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Stokes flow past three spheres: An analytic solution

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Stokes flow past three spheres is solved analytically by use of (new) addition theorems that transform vector harmonics between different coordinate centers. The extension to other special configurations of \(N\) spheres is indicated. The method is illustrated for uniform flow past three spheres in a triangular cluster, with good agreement with experimental sedimentation velocities.

I. INTRODUCTION

Hydrodynamic interaction between particles plays an important role in determining the rheological (and other transport properties) of dispersed multiphase systems, especially when the volume fraction of the dispersed phase is increased from the dilute limit. At the same time, the rigorous and completely general description of particle–particle interactions is limited to pair interactions. Beyond two particles, published results are rather limited, but include Kynch’s method-of-reflections results for well-separated three-sphere configurations,\(^1\) boundary collocation results for sedimentation of three spheres in a vertical plane,\(^2\) and multiparticle assemblies of rather restricted configurations (linear assembly of \(N\) spheres, periodic arrays, etc.).

The present work introduces a technique for multiparticle Stokes flow problems that involves the analytical transformation of basis functions between coordinates based on different spheres by the use of addition theorems for spherical harmonics. A detailed exposition of this method for two-particle problems, using the addition theorems in Hobson,\(^3\) is available elsewhere.\(^4,5\) The present work introduces new addition theorems that are crucial to the extension of the method to \(N\)-sphere geometries. The merits of the technique are that the end results are expressed in a semianalytical form (as a Laurent series in particle–particle separations); the method is particularly useful when the \(N\)-sphere configuration possesses a high degree of symmetry, and the method converges even when the particles are close. The theoretically predicted sedimentation velocity of a cluster formed by three touching spheres agrees within experimental error, with the recent results of Lasso and Weidman.\(^6\)

The method consists of three steps. In the first step, we pick a suitable velocity representation, usually taking into account symmetries in the particle geometry. Up to this point, the method is identical to the boundary collocation approach as used by Ganatos et al.\(^2\) The second step involves the construction of addition theorems for the vector harmonics that arise in the velocity representation. In the third step, we collect terms in the various vector harmonics and use the boundary condition to arrive at equations for the coefficients in the velocity representation. There is an optional fourth step that involves the expansion of the coefficients in a Laurent series in the particle–particle separation. This last step is worthwhile when the end result involves further manipulations of the resistance functions as in ensemble averaging over particle configurations.

In comparison with the boundary collocation method, the present approach suffers in that actual computations are restricted to special geometries. However, for those special geometries, the end result is an analytic expression readily amenable to further algebraic manipulations. This method also furnishes a starting point for a fundamental investigation on the unsolved problem of the optimal location of collocation points in the boundary collocation method in nonaxisymmetric flows.

II. THE DRAG ON THREE SPHERES IN A UNIFORM STREAM

We illustrate the essential points of the method by applying it to the calculation of the velocity field (and other useful results such as the hydrodynamic drag and torque) for a cluster of three identical spheres in a uniform stream \(U\). The governing equations for the pressure \(p\) and velocity \(v\) are the Stokes equation

\[
\mu \nabla^2 v = -\nabla p, \tag{1}
\]

where \(\mu\) is the viscosity, and the equation of continuity

\[
\nabla \cdot v = 0. \tag{2}
\]

The boundary conditions are

\[
v = 0 \quad \text{on} \quad |x - x_a| = a, \quad |x - x_b| = a, \quad |x - x_c| = a,
\]

\[
v = U \quad \text{as} \quad |x| \to \infty, \tag{3}
\]

where \(a\) is the sphere radius and the \(x_a, \alpha = 1, 2, 3\), denote the position of the sphere centers. The relative position vectors \(x - x_a\) will be denoted by \(r_a\) since they appear frequently and we will also use the notation \(r_a = |r_a|\).

A. The velocity representation

Because everything is linear in \(U\), the pressure, which is a harmonic, must be of the form \(p = U \cdot \nabla \psi\), where \(\psi\) is a harmonic (but independent of \(U\)). Now harmonics can be built from Maxwell poles, and this particular geometry suggests a basis at one of the spheres, say sphere 1, which for harmonics of degree \(-n-1\), consists of \(m\) poles in the direction of sphere 2 and \(-m\) poles in the direction of sphere 3, with \(m = 0, 1, 2, \ldots, n\). Thus the pressure may be written as a multipole expansion about the sphere centers, i.e.,

\[
p = \mu \sum_{n=1}^{m} \sum_{m=0}^{n-1} \left[ p_{-n-1}^+(r_1) + p_{-n-1}^-(r_2) + p_{m}^-(r_3) \right], \tag{4}
\]

with
\[ p_{n-1}^m (r_1) = 2a_{mn} (U \cdot \nabla) \frac{(d_{12} \cdot \nabla)^{n-m-1} (d_{13} \cdot \nabla)^m}{(n-m-1)! m!} \frac{1}{r_1}, \]

and analogous terms for the expansions about \( x_2 \) and \( x_3 \). We have introduced the notation \( d_{pq} \) for \( (x_{pq} - x_i) / |x_{pq} - x_i| \).

