

As a second rather important illustration of the first diagonalization method, recall the example developed in (3.8) through (3.11). This multi-input example demonstrated how a poor choice of Lyapunov function in conjunction with a naive choice of controller led to an inconsistent solution of the equations defining the switched feedback gains needed to drive the state trajectory of the plant (3.8) to the switching surface of (3.9). The use of a diagonalization method circumvents this difficulty by converting the problem to one where the intuitive choice of Lyapunov function actually will work.

Using the first diagonalization method with $Q = I$ and with $B$ defined in (3.8b) and the switching surface $S$ defined in (3.9), then as per (6.2), the new control has the form

$$
u^* = Q^{-1}[SB]u = \begin{bmatrix}
-1 & 2 \\
1 & -1
\end{bmatrix}u = Kx
$$

(6.31)

DeCARLO et al.: CONTROL OF NONLINEAR MULTIVARIABLE SYSTEMS
The system from an initial condition onto the surface \( \sigma_1 = 0 \). The second control then drives the system onto the intersection of \( \sigma_1 = 0 \) and \( \sigma_2 = 0 \), while \( u_1 \) maintains a sliding mode on \( \sigma_1 = 0 \). The third control \( u_3 \) drives the system along the intersection of the surfaces \( \sigma_1 = 0 \) and \( \sigma_2 = 0 \) to the intersection of the first three switching surfaces. This hierarchy of controls is continued until the last control \( u_m \) drives the system to a sliding mode on the intersection of all the \( m \) switching surfaces.

Design of the control entry \( u_1 \) presupposes i) existence of a sliding mode on \( \sigma_1 = 0, j = 1, \ldots, k - 1 \) for any possible value of the controls \( u_1 \) through \( u_m \), and ii) knowledge of the system structure in these sliding modes. Since all controls \( u_k, k < m \), depend on the values taken on by the control \( u_m \), \( u_m \) must precede the design of \( u_{m-1}, u_{m-2}, \ldots, u_1 \). In addition, design of the control \( u_2 \) presupposes the system structure obtained assuming a sliding mode exists on \( \sigma_1 = 0 \). This system structure results by replacing \( u_1 \) with the Utkin-Draženović equivalent control \( u_{eq} \). Call the resulting system structure \( \Sigma \).

To determine \( \Sigma \), consider \( (\partial \sigma/\partial x) \partial \sigma = 0 \), which, using (3.1), implies that

\[
 u_{eq} = - \left[ \frac{\partial \sigma}{\partial x} b_j \right]^{-1} \left( \frac{\partial \sigma}{\partial x} f + \frac{\partial \sigma}{\partial x} [b_2, \ldots, b_m] \begin{bmatrix} u_2 \\ \vdots \\ u_m \end{bmatrix} \right)
\]  

(6.35)

where \( b_j \) is the \( j \)th column of \( B(t, x) \). Note that this relationship requires \((\partial \sigma/\partial x) b_j \neq 0 \). In fact, it is necessary to assume that \((\partial \sigma/\partial x) b_j \neq 0 \) for all \( i \). Substituting (6.35) into (3.1) produces the equivalent system model

\[
 \dot{x} = f'(t, x) + B'(t, x) \begin{bmatrix} u_2 \\ \vdots \\ u_m \end{bmatrix}
\]  

for an appropriately formed \( f'(t, x) \) and \( B'(t, x) \).

Design of \( u_1 \) presupposes the system structure obtained by supposing a sliding mode exists on \( \sigma_1 = 0 \) for the system structure \( \Sigma \). This implies a sliding mode exists on \( \sigma_1 = \sigma_2 = 0 \). Call the resulting structure \( \Sigma \). Of course \( \Sigma \) is specified by replacing \( u_2 \) in \( \Sigma \) by \( u_{eq} \). In general, \( u_{k+1} \) is designed supposing a sliding mode exists on \( \sigma_1 = 0 \) for the system structure \( \Sigma^{k-1} \) and hence on \( \sigma_1 = 0, j = 1, \ldots, k \). The new system structure is \( \Sigma \) and is denoted by the dynamics

\[
 \dot{x} = f^k(t, x) + B^k(t, x) \begin{bmatrix} u_2 \\ \vdots \\ u_m \end{bmatrix}
\]  

(6.36)

Before designing the first control \( u_m \) it is clearly necessary to sequentially determine the set of equivalent systems \( \{ \Sigma^0, \Sigma^1, \ldots, \Sigma^{m-1} \} \). Given \( \Sigma^{m-1}, u_m \) has gains chosen to satisfy the reachability and existence conditions for a sliding mode on \( \sigma_m = 0 \). After this, one presumes the system structure \( \Sigma^{m-2} \) and proceeds to find gains for \( u_{m-1} \) so that a sliding mode exists on \( \sigma_{m-1} = 0 \) given the gains computed for \( u_m \).

