Stability analysis of the CMFD scheme with linear prolongation

Qicang Shen a,*, Yunlin Xu b, Thomas Downar a

a Department of Nuclear Engineering & Radiological Science, University of Michigan, Ann Arbor, MI 48105, USA
b School of Nuclear Engineering, Purdue University, West Lafayette, IN 47906, USA

ARTICLE INFO

Article history:
Received 27 November 2018
Received in revised form 28 January 2019
Accepted 5 February 2019

Keywords:
Fourier analysis
lpCMFD
SlpCMFD
odCMFD
Neutron transport

ABSTRACT

The stability analysis was performed on a new CMFD method (lpCMFD) for both k-eigenvalue problem (EVP) and transient fixed source problem (TFSP), and an unconditionally stable variant of lpCMFD method was presented in this paper. The lpCMFD is a new CMFD method which updates the fine cell scalar flux via linear prolongation of local and neighboring coarse mesh fluxes. The stability analysis was performed on the Sn-lpCMFD scheme using the Fourier analysis approach. The theoretical results agree well with numerical results. The lpCMFD presented here is shown to be more stable than the conventional CMFD (fpCMFD) method where flat prolongation is used. The unconditionally stable variant of lpCMFD is developed based on the fine cell optical thickness threshold and the insights provided by the optimal diffusive CMFD (odCMFD). The new method is referred to as stabilized CMFD with linear prolongation (SlpCMFD) and is shown to converge faster than all mainstream CMFD methods.

1. Introduction

Because of its simplicity and numerical efficiency, the Coarse Mesh Finite Difference (CMFD) (Smith, 1983) method has been widely used to accelerate solutions to the neutron transport equation (Kochunas et al., 2013). However, in both theoretical analysis and numerical calculations, it has been shown that the CMFD method could be unstable in solving practical large scales problems (Jarrett et al., 2016; Keady and Larsen, 2015; Zhu et al., 2017). Therefore, research has continued on CMFD and several modified methods have been proposed to resolve the stability issues with conventional CMFD and improve the convergence rate.

The most popular category of methods investigated to address the stability issue is to use some type of relaxation with the CMFD method. Among the most successful methods in this category have been the partial current-based CMFD (pCMFD) (Cho et al., 2003), which was adopted by nTRACER (Jung and Joo, 2009) and the optimally diffusive CMFD (odCMFD) (Zhu et al., 2016) which is used in MPACT (Kochunas et al., 2013). pCMFD is equivalent to a CMFD method in which an additional term $\Delta/4$ (where $\Delta$ denotes the coarse mesh size) is added to the diffusion term, whereas in the odCMFD method, the additional adaptive term $\theta A$ (denote denotes an adaptive number) is used. It was demonstrated through linearized Fourier analysis that the method becomes not just more

stable but also more efficient since the spectral radius of odCMFD is smaller when compared with either the CMFD or pCMFD methods (Jarrett et al., 2016).

However, for this category of CMFD methods, the scalar flux of the fine cells in the same coarse mesh is corrected with the same factor. As a result, after being updated by the CMFD results, the fine cell scalar flux becomes discontinuous at the boundary of coarse mesh (Wang and Xiao, 2018). Recently a new category of methods was proposed to address this issue (Li et al., 2015). This group of methods can be considered as high-order prolongation CMFD (hpCMFD) method which use the fluxes of both the local coarse meshes as well as the neighboring coarse meshes to update the fine cell scalar flux, using a prolongation method which provides for a continuous flux at the mesh boundary. One promising hpCMFD method has been the implementation of a linear prolongation method for the CMFD scheme by Wang and Xiao (2018). Their method referred to as lpCMFD was shown to perform better than the conventional CMFD scheme for preliminary 2D numerical applications and a preliminary Fourier analysis showed that lpCMFD has comparable performance with pCMFD and odCMFD for fixed source problems.