Now for any harmonic \( \psi \), the velocity field associated with the pressure \( 2 \mu U \cdot \nabla \psi \) is \( x \cdot U \nabla \psi - U \psi \). We may add to this velocity field vector harmonics that represent the homogeneous solution of (1). In general, these vector harmonics may be written as a combination of the vector field \( \nabla \Phi \) and a toroidal field \( \nabla \times (x \nabla \psi) \), but, again, we exploit the linear dependence on \( U \) and write these harmonics as \( \nabla (U \cdot \nabla) \Phi \) and \( (dU - Ud) \cdot \nabla \psi \). After incorporation of the required Maxwell poles and the particular solution derived above, we obtain the representation

\[ v = U + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \left[ v_{n-1}^m (r_1) + v_{n-1}^m (r_2) \right] + \sum_{m=1}^{\infty} \left[ v_{n-1}^m (r_3) \right], \]

with

\[ v_{n-1}^m (r_1) = a_{mn} (r_1^2 U \cdot \nabla - U) \frac{(d_{12} \cdot \nabla)^{n-m-1} (d_{13} \cdot \nabla)^m}{(n-m-1)! m!} \frac{1}{r_1} + b_{mn} \nabla (U \cdot \nabla) \frac{(d_{12} \cdot \nabla)^{n-m-1} (d_{13} \cdot \nabla)^m}{(n-m-1)! m!} \frac{1}{r_1} + c_{mn} (d_{12} \times U) \times \nabla \frac{(d_{12} \cdot \nabla)^{n-m-1} (d_{13} \cdot \nabla)^m}{(n-m-1)! m!} \frac{1}{r_1}. \]

The key step involves the setting of the boundary condition at the surface of the spheres. For instance, if we consider the boundary condition on sphere 1, then as in the single-sphere problem, the functions expanded about \( x_1 \) can be rewritten and collected in terms of vector harmonics about \( x_1 \). This is the trivial step. The difficulties come from the functions expanded about \( x_2 \) and \( x_3 \). As in potential theory, addition theorems are required to reexpand the \( x_2 \)- and \( x_3 \)-centered basis functions in terms of those at \( x_1 \).

**B. Addition theorems**

We start with the Legendre expansion,

\[ \frac{1}{r_2} = \sum_{n=0}^{\infty} \frac{r_2^{n+1}}{|x_1 - x_2|^{n+1}} \frac{1}{n!} \frac{(d_{12} \cdot \nabla)^n}{r_1}. \]

We may differentiate this expression to obtain the addition theorem for poles in the direction of an arbitrary unit vector \( e \),

\[ (e \cdot \nabla)^j \frac{1}{r_2} = \sum_{n=0}^{\infty} \frac{r_2^{n+1}}{|x_1 - x_2|^{n+1}} \frac{1}{n!} \frac{(d_{12} \cdot \nabla)^n}{r_1}, \]

where \( \nabla_2 \) denotes the nabla operator with respect to \( x_2 \). We note that since \( \nabla_2 \) operates on both \( |x_1 - x_2| \) and \( d_{12} \), we must employ the Leibnitz product rule. The resulting expression (after some simplification) may be expressed as

\[ \frac{(-e \cdot \nabla)^j}{l!} \frac{1}{r_2} = \sum_{n=0}^{\infty} \frac{r_2^{n+1}}{|x_2 - x_1|^{n+1}} \frac{1}{n!} \frac{(-1)^{2n!}}{(2n)!} |x_2 - x_1|^{-(n+l+1)} \frac{(d_{12} \cdot \nabla)^{n-s}_{l+s}}{(l+s)!} \frac{1}{r_1}, \]

where \( P_m^{(n)} \) is the \( m \)th derivative of the Legendre polynomial of degree \( n \), \( P_n^{(z)}(z) \), evaluated at \( z = e \cdot d_{12} \).