To see how the existence and reachability conditions are determined at the \((k + 1)\)th step, realize that a sliding mode exists and is reachable on \( \sigma_{k+1} = 0 \) provided \( \sigma_{k+1} < 0 \) is chosen so that

\[
 \sigma_{k+1} + a_{k+1} < 0
\]  

(6.37)
for all values of $u_{k+2}, \ldots, u_m$. Observe that $\phi_{k+1}$ has the form
\[
\phi_{k+1} = \nabla \phi_{k+1} f^k + \sum_{i=k+2}^{m} \nabla \phi_{k+1} b_i u_{i+1},
\]
where $b_i$ is the $i$th column of $B(t, x)$ and $\nabla \phi_{k+1} = (\partial \phi_{k+1})/\partial x$.

To insure the existence of a sliding mode on $\phi_{k+1} = 0$, (6.37) must hold for all $u_{k+1}^i, i = k+2, \ldots, m$. Specifically, this condition has the form
\[
\nabla \phi_{k+1} b_i u_{k+1} < \min_{u_{k+1}} \left[ -\nabla \phi_{k+1} f^k + \sum_{i=k+2}^{m} \nabla \phi_{k+1} b_i u_{i+1} \right], \quad \text{if } \phi_{k+1} > 0
\]
\[
\nabla \phi_{k+1} b_i u_{k+1} > \max_{u_{k+1}} \left[ -\nabla \phi_{k+1} f^k + \sum_{i=k+2}^{m} \nabla \phi_{k+1} b_i u_{i+1} \right], \quad \text{if } \phi_{k+1} < 0.
\]

(6.38a)
(6.38b)

The maximal and minimal values in (6.38) indicate that $\phi_{k+1} < 0$ no matter which of the two values $u_{k+1}^i$ or $u_{k+1}^j$ is taken on by the components of the control $u_i (i = k+2, \ldots, m)$.

Summarizing, we introduce a hierarchy of controls whereby $u_1$ guarantees motion in a sliding mode along $\phi_1 = 0$ for any value of $u_2, \ldots, u_m$. The second component $u_2$ guarantees motion along the intersection of $\phi_1 = 0$ and $\phi_2 = 0$ for any values of $u_3, \ldots, u_m$ and so on. The significance of this method is that a sufficient condition for a sliding mode is obtained from the sliding mode existence condition for the scalar case.

Example 6.39: To clarify the method of control hierarchy, consider the state model
\[
x = Ax + Bu
\]
where
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 2 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & -2
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Let $\sigma = [\sigma_1, \sigma_2, \sigma_3]^T = Sx = 0$ where
\[
S = \begin{bmatrix}
1 & 2 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\]
Following the preceding algorithm, first find $\phi_1 = S_1^1Ax + S_1^2Bu = 0$ where $S_1^1$ is the first row of $S$. Hence
\[
S_1^1A = [2, 5, 2, 4, -1, -2], \quad S_1^2B = [2, 1, 1].
\]
Solving for $u_{1m}$ in terms of $x, u_2$ and $u_3$ yields
\[
u_{1m} = -\frac{1}{2}[2, 5, 2, 4, -1, -2]x - \frac{1}{3}[u_2, u_3].
\]
Inserting $u_{1m}$ into $\dot{x} = Ax + Bu$, yields $\dot{x} = A^1x + B^1u^1$, where
\[
A^1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1/2 & -1 & -2 & 1/2 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 2 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & -2
\end{bmatrix}
\]
\[
B^1 = \begin{bmatrix}
0 & 0 \\
-1/2 & -1/2 \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\]
\[
u^1 = \begin{bmatrix}
u_2 \\
u_3
\end{bmatrix}
\]
Next, solve $\phi_2 = S_2^1A^1x + S_2^2B^1u^1 = 0$ for $u_{2m}$ in terms of $x$ and $u_3$, where $S_2^1A^1 = [0, -1/2, 1, 1, -1/2, 0], S_2^2B^1 = [1/2, 1/2, 1/2, 1/2, 1/2]$, and $u_{2m}$ is given by
\[
u_{2m} = [0, 1, -2, -2, 1, 0]x - u_3,
\]
Inserting $u_{2m}$ into $\dot{x} = A^1x + B^1u^1$ yields $\dot{x} = A^2x + B^2u^2$ where
\[
A^2 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & -2
\end{bmatrix}
\]
\[
B^2 = \begin{bmatrix}
0 \\
0 \\
0 \\
-1 \\
0 \\
1
\end{bmatrix}
\]
Now that $A^1, A^2, B^1, B^2$ are known, the second half of the algorithm is used to determine the controls. Starting with $k = 2$, (6.38) gives
\[
S_2^1b_1^2u^2 < -S_2^1A^2x
\]
\[
S_2^1b_1^2u^2 > -S_2^1A^2x.
\]
Since $S^2b^2_1 = 1$, and $S^2A^2 = [0, 1, 0, 1, -1, -2]$, to satisfy the above inequalities it is sufficient that

$$u_3 = \begin{cases} 
  u_3^+ < -S^2A_3^2x, & \text{if } x_3 > 0 \\
  u_3^- > -S^2A_3^2x, & \text{if } x_3 < 0.
\end{cases}$$