In the research performed here, an extensive Fourier analysis was carried out to investigate the stability of lpCMFD for both k-eigenvalue problems (EVP) and transient fixed source problems (TFSP) (Zhu et al., 2016). The objective was to investigate the parameter space for the stability of the method and to develop an unconditionally stable variant of lpCMFD method.
The theoretical basis for the transport equations accelerated by lpCMFD is demonstrated in the next section for the one-dimensional problem. In Section 3, the equations are linearized and a Fourier analysis is performed. The results are shown in Section 4. Work continues on developing an unconditionally stable variant of lpCMFD in Section 5 and a summary and conclusions are provided in Section 6.

2. CMFD method with linear prolongation

The conventional CMFD which is denoted as fpCMFD in this paper, is well established (Keady and Larsen, 2015; Zhu, 2016), and there are only minor differences between this method and the lpCMFD method which is analyzed here. We first review the standard CMFD accelerated scheme and then discuss the enhancement of lpCMFD. For brevity, the model problem investigated in this paper is a one-speed one-dimension (1D) homogeneous isotropic problem with periodic boundary condition and the transport solver is assumed to be an $S_N$ solver.

2.1. CMFD accelerated transport iteration scheme

The overall flowchart of CMFD accelerated iteration scheme is shown in Fig. 1. In general, the CMFD accelerated scheme is composed of 4 steps:

1. Perform transport iterations to solve the neutron transport equation (NTE).
2. Solve CMFD equations. The coefficients of CMFD equation are homogenized from NTE.
3. Update the fine cell scalar flux (and $k_{eff}$).
4. Perform next neutron transport iteration or finish the calculation.

2.1.1. The neutron transport equation

The basis of the reactor simulation is to solve neutron transport equations. Eq. (1) is the generalized expression for one-speed one-dimension homogeneous neutron transport problem:

$$\frac{d\phi^{i+1}(x,\mu)}{dx} + \Sigma_r \phi^{i+1}(x, \mu) = \frac{\Sigma_f}{2} \phi^{i+1}(x) + W^i(x) + \frac{Q(x)}{2},$$

$$\phi(x) = \int \phi(x, \mu) d\mu.$$  \hspace{1cm} (1a)

where

1. $i + \frac{1}{2}$ = superscripts that denote the CMFD iteration index and the intermediate transport iteration index between two successive CMFD iterations
2. $W$ = the source related with fission, and time dicretization when transient source is taken into account (Zhu et al., 2017).
3. $Q(x)$ = the external source.

Define

$$\Sigma_r = c \Sigma_t,$$  \hspace{1cm} (2a)

$$W = w \Sigma_t,$$  \hspace{1cm} (2b)

$$\epsilon = 1 - c - w.$$  \hspace{1cm} (2c)

In practice, $\epsilon$ is a non-negative value to ensure the positive flux result. The expressions for $W$ and $Q$ in eigenvalue problem (EVP) and transient fixed source problem (TFSP) are different. In EVP, they can be formulated as:

$$W^i = \frac{\Sigma_f}{\epsilon},$$

$$Q(x) = 0.$$  \hspace{1cm} (3)

In the TFSP, they can be written as

$$W^i = \frac{\Sigma_f}{\epsilon} \left(1 - \sum_{l=1}^{\infty} \frac{k_{en}^l}{\epsilon} \right) - \frac{1}{\epsilon},$$

$$Q(x) = \sum_{l=1}^{\infty} \frac{\phi^{i-1}(x)}{\epsilon}.$$  \hspace{1cm} (4)

where

- $i$ = the index of the delayed precursor group
- $n$ = the index of time step (Zhu et al., 2017).