The multipoles employed in the pressure and velocity representations require poles in two distinct directions. The addition theorem for such poles are obtained by successive operations on both sides of Eq. (10) with \( d_{12} \cdot \nabla \). [We choose this particular operation because \( (d_{12} \cdot \nabla)^2 d_{12} = 0 \).] We thus arrive at the following addition theorem:

\[ \frac{(-d_{12} \cdot \nabla)^k}{l!} \frac{1}{r_2} = \frac{1}{k!} \frac{1}{l!} \sum_{n=0}^{\infty} \frac{(-1)^{2n!}}{(2n)!} |x_2 - x_1|^{-(n+k+l+1)} \frac{r_2^{n+1}}{|x_2 - x_1|^{n+k+l+1}} \frac{P_{l+s}^{(n)}}{(l+s)!} \frac{(d_{12} \cdot \nabla)^{n-s}_{l+s}}{(l+s)!} \frac{1}{r_1}. \]

This appears to be a new result. The addition theorems for the multipoles in the velocity representation may be obtained by setting \( e = d_{32} = d_{12} - d_{13} \).

**C. The symmetric three-sphere cluster**

The preceding discussion furnishes the basis for the development of a general solution algorithm for Stokes flow about three spheres in any configuration. We now restrict the particle geometry to the highly symmetric configuration in which the sphere centers form an equilateral triangle. Furthermore, the uniform stream is directed perpendicular to the plane of the sphere centers. Thus the geometry is completely specified by a single parameter \( R / a \), where \( R \) is the center-to-center separation. This special configuration includes the recent sedimentation experiments with horizontally settling three-sphere clusters.}

We digress briefly at this juncture to point out the similarities and differences between the Laplace and Stokes problems. For the scalar Laplace equation, the addition theorems of the preceding section furnish the required set of transformation relations. On the other hand, the solution procedure for the Stokes and vector potential equations includes the selection of a
complete basis set of vector harmonics and the development of the corresponding addition theorems. For the problem at hand, we select the vector harmonics

\[
\mathbf{v}(U \cdot \mathbf{v}) (d_{12} \cdot \mathbf{v})^{n-m-1} (d_{13} \cdot \mathbf{v})^m \frac{1}{(n-m-1)! \ m! \ r_1}, \quad \mathbf{u}(d_{12} \cdot \mathbf{v})^{n-m-1} (d_{13} \cdot \mathbf{v})^m \frac{1}{(n-m-1)! \ m! \ r_1},
\]

and

\[
(d_{12} \times \mathbf{u}) \times \nabla (d_{12} \cdot \mathbf{v})^{n-m-1} (d_{13} \cdot \mathbf{v})^m \frac{1}{(n-m-1)! \ m! \ r_1} + (d_{13} \times \mathbf{u}) \times \nabla (d_{13} \cdot \mathbf{v})^{n-m-1} (d_{12} \cdot \mathbf{v})^m \frac{1}{(n-m-1)! \ m! \ r_1},
\]

as the basis because they appear in the velocity representation [Eq. (7)]. We will define

\[
\psi_n^m (r_1) = \frac{(d_{12} \cdot \mathbf{v})^{n-m-1} (d_{13} \cdot \mathbf{v})^m \frac{1}{m! \ r_1}}{(n-m-1)! \ r_1}
\]

to save space.

We note that when matching the boundary condition, the one term in Eq. (7) that is not a harmonic must be represented in terms of the preceding basis elements. The required formula is of the form

\[
(2n+3) \times \mathbf{u} \cdot \nabla (d_{12} \cdot \mathbf{v})^{n-m} (d_{13} \cdot \mathbf{v})^m (1/r)
\]

\[
= -r^2 \mathbf{v}(U \cdot \mathbf{v}) (d_{12} \cdot \mathbf{v})^{n-m} (d_{13} \cdot \mathbf{v})^m (1/r) + (n-m)(n-m-1) \mathbf{v}(U \cdot \mathbf{v}) (d_{13} \cdot \mathbf{v})^{n-m-2} (d_{12} \cdot \mathbf{v})^m (1/r)
\]

\[
+ 2m(n-m) \mathbf{v}(U \cdot \mathbf{v}) (d_{13} \cdot \mathbf{v})^{n-m-1} (d_{12} \cdot \mathbf{v})^m (1/r) + m(m-1) \mathbf{v}(U \cdot \mathbf{v}) (d_{13} \cdot \mathbf{v})^{n-m} (d_{12} \cdot \mathbf{v})^m (1/r)
\]

\[
- 2(n-m)(d_{12} \times \mathbf{u}) \times \nabla (d_{12} \cdot \mathbf{v})^{n-m-1} (d_{13} \cdot \mathbf{v})^m (1/r) - 2m(d_{13} \times \mathbf{u}) \times \nabla (d_{12} \cdot \mathbf{v})^{n-m} (d_{13} \cdot \mathbf{v})^m (1/r)
\]

and may be verified by mathematical induction on \(m \) and \(n\).