To determine the second control we again apply (6.38) obtaining

$$S^3b^3_1u_2^+ < \min_{\alpha_3} [-S^3A_2^3x - S^3b^3_1u_3]$$

and

$$S^3b^3_1u_2^- > \max_{\alpha_3} [-S^3A_2^3x - S^3b^3_1u_3].$$

From our previous calculations we have $S^2b^2_1 = 1/2$, $S^2b^2_2 = 1/2$, and $S^2A^2 = [0, -1/2, 1, 1, -1/2, 0]$. To satisfy (6.38) it is sufficient that

$$u_2 = \begin{cases} 
  u_2^+ < -(2S^2A_2^2x + u_2^-), & \text{if } x_2 > 0 \\
  u_2^- > -(2S^2A_2^2x + u_2^+), & \text{if } x_2 < 0.
\end{cases}$$

Finally the first control $u_1$ must satisfy

$$S^2b^2_1u_1^+ < \min_{\alpha_2} [-S^2A_1x - S^2b^2_2u_2 - S^2b^2_3u_3]$$

and

$$S^2b^2_1u_1^- > \max_{\alpha_2} [-S^2A_1x - S^2b^2_2u_2 - S^2b^2_3u_3],$$

where $B(t, x) = [b_1, b_2, b_3]$. Since $S^2b^2_1 = 2, S^2b^2_2 = 1, and S^2A = [2, 5, 2, 4, -1, -2]$ to satisfy the above inequalities it is sufficient that

$$u_1 = \begin{cases} 
  u_1^+ < -0.5(S^2A_1^2x + u_1^+ + u_1^-), & \text{if } x_1 > 0 \\
  u_1^- > -0.5(S^2A_1^2x + u_1^+ + u_1^-), & \text{if } x_1 < 0.
\end{cases}$$

In this example the control hierarchy is $u_1$ then $u_2$ and then $u_3$, but is it the optimum order? Is there another order in which the control gains are smaller? The answer depends on the initial condition.

Other Approaches

In addition to the diagonalization methods and hierarchical control method mentioned above, other approaches are possible. In theory an infinite variety of control strategies of the form (6.1) are possible. An alternative structure for the control of (6.1) is

$$u_i = u_{i\text{eq}} + u_{i\text{N}} \quad (6.40)$$

where $u_{i\text{eq}}$ is the $i$th component of the equivalent control (which is continuous) and where $u_{i\text{N}}$ is the discontinuous or switched part of (6.1). For controllers having the structure of (6.40), the following is true:

$$\sigma(x) = \frac{\partial \sigma}{\partial x} x = \frac{\partial \sigma}{\partial x} [f(t, x) + B(t, x)[u_{i\text{eq}} + u_{i\text{N}}]]$$

$$= \frac{\partial \sigma}{\partial x} [f(t, x) + B(t, x)u_{i\text{eq}}] + \frac{\partial \sigma}{\partial x} B(t, x)u_{i\text{N}}$$

$$= \frac{\partial \sigma}{\partial x} B(t, x)u_{i\text{N}}.$$

Let us assume that $(\partial \sigma/\partial x) B(t, x) = I$, the identity. Then $\sigma(x) = u_{i\text{N}}$. This condition allows an easy verification of the sufficiency conditions for the existence and reachability of a sliding mode, i.e., the condition that $\sigma, \dot{\sigma} < 0$ when $\sigma(x) \neq 0$. Below are five possible discontinuous control structures for $u_{i\text{N}}$.

1) Relays with constant gains:

$$u_{i\text{N}}(x) = \begin{cases} 
  \alpha_i \text{sgn} (\sigma(x)), & \sigma(x) \neq 0, \quad \alpha_i < 0 \quad (6.41) \\
  0, & \sigma(x) = 0.
\end{cases}$$

Observe that this controller will meet the sufficiency condition for the existence of a sliding mode since

$$\sigma, \dot{\sigma} = \alpha_i \text{sgn} (\sigma(x)) < 0, \quad \text{if } \sigma(x) \neq 0.$$}

For an example of $\sigma_i(x)$ consider $\alpha_i(x) = \beta_i(\sigma_i^2(x) + \gamma_i)$ with $\beta_i < 0$, $\gamma_i > 0$, where $k$ is a natural number.