2.1.2. The CMFD equations

CMFD is used to accelerate the iterative solution of the transport equations. The CMFD begins with the neutron balance equation. In 1D, one group homogeneous case, it is written as shown in Eq. (5).

$$\frac{J^{i+1}_{j+\frac{1}{2}}} {\Delta x} + \Sigma_i \phi^{i+1}_{j+\frac{1}{2}} = \Sigma_f \phi^{i+1}_{j+\frac{1}{2}} + W^i_{j+\frac{1}{2}} \phi^{i+1}_{j+\frac{1}{2}} + Q_{j+\frac{1}{2}},$$  \hspace{1cm} (5a)

$$J^{i+1}_{j+\frac{1}{2}} = -D_{j+1} \left( \phi^{i+1}_{j+\frac{1}{2}} - \phi^{i+1}_{j+1} \right) + \bar{D}^{i+1}_{j+\frac{1}{2}} \left( \phi^{i+1}_{j+\frac{1}{2}} + \phi^{i+1}_{j+1} \right),$$  \hspace{1cm} (5b)

$$\bar{D}^{i+1}_{j+\frac{1}{2}} = \frac{J^{i+1}_{j+\frac{3}{2}} + D_{j+1} \left( \phi^{i+1}_{j+2} - \phi^{i+1}_{j+1} \right)} {\phi^{i+1}_{j+1} + \phi^{i+1}_{j+2}}.$$  \hspace{1cm} (5c)

In the CMFD equations,

- $J_{j+\frac{1}{2}}$ = the current at the left boundary of the $j_{th}$ coarse mesh (CM),
- $\bar{D}^{i+1}_{j+\frac{1}{2}}$ = the nonlinear correction term to preserve the high-order neutron current from transport solution,
- $\phi^{i+1}_{j}$ = the cell averaged flux of $j_{th}$ coarse mesh. It is also the solution of CMFD equations,
- $\phi^{i+1}_{j+\frac{1}{2}}$ = the cell averaged flux of $j_{th}$ coarse mesh from transport calculation,
- $\Delta x$ = the coarse mesh size.

![Fig. 1. Flowchart of CMFD accelerated transport calculations.](image-url)
The coefficients are homogenized with the NTE solutions from (1). In this paper, the coarse mesh is uniform, and as a result, the \( \Delta \) is fixed for all the coarse meshes. Define the range of \( j_\Delta \) coarse mesh as \( C_j \). Therefore, the homogenization process is written as:

\[
\phi_{j \Delta}^{1/2} = \int_{C_j} \phi(x) dx / \Delta, \tag{6a}
\]

\[
j_{j \Delta}^{1/2} = \int \mu \varphi(x) \, d\mu, \quad x = (j - 1)\Delta, \tag{6b}
\]

\[
D_{j, j+1} = \frac{2J_{j, j+1}}{D_{j, j+1} + D_{j+1, j}} = \frac{1}{\Sigma_i \Delta}, \tag{6c}
\]

\[
Q_j = \int_{C_j} g(x) dx / \Delta, \tag{6d}
\]

\[
\Sigma_l = \int_{C_j} \Sigma_i \varphi(x) dx / \Delta = \Sigma_i. \tag{6e}
\]

### 2.1.3. Enhancement of IpCMFD

The IpCMFD and fpCMFD methods differ only in the process of using CMFD solutions to update the transport solutions, i.e., fine cell scalar flux. In fpCMFD, the fine cell flux in the same coarse cell is corrected with the same factor as Eq. (7) suggested.

\[
\phi^{1+1}(x) = \frac{\Phi_{j \Delta}^{1+1}}{\Phi_{j \Delta}^{1+1/2}} \phi_{j \Delta}^{1+1/2}(x), \quad x \in C_j. \tag{7}
\]

The issue with this method is that the fine cell flux becomes discontinuous after being updated as shown in Fig. 2, where TS denotes the transport and CM denotes the coarse mesh. It should be noted this is still an issue with the pCMFD, odCMFD and other variants of CMFD methods which use the same type of updating technique.