The addition theorem for the first basis element is

\[
\mathbf{v}(U \cdot \mathbf{v}) (d_{12} \cdot \mathbf{v})^{l-k-1} (d_{13} \cdot \mathbf{v})^k \frac{1}{(l-k-1)! \ k! \ r_2}
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{s=m}^{n} \left[ Q^{(1)}_{\text{klmn}} \mathbf{v}(U \cdot \mathbf{v}) \psi_{n+2}^m (r_1) + Q^{(2)}_{\text{klmn}} \mathbf{w}(U \cdot \mathbf{v}) \psi_{m+n}^m (r_1) + Q^{(3)}_{\text{klmn}} \mathbf{w}(U \cdot \mathbf{v}) \psi_{n-m+1}^m (r_1) + Q^{(4)}_{\text{klmn}} (d_{12} \times \mathbf{u}) \nabla \psi_{n-1}^m (r_1) + Q^{(5)}_{\text{klmn}} (d_{13} \times \mathbf{u}) \nabla \psi_{m+1}^m (r_1) \right].
\]

The upper limit of the \(s \) summation is given by \(S = \max \{n-1, l-k-1 \} \) and

\[
Q^{(1)}_{\text{klmn}} = 2^{n-1} p_{n-1}^{(n+1)} (1-1) s-m \frac{n-1}{(n-1)! \ k! \ s-m},
\]

\[
Q^{(2)}_{\text{klmn}} = 2^{n-1} \left[ \frac{1}{2} \frac{p_{n-1}}{p_{n-l-k-1-1-s}} - (2n+l-k-2-s) \frac{p_{n-l-k-2-s}}{p_{n-l-k-1}} \right] (n-s-1) - \frac{p_{n-l-k-1}}{p_{n-l-k-1}}
\]

\[
\times (1)^{s-m} \frac{n+m}{(n+1)! \ k! \ s-m},
\]

\[
Q^{(3)}_{\text{klmn}} = \left( 1 \right)^{s-m} \frac{n+m}{(n+1)! \ k! \ s-m},
\]

\[
Q^{(4)}_{\text{klmn}} = 2^{n-1} \left( 2n+l-k-2-s \right) \frac{p_{n-l-k-2-s}^{n-l}}{p_{n-l-k-1}} - \frac{1}{2} \frac{p_{n-l-k-2-s}}{p_{n-l-k-1}} \left( 1 \right)^{s-m} \frac{n+m}{(n+1)! \ k! \ s-m},
\]

\[
Q^{(5)}_{\text{klmn}} = 2^{n-1} \frac{p_{n-2}}{p_{n-1}} \left( 1 \right)^{s-m} \frac{n+m}{(n+1)! \ k! \ s-m},
\]

with the Legendre functions in (15a)–(15e) evaluated at \( z = \frac{1}{k} \).

We now examine the remaining terms in the velocity representation. The transformation for

\[
(r_2 \cdot \mathbf{u}) \nabla (d_{13} \cdot \mathbf{v})^{l-k-1} (d_{12} \cdot \mathbf{v})^k \frac{1}{(l-k-1)! \ k! \ r_2},
\]

requires only the addition theorem for \( (d_{23} \cdot \mathbf{v})^{l-k-1} (d_{12} \cdot \mathbf{v})^k (1/r_2) \) because \( r_2 \cdot \mathbf{u} = r_1 \cdot \mathbf{u} \) since \( d_{12} \cdot \mathbf{u} = 0 \). The remaining transformation formula, for

\[
(d_{23} \times \mathbf{u}) \times \nabla (d_{23} \cdot \mathbf{v})^{l-k-1} (d_{12} \cdot \mathbf{v})^k \frac{1}{(l-k-1)! \ k! \ r_2},
\]

is derived by using the relation \( (d_{23} \times \mathbf{u}) \times \nabla = (d_{13} \times \mathbf{u}) \times \nabla - (d_{12} \times \mathbf{u}) \times \nabla \) and the addition theorems for the appropriate harmonics. This completes step 2.