2) Relays with state dependent gains:

$$u_{i\text{N}}(x) = \begin{cases} 
  \alpha_i \text{sgn} (\sigma(x)), & \sigma(x) \neq 0, \quad \alpha_i(\cdot) < 0 \\
  0, & \sigma(x) = 0.
\end{cases}$$

Again it is straightforward to check that

$$\sigma, \dot{\sigma} = \alpha_i \text{sgn} (\sigma(x)) < 0, \quad \text{if } \sigma(x) \neq 0.$$}

3) Linear feedback with switched gains:

$$u_{i\text{N}}(x) = \psi_i x; \quad \psi_i = \begin{cases} 
  \alpha_{i\mu} \sigma_i(x) > 0 \quad \text{if } \sigma_i(x) > 0, \\
  0 \quad \text{if } \sigma_i(x) \leq 0.
\end{cases}$$

4) Linear continuous feedback:

$$u_{i\text{N}}(x) = \alpha_i \sigma_i(x) \text{ and } \alpha_i < 0.$$}

The condition for the existence of a sliding mode is $\sigma, \dot{\sigma} = \alpha_i(\cdot) < 0$ or more generally

$$u_{i\text{N}}(x) = -I_{n\times m} \text{ where } I \in R^{n\times m} \text{ is positive definite constant matrix. The condition for the existence of a sliding mode is easily checked}$$

$$\sigma^T(x) \dot{\sigma}(x) = -\sigma^T(x) L(x) \sigma(x) < 0, \quad \text{if } \sigma(x) \neq 0.$$}

5) Univector nonlinearity with scale factor:

$$u_{i\text{N}}(x) = \frac{\rho(x) \sigma(x)}{\|\sigma(x)\|}, \quad \rho < 0.$$}

The existence conditions are

$$\sigma^T(x) \dot{\sigma}(x) = \|\sigma(x)\| \rho < 0, \quad \text{if } \sigma(x) \neq 0.$$}

Given a nonlinear system, if a linear behavior is required in a sliding mode then one may need to use a nonlinear switching surface. A good control for this purpose is the controller of (6.41). To illustrate the above point in the context of this control structure consider the following example.

Example 6.42: Consider a simple robotic manipulator driven by a dc armature control dc motor [72] modeled by the following equations

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_3) + x_3 \\ x_2 + x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u \overset{\Delta}{=} f(x) + Bu. \quad (6.43)$$
For any switching surface \( \sigma(x_1, x_2, x_3) = 0 \), development of the control requires \( (\sigma / \sigma x) B \) be nonsingular. The structure of \( B \) then implies that \( \sigma / \sigma x \neq 0 \). Without loss of generality, we set \( \sigma / \sigma x = 1 \). Hence a general structure for a nonlinear switching surface in this example (similar to (5.19)) is

\[
\sigma(x) = x_3 + \sigma(x_1, x_2) = 0.
\]

Using a control structure of (6.40) and (6.41) yields

\[
u = u_{eq} + \alpha \operatorname{sgn} [\sigma(x)]. \quad \alpha < 0.
\]

Note that the above control somewhat resembles a control strategy resulting from the quadratic form-based design of Utkin [1].

It is easy to check that

\[
\sigma \phi = \alpha \frac{\phi^2}{|\phi|} < 0
\]

since \( \phi = \sigma / |\phi| \). Thus as expected we are assured the existence of a sliding mode.

The next step is to determine the structure of the switching surface and the equivalent system. Suppose we desire the system to exhibit a linear behavior in a sliding mode described by the equation

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
-x_1 - a_1 x_2
\end{bmatrix}
\]

for appropriate \( a_1 \) and \( a_2 \) both positive. Computing \( u_{eq} \) according to (5.1) results in

\[
u_{eq} = -\frac{\partial \sigma_1}{\partial x_1} a_1 x_1 - a_2 x_2 \begin{bmatrix} x_2 \\ \sin(x_1) + x_3 \\ x_2 + x_3 \end{bmatrix}.
\]

Substituting the above into the plant dynamics implies that on \( \sigma(x) = 0 \)

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
\sin(x_1) - \sigma(x_1, x_2)
\end{bmatrix}
\]

which follows according to (5.20). Comparing (6.44) and (6.45) requires that

\[
\sigma(x_1, x_2) = \sin(x_1) + a_1 x_1 + a_2 x_2
\]

a nonlinear switching surface, i.e., the desired linear behavior with the controller structure of (6.40), (6.41) requires the use of a nonlinear switching surface. For other examples of using a nonlinear switching surface see for example [73].

As a final point, the type of controller discussed in this subsection is further developed in the next section to account for the problem of uncertain parameters in the plant model.