In IpCMFD, the updating process is designed to preserve the continuity of the flux at the boundary of the coarse mesh. The updating process starts with calculating the “flux correction” at the center point of the coarse mesh. It is the weighted average of the flux corrections from the coarse meshes which own the boundary face. Then the flux of a fine cell will be updated with the addition of the linear prolongation of all the flux corrections at the center point of boundary faces of the coarse cell inside with the fine cell inside. Since the updated scalar flux is continuous both at the boundary and inside the coarse mesh cell, the flux distribution in the whole problem is continuous as illustrated in Fig. 3.

For the 1D case, the flux correction at the \( j \Delta \) coarse mesh boundary is calculated by

\[
\delta_{j \Delta}^{1+1} = \frac{\Phi_{j \Delta}^{1+1} - \Phi_{j \Delta}^{1+1/2} + \Phi_{j \Delta}^{1} - \Phi_{j \Delta}^{1+1/2}}{2}. \tag{8}
\]

And the scalar flux of fine cell is updated by:

\[
\phi^{1+1}(x) = \phi^{1+1/2}(x) + \text{Linear Interpolation} \left( \delta_{j \Delta}^{1+1/2}, \delta_{j \Delta}^{1+1} \right) , \quad \text{for } x \in C_j. \tag{9}
\]

More details about IpCMFD prolongation process can be found in Wang and Xiao (2018).

#### 2.2. Simplified discrete iteration scheme

##### 2.2.1. Discrete NTE

The discretized neutron transport equation can be solved numerically by imposing \( S_N \) discretization on the polar angle which gives the NTE as:

\[
\mu_p \frac{\phi_{k+1/2}^{l+1m} - \phi_{k-1/2}^{l+1m}}{h} + \sum_{p} \phi_{k,p}^{l+1m} = \sum_{k=1}^{M} \phi_{k+1/2}^{l+1m-1} + W \phi_{k}^{l+1m} + Q_k, \tag{10a}
\]

\[
\phi_{k}^{l+1m} = \sum_{p=1}^{P} W_p \phi_{k,p}^{l+1m}, \tag{10b}
\]

where

- \( k \) = integer index which represents the fine cell center flux,
- \( K \) = integer which represents the number of fine cells,
- \( k \pm 1/2 \) = index which represents the fine cell edge value,
- \( m \) = integer index which represents the transport sweep index of the transport iteration,
- \( M \) = integer which represents the number of transport sweep per outer iterations,
- \( p \) = integer index which represents the discrete angle,
- \( P \) = integer which represents the total number of discrete angles,
- \( h \) = fine cell size.

---

**Fig. 2.** Flux distribution of the \( S_N \)-fpCMFD scheme.

**Fig. 3.** Flux distribution of the \( S_N \)-IpCMFD scheme.
Closure of the discretized $S_n$ equation is finalized with the auxiliary equations for spatial differencing as defined in Eq. (11):

$$\phi^{l,m}_{k} = \frac{1 + \alpha_p}{2} \phi^{l-1,m}_{k} + \frac{1 - \alpha_p}{2} \phi^{l+1,m}_{k}.$$  

The step-characteristics (SC) differencing is investigated in this paper to approximate the Method of Characteristic (MOC) which is used in mainstream deterministic transport codes, and the corresponding $\alpha_p$ therefore is defined as:

$$\alpha_p = \coth \left( \frac{\Sigma_p h}{2 \mu_p} \right) - \frac{2 \mu_p}{\Sigma_p h}.$$  