### VII. Overview of Uncertain System Theory, VSC, and Chattering

#### Introduction

The purposes of this section are the exposition of VSC for uncertain systems, unification of the theories of VSC and deterministic methods of controlling uncertain systems, and a discussion of chattering. The motivation for exploring uncertain systems is the fact that model identifica-

fication of real-world systems introduces parameter errors. Hence models contain uncertain parameters which are often known to lie within upper and lower bounds. A whole body of literature has arisen in recent years concerned with the deterministic stabilization of systems having uncertain parameters lying within known bounds. Such control strategies are based on the second method of Lyapunov.

On the other hand, VSC controllers are based on the Generalized Lyapunov Second Method. Hence, one expects some fundamental links in the two theories. Using the control structures described in Section VI under the heading “Other Approaches” we will establish these links.

First, a description of the uncertain plant and a brief review of the basic definitions of deterministic control of uncertain systems will be given. Following this we will outline the VSC approach to uncertain system control. This will then lead us to the expected fundamental connections. Lastly, we will focus on improving controller performance by reducing or eliminating chattering through the introduction of the so-called boundary layer controllers. This, in fact, is a natural outgrowth of the theory of uncertain systems although it was developed independently in the VSC context [75], [76], [30], [7].

### Deterministic Control of Uncertain Systems

To represent uncertainties in the plant from parameter uncertainties consider the following state dynamics

\[
\dot{x}(t) = \begin{bmatrix} f(t, x(t)) + \Delta f(t, x(t), r(t)) \\ \Delta B(t, x(t)) + \Delta B(t, x(t), r(t)) \end{bmatrix} u(t)
\]

where \( r(t) \) is a vector function (Lebesgue measurable) of uncertain parameters whose values belong to some closed and bounded set. The formulation presumes no statistical information on the uncertainties. The plant uncertainties \( \Delta f \) and \( \Delta B \) (arising from \( r(t) \)) are required to lie in the image of \( B(t, x) \) for all values of \( t \) and \( x \). This requirement is the so-called matching condition [47], [50], [51], [61]. Assuming the satisfaction of the matching conditions, it is possible to lump the total plant uncertainty into a single vector \( e(t, x(t), r(t), u(t)) \) and represent the uncertain plant as

\[
\dot{x} = f(t, x) + B(t, x) u + B(t, x) e(t, x, r, u)
\]

\( x(t_o) = x_0 \).

With regard to a stabilization analysis of the above model (7.2) the following definitions are pertinent:

**Definition 2:** Let \( x(\cdot) : [t_0, \infty) \rightarrow \mathbb{R}^n \) be a solution of (7.2). \( x(t) \) is uniformly bounded if for each \( x_0 \) there is a positive finite constant, \( d(x_0) \), \( 0 < d(x_0) < \infty \) such that \( \| x(t) \|_2 < d(x_0) \) for all \( t \in [t_0, \infty) \) where \( \| \cdot \|_2 \) is the usual Euclidean vector norm [47], [68].

**Definition 3:** Solutions to (7.2) are uniformly ultimately bounded with respect to some closed bounded set \( S \subset \mathbb{R}^n \) if for each \( x_0 \) there is a non-negative constant \( T(x_0, S) < \infty \) such that \( x(t) \in S \) for all \( t > t_0 + T(x_0, S) \).

The problem is to find a state feedback \( u(t, x(t)) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^m \) such that for any initial condition \( x_0 \) and for all uncertainties \( r(t) \) a solution \( x(\cdot) : [t_0, \infty) \rightarrow \mathbb{R}^n \) of (7.2) exists and every such solution is uniformly bounded.

The literature contains two main approaches for the solution of the above stabilization problem, the so-called min-max controller discussed by Gutman and Palmor [50] and
the Corless-Leitmann approach [47]. These approaches begin with a nominal system defined by
\[
x = f(t, x) \quad x(t_0) = x_0 \tag{7.3}
\]
assuming that \(x = 0\) is an equilibrium point, i.e., \(f(t, 0) = 0\) for all \(t\). Both approaches require this nominal system to be uniformly asymptotically stable, i.e.,

1) for any \(\varepsilon > 0\), there is a \(\delta(\varepsilon) > 0\) such that a trajectory starting within a \(\delta(\varepsilon)\)-neighborhood of \(x = 0\) remains for all subsequent time within the \(\varepsilon\)-neighborhood of \(0\);
2) there is a \(\delta_i\) such that a trajectory originating within a \(\delta_i\)-neighborhood of \(x = 0\) tends to zero as \(t \to \infty\).

It turns out that if there exists a (Lyapunov) function \(V(t, x)\) with \(V(t, x) = 0\) when \(x = 0\) and \(V(t, x) > 0\) when \(x \neq 0\), then the nominal system (7.3, 7.4) is uniformly stable. This background sets up a discussion of the two methods of stabilization of uncertain systems. The min-max control method comes first.

In the min-max approach one assumes a stable nominal system (7.3) with Lyapunov function \(V(t, x)\). A Lyapunov function candidate for the closed loop plant (7.2), with \(u = u(t, x)\), is again \(V(t, x)\). The objective is to choose \(u(t, x)\) to make the derivative of the Lyapunov function negative on the trajectories of the closed loop system, i.e., choose \(u = u(t, x)\) such that
\[
\dot{V}(t, x) = \nabla_V^T \nabla V(t, x) \cdot f(t, x) + \nabla_V^T \nabla V(t, x) B(u + \varepsilon) < 0. \tag{7.5}
\]

Since (7.5) holds if \(u = u(t, x)\) is chosen such that
\[
\min_{u} \left( \nabla_V^T \nabla V(t, x) B(u + \varepsilon) \right) \leq 0 \tag{7.6}
\]
for all \((t, x) \in R \times R^n\) and all admissible controls and admissible uncertainties. Assuming \(B^T(t, x) \nabla_V V(t, x) = 0\), the control
\[
u_n(t, x) = \frac{\rho(t, x)}{\|B^T(t, x) \nabla V(t, x)\|^2} \tag{7.7}
\]
where \(\rho(t, x)\) is a scalar function satisfying \(\rho(t, x) \geq \|e(t, x, r, u)\|_2\) can be shown by direct substitution to satisfy (7.6).

If \(B^T(t, x) \nabla_V V(t, x)\) is zero then take
\[
u \in \{u \in R^n \mid \|u\| \leq \rho(t, x)\}. \tag{7.8}
\]
The reader should note that the set
\[
\{u(t, x) = 0 \mid \|u\| = \rho(t, x)\} \tag{7.9}
\]
can be thought of as a switching surface. One of the main goals of this section is to show that the controller of (7.7) can be made to behave as a VSC controller with the switching surface (7.9).

A close inspection of (7.7) reveals that this control is discontinuous in the state since, for example, in the single input case it reduces to \(u = -\text{sgn} (B^T(t, x) \nabla V(t, x) \rho(t, x))\).

Since the above control is discontinuous it may excite unmodeled high-frequency dynamics of the plant. To avoid this problem, it is necessary to modify this controller by introducing the so-called boundary layer controller which continuously approximates the discontinuous action of (7.7) in a neighborhood of the switching surface (7.9). Let \(p(t, x)\) be any continuous function such that \(p(t, x) = -\frac{|u(x)|}{\|u\|} \rho(t, x)\) when \(\|u\| = \varepsilon\). Then the structure of the boundary layer controller is
\[
u = u(t, x) = \begin{cases} -\frac{1}{\|u\|} \rho(t, x), & \text{if } \|u\| \geq \varepsilon \\ p(t, x), & \text{if } \|u\| < \varepsilon. \end{cases} \tag{7.10}
\]

Unfortunately, this controller does not guarantee asymptotic stability but rather uniform ultimate boundedness as per Definition 3. For a proof of this fact in a slightly modified context see Corless and Leitmann [47]. This completes our brief review of the deterministic control of uncertain systems.

The VSC Approach to the Control of Uncertain Systems

Again consider the uncertain plant as described in (7.2). In the VSC approach it is not necessary for the nominal system (7.3) to be stable. However, the equivalent system, i.e., the restriction of (7.3) to the switching surface \(\sigma(t, x) = 0\), must be asymptotically stable.

The VSC control structure for plant (7.3) will be
\[
u = u_n + u_\infty \tag{7.11}
\]
where \(u_\infty\) is the equivalent control for (7.3) assuming all uncertainties \(e(t, x, r, u)\) are zero and \(u_n\) is to be designed to account for nonzero uncertainties. Recall from Section VI under the heading “Other Approaches” that the structure of (7.11) makes determination of the reachability and existence conditions for a sliding mode more straightforward to compute.

Proceeding in the usual fashion, with the switching surface \(\sigma(t, x) = 0\), one may compute
\[
u_\infty = -\frac{\partial \sigma}{\partial x} B \left( \frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial x} \right) \tag{7.12}
\]
assuming as usual that \([\partial \sigma/\partial x] B\) is nonsingular and that \(e(t, x, r, u) = 0\). It is now necessary to account for uncertainties and develop an expression for \(u_\infty\).

To develop this thread assume as in the previous subsection that
\[
\|e(t, x, r, u)\|_2 \leq \rho(t, x) \tag{7.13}
\]
where \( p(t, x) \) is a non-negative scalar valued function. Also introduce the scalar valued function
\[
\dot{p}(t, x) = \alpha + p(t, x) \tag{7.14}
\]
where \( \alpha > 0 \). This particular structure simplifies some of the derivations.