### 2.2.2. Discrete lpCMFD

The form of neutron balance equations of CMFD method is unchanged after spatial differetiation is applied. However, for the homogenization process Eq. (6), the equations are rewritten as:

$$\phi^{l,\frac{1}{2}}_j = \frac{1}{q} \sum_{k \in C_j} \phi^{l,\frac{1}{2}}_k,$$  

$$f^{l,\frac{1}{2}}_j = \sum_{k \in C_j} w_{kp} \mu_p \phi^{l-1,\frac{1}{2}}_k,$$  

$$D^{l+1}_j = \frac{1}{3 \Sigma_j \Delta},$$  

$$Q_j = \frac{1}{q} \sum_{k \in C_j} Q_k,$$  

$$\Sigma_i = \Sigma,$$

where $q$ is the number of fine cells per coarse cell and is defined by:

$$q = \frac{2 \pi}{\Delta}.$$  

With the linear prolongation, the scalar flux of the fine cell in the coarse mesh $C_j$ is updated as:

$$\phi^{l+1}_1 = \phi^{l,\frac{1}{2}}_k + \frac{k - \frac{1}{2} - (j - 1)q}{q} \phi^{l,\frac{1}{2}}_k + \frac{j - \frac{1}{2} - (k - 1)q}{q} \phi^{l,\frac{1}{2}}_k.$$  

### 3. Simplified Fourier analysis

The Fourier analysis is a standard technique regularly used to investigate the stability of CMFD acceleration (Ready and Larsen, 2015; Jarrett et al., 2016). The standard procedure begins with defining a monoenergetic homogeneous infinite media problem. The equations are then linearized near the solutions and the error terms are treated as a summation of Fourier modes. Eventually the coefficients of the Fourier modes are solved and the largest value is the spectral radius associated modes. Eventually the coefficients of the Fourier modes are solved and the largest value is the spectral radius associated

$$\lambda = 2 \pi \frac{\Sigma}{\Delta},$$

Define

$$\theta = \frac{\Sigma}{\Delta}.$$  

The bold vector $E$ is the error vector for the fine cell scalar flux, which is defined as:

$$E^l = \left( E_{q(j-1)+1}, E_{q(j-1)+2}, \ldots, E_{qj} \right)^T.$$  

And the error vector of the CMFD-calculated coarse mesh flux is:

$$R^{l+1} = \frac{(2 - 2 \cos \theta) \mathbf{C} - 3 \Sigma \Delta (\epsilon^d - 1) \mathbf{G}}{-2 - 2 \cos \theta + 3(\Sigma \Delta)^2 \epsilon} E^l = \mathbf{GE}^l.$$  

Here $\mathbf{C}$ and $\mathbf{G}$ is the error transition matrix defined with the relation:

$$C^{l+1} = \frac{1}{q} \begin{pmatrix} 1 & 1 & 1 & \cdots \end{pmatrix} \mathbf{H}^d E^l = \mathbf{CE}^l,$$  

$$G^{l+1} = \begin{pmatrix} 1 & 0 & 0 & \cdots \end{pmatrix} \frac{1}{2} \sum_{p=1}^p \omega_p \mathbf{Y}_p^{-1} \mathbf{S}^d + W E^l = \mathbf{GE}^l.$$  

Finally, the error of angular flux after being updated at the end of the outer iteration is

$$E^{l+1} = \left( \tilde{H}^d + \mathbf{p}' \left( \mathbf{R} - \mathbf{C} \right) \right) E^l = \mathbf{TE}^l.$$  

where

$$\mathbf{p}' = (\delta_1, \delta_2, \ldots, \delta_r, \ldots, \delta_r),$$  

$$\delta_r = \frac{2q + (2q - 2r + 1)e^{-\mu} + (2r - 1)e^{\mu}}{4q}.$$  

And $\tilde{H}^d$ is defined in the relation:

$$F^{l+1} = \tilde{H}^d E^l.$$  

$T$ is defined as error transition matrix of the vector $E$ between two successive outer iterations. Since $T$ in Eq. (23) is based on the Fourier frequency, the reduction rate for the error model of frequency $\lambda$ can be expressed as:

$$o(\lambda) = \max(\text{abs}(\text{eig}(T(\lambda)))).$$
The spectral radius for a typical problem is the reduction rate of the slowest converging model over the whole Fourier frequency domain, and is written as:

\[ \rho = \max(\omega(\lambda)). \]  

(27)

3.2. Comparison with error transition matrix of other variants of CMFD

Zhu et al. (2017) used Fourier analysis to perform a thorough study on the theoretical convergence of the variants of CMFD method in the eigenvalue problem. The final expressions of the error transition matrices in different variants have the same form, with the only difference being how the \( \beta \) vector is defined as shown in:

\[ \beta = \beta(\Sigma_A) P_t. \]  

(28a)

For the variants of fpCMFD, the \( P_t \) stays the same, which is defined as:

\[ P_t = (1, \ldots, 1, \ldots, 1), \]  

(29)

indicating that the flux of the fine cells in the same coarse mesh will be corrected with the same factor. The \( \beta \) varies as Eq. (28a) indicated:

\[ \beta(\Sigma_A) = \begin{cases} 1, & \text{fpCMFD w/o relaxation} \\ \theta_R, & \text{fpCMFD w/ relaxation factor } \theta_R \\ \text{pCMFD and odCMFD} \end{cases} \]  

(30)

Theoretical investigation from the Fourier analysis, the number of coarse meshes must be integer.

Table 1 lists the parameters involved in the validation. The model uses a uniform fine mesh of dimension \( h = 0.2 \) cm. For the theoretical investigation from the Fourier analysis, the number of coarse mesh \( J \) is 1000 and for the numerical investigation, the model size \( X \) is 1000 cm. The coarse mesh size \( \Delta \) in the numerical investigation varied as 1,2,5,10,20 cm because only the number of coarse meshes must be integer.

Despite the difference in the parameters used, Fig. 4 shows the theoretical results agree very well with the code numerical results. When the coarse mesh optical thickness \( \Sigma_A \) is 1, the numerical results converge very quickly which results in a large error in the estimate of the spectral radius of the numerical solution. However, the overall agreement between theoretical and numerical investigation results indicates that the theoretical results are sufficient to investigate the properties of lpCMFD. In the following investigations, only the Fourier analysis results are used. Moreover, by comparing the plots in Fig. 4a and Fig. 4b, it is apparent that when \( M = 1 \) the spectral radius is independent on scattering ratio \( c \) in lpCMFD, which have been observed for fpCMFD in Keady and Larsen (2015). This consistency comes from the point that the prolongation is independent of the scattering ratio. It is also evident that the spectral radius is increasing when the scattering ratio is decreasing.

4.1.2. Comparison between lpCMFD and fpCMFD

The performance of lpCMFD and fpCMFD can be compared using the results in Fig. 5. The parameters used in comparison are listed in Table 2. When the coarse cell optical thickness \( \Sigma_A > 5 \), for fpCMFD, the spectral radius is greater than 1, the method is unstable, while for lpCMFD, the spectral radius is less than 1. The lpCMFD method is stable even when the coarse mesh optical thickness is as large as 40. These results suggest that linear prolongation makes CMFD much more stable, thus there is a much larger parameter space where lpCMFD is stable.

It is also apparent that when \( M \) is increasing, the spectral radius is decreasing and converges to a constant. The reason for this is that by that as the number of transport sweep increases, the angular flux becomes more accurate and high-frequency error due to fine cell discretization becomes negligible for each outer iteration. However, the low-frequency error on the coarse mesh dominates and as a result, the spectral radius reduces to a value which is determined by the coarse mesh.

4.1.3. Fine mesh optical thickness threshold in lpCMFD

In most practical simulations, the size of fine cell is always adjusted to meet the accuracy requirement on discretization. Therefore, it’s worthwhile to find the stable space for fine cell size. First, the effect of the number of fine cells per coarse mesh \( q \) is investigated and is shown in Fig. 6. \( q \) ranges from 1 to 100, and \( M \) is set to be 1 (Table 3) because it is the least time consuming option in numerical simulations and is the most unstable case as indicated in Fig. 5. The number of coarse meshes \( J \) is fixed as 100.