Before specifying the control structure, we choose the most simple generalized Lyapunov function
\[
V(t, x) = \frac{1}{2}a^T(t, x) \alpha(t, x). \tag{7.15}
\]
As usual, in order to insure the existence of a sliding mode and attractiveness to the surface, it is sufficient [1] to choose a variable structure controller so that
\[
\frac{dV}{dt}(t, x) \triangleq V = a^T(t, x) \dot{\alpha} < 0 \tag{7.16}
\]
whenever \( \alpha(t, x) \neq 0 \) where
\[
\dot{\alpha}(t, x) = \frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial x} \dot{x}. \tag{7.17}
\]

The controller form given in (6.41) in conjunction with the controller of (7.7) suggests the VSC form
\[
u(t, x) = \nu_0(t, x) + \nu_N(t, x) = \frac{B^T(t, x) \nabla V(t, x)}{\|B^T(t, x) \nabla V(t, x)\|_2} \dot{\alpha}(t, x) \tag{7.18}
\]
when \( \alpha(t, x) \neq 0 \) and where
\[
\nabla V(t, x) = \begin{bmatrix} \frac{\partial \alpha}{\partial x} \\ \frac{\partial \alpha}{\partial t} \end{bmatrix} \dot{\alpha}(t, x) \tag{7.19}
\]
where \( \nabla V(t, x) \) is the gradient of the generalized Lyapunov function (7.15). This is different from the \( V(t, x) \) used to develop (7.7). If \( \alpha(t, x) = 0 \), then set \( u(t, x) = \nu_0(t, x) \).

In order to verify the validity of this controller notice that (suppressing \( t \) and \( x \) arguments)
\[
\dot{V} = a^T(t, x) \frac{\partial \alpha}{\partial t} + a^T(t, x) \frac{\partial \alpha}{\partial x} (f + Bu + Be).
\]
(7.20)
Substituting (7.18) into (7.20) and manipulating produces
\[
\dot{V} = a^T(t, x) \frac{\partial \alpha}{\partial t} + a^T(t, x) \frac{\partial \alpha}{\partial x} (f - \alpha^T \frac{\partial \alpha}{\partial x} f - a^T \frac{\partial \alpha}{\partial t} \frac{\partial \alpha}{\partial t})
\]
and
\[
\leq -\alpha \|B^T(t, x) \frac{\partial \alpha}{\partial x} \| \|a\| \tag{7.21}
\]
verifying the negative definiteness of \( \dot{V} \). This establishes attractiveness to the switching surface.

**A Comparison of the VSC and Deterministic Controllers**

The similarity of the structure of (7.7) and (7.18) is clear in view of the \( \nu_N \) term. The controllers, however, generate different responses with respect to different switching surfaces. This is best seen by viewing Fig. 12. Fig. 12(a) illustrates a typical trajectory of the closed loop system (7.2) driven by the controller (7.7). Notice that the control switches with respect to the surface \( B^T \nabla V, V = 0 \) where \( V \) is the Lyapunov function of the nominally stable system. In some cases for sufficiently large \( p(t, x) \), the response will behave as a VSC response in a vicinity of the origin. In fact, if \( p(t, x) \) is sufficiently large the controller exhibits the standard VSC response once the trajectory intercepts the surface \( B^T \nabla V, V = 0 \) [50].

In Fig. 12(b) we have the usual VSC response with respect to the user-chosen switching surface \( \alpha(t, x) = 0 \). In addition to stabilization of the closed loop system one may obtain tracking properties as implicitly built into the user-designed switching surface.

**Chattering**

The VSC controllers developed in Section VI and the uncertain system controllers of this section assure the desired behavior of the closed loop system. These controllers, however, require an infinitely (in the ideal case) fast switching mechanism. The phenomenon of nonideal but fast switching was labeled as chattering (actually the word stems from the noise generated by the switching element). The high frequency components of the chattering are undesirable because they may excite unmodeled high-frequency plant dynamics which could result in unforseen instabilities. The boundary layer controller of (7.10) helps to eliminate the effects of chattering. Let us now refine this notion of boundary layer and boundary layer controller.

Define the set
\[
\{ x \mid |\alpha(x)| \leq \epsilon, \epsilon > 0 \}
\]
as the so-called boundary layer of thickness 2\( \epsilon \). Consider the control law (suppressing \( t \) and \( x \) arguments)
\[
u = \begin{cases} 
\nu_0 - \frac{B^T \nabla V}{\|B^T \nabla V\|_2} \dot{\alpha}, & \text{if } |\alpha| \geq \epsilon \\
\nu_0 + \rho, & \text{if } |\alpha| < \epsilon 
\end{cases} \tag{7.22}
\]
where $u_{eq}$ is given by (7.12) and where $p = p(t, x)$ is any continuous function such that

$$p(t, x) = -\frac{B^T \nabla_x V}{\|B^T \nabla_x V\|} \dot{\rho},$$

whenever $|e| = \epsilon$ and $|p| \leq \dot{\rho}$. This control guarantees attractiveness to the boundary layer and inside the boundary layer, (7.22) offers a continuous approximation to the discontinuous control action of (7.18). As shown in Corless and Leitmann [47], one is not guaranteed asymptotic stability but ultimate boundedness of trajectories to within a neighborhood of the origin depending on $\epsilon$.