As indicated in Fig. 6, the spectral radius is less than 1 for any coarse mesh size if the number of fine cells is large enough. Additionally, from the result of varying \( q \), it is apparent that there is a threshold number \( q_1 \) at which the spectral radius is less than 1.

Table 1

<table>
<thead>
<tr>
<th>Parameters Used in Section 4.1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X(\text{cm}) )</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>Theoretical</td>
</tr>
<tr>
<td>Numerical</td>
</tr>
</tbody>
</table>
for each coarse mesh optical thickness $\Sigma \Delta$. It can also be concluded that, there is a fine cell optical thickness threshold $\Sigma_{\text{thop}}$ that makes the spectral radius less than 1 when $\Sigma h < \Sigma_{\text{thop}}$.

A more accurate threshold was determined by continuously varying the optical thickness of fine cells (Table 4). The threshold at which the spectral radius equals to 1 for different number of fine cells per coarse mesh is shown in Fig. 7. The result shows that:

1. The threshold is almost a constant for different number of fine cells.
2. With the number of coarse mesh increasing, the threshold decreases and converges to a constant which is around 2.46.

It should be noted that $M = 1$ is the most unstable option. Therefore, it can be concluded that there is a fine cell optical thickness threshold for which the lpCMFD is stable no matter how large the size of coarse mesh is.

### 4.2. Performance in transient fixed source problem

As the derivation of Eq. (21) shows, the spectral radius is not affected by the source, but affected by the scattering ratio $c$ and residual $\epsilon$. And the range of $\epsilon$ is constrained by $c$. As a result, to determine the performance of lpCMFD for the transient fixed source problem, the relation between spectral radius and $\epsilon$ was investigated for the case of a fine cell of 0.2 (Table 5). The results are shown in Fig. 8 where it is apparent that with $\epsilon$ increasing, the spectral radius is decreasing, which indicates that the lpCMFD for TFSP is more stable than for EVP. And lpCMFD performs better than fpCMFD. It should be noted that, when $\epsilon$ increases to $1 - c$, the spectral radius can reduce to around 0. This agrees with the results in Zhu et al. (2017). However, the investigation was performed for a 1G problem, whereas for practical multi-group cases, the spectral radius could be much larger. Moreover, $\epsilon = 1 - c$ means that the time step should be small enough to offset
the influence of delay neutron source, which is impractical for simulations.

The performance was therefore assessed for a more practical case where \( dt = 1 \) ms. The results are shown in Fig. 9. For variations in \( c \) and \( M \), spectral radius of lpCMFD is less than that of fpCMFD. With \( M \) increasing, the spectral radius is decreasing and converges to minimum. In the parameter space investigated here, all the spectral radii of the lpCMFD are less than 1.

The eigenvalue problem is a special case of the transient fixed source problem with \( \epsilon = 0 \), which has the largest spectral radius. As a result, the threshold \( \Sigma h = 2.46 \) is also applicable for the transient fixed source problem. However, for most transient fixed source problems, the spectral radius is less than that of the eigenvalue problem with the same scattering ratio and geometry, since \( \epsilon \) is not zero.

5. Unconditionally stable version of lpCMFD (SlpCMFD)

As mentioned above, the lpCMFD is a CMFD method with spatially high-order prolongation which performs better than fpCMFD. The odCMFD is a fpCMFD method with flux relation which is unconditionally stable. As a result, it is possible to combine the idea of flux relaxation and spatially high-order prolongation together to develop an unconditionally stable version of lpCMFD. The relaxation is implemented using the adaptive artificially diffusive factor approach, which is the same approach used to develop the odCMFD scheme (Zhu et al., 2016).

5.1. Development of SlpCMFD

The development process aims at making lpCMFD scheme stable in the most unstable cases with various of number of fine cells per coarse mesh \( q \). The parameters involved in the development are listed in Table 6.

The Fig. 10 shows the effect the artificially diffusive factor applied in lpCMFD. For the region where \( \Sigma h < 2.46 \), for larger values of \( \theta \), the spectral radius increases and the rate of convergence of the problem is reduced. For the region where \( \Sigma h > 2.46 \), a large \( \theta \) will make the lpCMFD scheme turns to be stable.

As a result, an unconditionally stable lpCMFD can be developed by using optimal artificially diffusive factor \( \theta_{ad} \) defined in

\[
\theta_{ad}(\Sigma h) = \begin{cases} 
0, & \Sigma h < 1 \\
0.25(\Sigma h - 1), & 1 \leq \Sigma h < 2 \\
0.25, & \Sigma h \geq 2 
\end{cases}
\]

In numerical simulations, the numerical error will affect the stability of the scheme. As a result, the criteria used for determining the optimal artificially diffusive factor in Eq. (32) is set to be 2 rather than 2.46. The lpCMFD with the optimal artificially diffusive factor is referred to as SlpCMFD here. It is evident in Fig. 10 that the spectral radius of SlpCMFD is the smallest in both small and large fine cell optical thickness regions, and SlpCMFD is unconditionally stable for any optical thickness \( \Sigma A \) and fine cell number per coarse mesh \( q \).

5.2. Comparison with mainstream CMFD methods

The spectral radius of SlpCMFD with lpCMFD and other mainstream variants of CMFD methods are compared in Fig. 11. As indicated in the figure, the SlpCMFD converges faster than any mainstream CMFD methods analyzed. It has comparable stability with the other unconditionally stable methods in small and large optical thickness region, but is more stable in intermediate optical thickness region.

Compared with odCMFD, SlpCMFD requires fewer FLOPS to calculate the artificially diffusive factor. One reason for this is that in odCMFD, a 6th-order polynomial is required to calculate \( \theta \). Another reason is that the \( \theta \) is dependent on the optical thickness of the coarse mesh. Therefore it is necessary to recalculate \( \theta \) every time the CMFD coefficients are homogenized from transport results. However, the drawback of SlpCMFD is that more FLOPS are required for linear prolongation. It should be noted that fine mesh thickness should be small enough in any practical application for discretization accuracy, which can in turn make the fine mesh optical thickness smaller than the threshold and reduce the frequency to calculate the \( \theta \).

6. Summary and conclusion

In the work presented here, a Fourier analysis of a \( S_N \)-CMFD scheme with linear prolongation was performed and the results from the theoretical derivation agree well with results from numerical simulation. In general, the results show that lpCMFD is much more stable than fpCMFD for both \( k \)-eigenvalue problem and transient fixed source problem.

In the \( k \)-eigenvalue problem, it was shown that there is a fine cell optical thickness threshold for which it is possible to make all calculations stable if the size of the fine cells is less than the
The same threshold holds for transient fixed source problem. It can be concluded that there is a fine cell optical thickness stability threshold for linear prolongation, and the optical thickness of fine mesh is a key factor which influences the stability of the method. This suggests that regions with strong absorption materials will contribute to the convergence instability.

To solve the instability feature of lpCMFD, a variant method was developed in this paper. The new method SlpCMFD was inspired by the optimal optical thickness dependent artificially diffusive method. It is shown that SlpCMFD is unconditionally stable and has a smaller spectral radius than the odCMFD and lpCMFD. The result is promising that the SlpCMFD has the potential to improve the performance of the current neutronic codes with CMFD acceleration.
Fig. 10. Spectral radius ($\rho$) vs. artificially diffusive factor ($\theta$).

Fig. 11. Comparison of SlpCMFD and conventional CMFD.
Future work will include the implementation of SlpCMFD in 3D transport neutronics codes MPACT and Proteus-MOCEX (Marin-Lafleche et al., 2013) and the investigation of its performance for large scale practical applications.

Acknowledgement

This research was supported by the Department of Energy Nuclear Energy University Program (NEUP).

References