The reader might persevere [30], [51] which offer an alternate class of controllers than those above. Specifically [50], [60], [61] offer a discussion of such controllers for linear plants.

Finally, the reader might note that in the control of dc motors chattering is of minimal concern since switching can occur in the high kilohertz range if not megahertz range due to advances in power electronics. This, of course, is well beyond the structural frequencies of mechanical systems involved.

VIII. APPLICATIONS

In [26] Young developed an adaptive VSC for an aircraft control. Calise and Krammer [13] also investigated VSC for aircraft control. An alternative approach to aircraft control using uncertain system controllers was proposed by Petersen [12]. Other applications to spacecraft control can be found in Sira-Ramirez and Dwyer in [63], [64].

In the area of robotic control, Young [27] developed an algorithm based on hierarchical VSC, later refined by Morgan and Ozguner [36]. Also Slotine and Sastry [7] used VSC for tracking control of robot manipulators. A model following VSC scheme was developed by Ambrosino et al. [20] and applied to a simple model of a robot manipulator. A similar application can be found in Bailey and Zolezzi [70]. The use of the Filippov method applied to robotic control can be found in Bartolini and Zolezzi [44], Paden and Sastry [74], and [70]. A combination of VSC and deterministic approach to the control of uncertain systems was proposed by Spong and Sira-Ramirez [54].

There are various applications of VSC to power systems. Among these are Young and Kwatny [6] who developed an overspeed protection control for an electric power generating plant, Hawley and DeCarlo [22], Lefebvre et al. [23] and Richter et al. [24]. This was later extended by Matthews and DeCarlo in [78], [81]. Also Sivaramakrishnan et al. [34] applied the VSC to the design of a variable structure load-frequency controller for a single area power system. In Benjamin and Kaufmann [32] an application of VSC to the dc motor position control is described. While Utkin and Orlov [19] treat the problem of distributed control using VSC.

In contrast to the above VSC applications other methods applicable to real-world systems can be found in Hunt et al. [66], [67] and Sain and Peczkowski [65]. A nice review paper on recent trends in nonlinear system feedback control is Kokotovic [58].

IX. CONCLUDING REMARKS

This paper has developed and surveyed the essential concepts of VSC. Recall that the design of a VSC has two steps: 1) design of the switching surface to assure the desired behavior of the plant in a sliding mode, and 2) development of the control law which forces the system’s trajectory to and maintains it on the sliding surface.

In discussing the multi-input case, we saw that the design process was complicated by the coupling of the controls through the switching surface. Several different methods were then developed (the diagonalization and hierarchical methods) to effectively decouple the controls and thus simplify the design process. Essentially, these techniques reduced the overall control problem to a series of single input problems.

After this, important connections to the theory and application of the deterministic control of uncertain system were introduced and developed. The two theories were seen to have a close alignment.

To illustrate that VSC theory is sufficiently advanced to allow the design of sophisticated control systems, a brief survey of VSC applications as found in the literature was given. The warnings of the previous sections combined with the application survey indicate that when VSC can be applied to a particular problem a very high-quality control system results. However, significant research is needed to successfully apply VSC to even more general classes of nonlinear systems, for example, systems nonlinear in the control $u = u(t, x)$ and systems in which the matching conditions are not satisfied.

Another problem with VSC is the need for complete state information. Development of switching surfaces and controllers based on measurable output signals represents an open problem and an area of important research. The development of nonlinear observers using VSC concepts is a step in this direction. See, for example, Walcott and Zak [69].

Another area filled with fertile research soil is the relationship of VSC with the recently fabricated Lie Algebraic approach to the control of nonlinear systems. Preliminary results in this direction have been published by Marino [57].

Other exciting open problems fall in the categories of tracking and output regulation [59], [80], discrete variable structure control [62] and large scale systems. In the large scale systems area a number of promising results (especially in the decentralized control framework) have been obtained by DeCarlo and co-workers [22]-[24], [31], [78]-[82].

A very interesting application of the VSC theory is in the design of Pulse-Width-Modulated (PWM) control strategies in nonlinear systems. In particular, Sira-Ramirez [83] established an equivalence between the sliding modes, resulting from a VSC strategy, and the response resulting from a PWM control law in nonlinear analytic systems. Specifically, under certain conditions the PWM controlled response represents an ideal sliding motion on an invariant manifold associated with an ideal “average” system. Details of these results and an application to the control design of switch-mode dc-to-dc power converter circuits can be found in [83].

Finally let us close with the point that this paper attempts to present to the reader possible solutions to some questions raised in the report [71].

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REFERENCES


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