We collect the three types of harmonics at the surface of sphere 1 and obtain three sets of equations for the coefficients \( a_{mn} \), \( b_{mn} \), and \( c_{mn} \). This system may be truncated and inverted to get at the (numerical) solution for the coefficients. However, we may follow the formalism of the method.
of reflections and expand each coefficient in a power series in the small parameter $a/R$, i.e.,

$$a_{mn} = A_{010} \delta_{n1} + \sum_{p=n}^{\infty} A_{mpn} \left( \frac{a}{R} \right)^p,$$

$$b_{mn} = B_{010} \delta_{n1} + \sum_{p=n}^{\infty} B_{mpn} \left( \frac{a}{R} \right)^p,$$

$$c_{mn} = \sum_{p=n+1}^{\infty} C_{mpn} \left( \frac{a}{R} \right)^p.$$  \hfill (16a, 16b, 16c)

When we substitute these expansions into the equations for the boundary condition at sphere 1, we find that $A_{010} = \frac{1}{2}$ and obtain recursion formulas for $A_{mpn}$, $B_{mpn}$, and $C_{mpn}$. The formulas are given explicitly in the Appendix.

The force and torque on one of the spheres, e.g., sphere 1, are obtained as

$$F = 6 \pi \mu a U \hat{a}_{01},$$

and

$$T = 8 \pi \mu a^2 (d_{12} + d_{13}) \times U (c_{01} - \frac{1}{2} a_{02}),$$

with

$$\frac{4}{3} a_{01} = \frac{4}{3} a_{01} \sum_{n=0}^{\infty} A_{01n} \left( \frac{a}{R} \right)^n = \sum_{n=0}^{\infty} (-1)^n F_n \left( \frac{2a}{R} \right)^n$$

and

$$\left( c_{01} - \frac{1}{2} a_{02} \right) = \sum_{n=2}^{\infty} \left[ C_{02n} - \frac{1}{2} A_{02n} \left( \frac{a}{R} \right)^n \right] = \sum_{n=2}^{\infty} (-1)^n T_n \left( \frac{2a}{R} \right)^n.$$ \hfill (17, 18, 19)

These coefficients have been computed to $n = 80$, but for the sake of brevity, Table I lists values only up to $n = 30$. (The higher-order coefficients are available from the author.) The drag on one sphere, scaled by $6 \pi \mu a U$, along with the result from the pairwise additive theory, is shown in Fig. 1.

III. RESULTS

A. Large separations

For large separations, the series converges rapidly and the first few terms dominate the behavior. As in the two-sphere problem, the leading-order correction of $O(R^{-1})$ to the single-sphere result comes from the reflection from the other spheres. This correction is pairwise additive, i.e., the $O(R^{-1})$ term is the sum of the 1–2 and 1–3 interactions. Three-sphere interaction terms are introduced at $O(R^{-2})$, therefore significant deviations from the pairwise additive theory will be present unless $R \gg a$.

B. Small separations

The results of the present analysis are most interesting for small sphere–sphere separations. For small sphere–sphere separations, an expansion in $a/R$ may not converge because higher-order terms in the series make significant contributions. For the two-sphere problems, this behavior manifests itself as “slow” convergence of the series. In fact, it can be shown quite rigorously that the coefficients in the two-sphere sum behave at large $n$ as the coefficients in the geometric series $(1 + 2a/R)^{-1}$ and the logarithmic series $\ln(1 + 2a/R)^{-1}$. Since singularities in these functions are in the left-half plane of $a/R$, one can successfully implement acceleration methods. However, the situation is considerably more complicated in the three-sphere problem.

One can make the following qualitative argument. For three spheres, the number of reflections grows as $2^n$ since each reflection has to be canceled on two other surfaces. Since the contribution from each reflection is of the same order in $a/R$, we deduce that the coefficients $F_n$ in the series expansion should behave as $C^n$ for some constant $C$. Indeed, the coefficients in Table I grow as

$$\ln(F_n) \sim -1.0014 + 0.07626n$$

for $41 < n < 80$, as shown in Fig. 2(a). [The deviation between the approximant and the actual values is plotted in Fig. 2(b). A regular pattern is obtained and this magnified scale shows what appears to be incipient growth of roundoff errors.] If we extrapolate this behavior to all $n > 80$, we obtain the geometric series for the function $(1.0 + 2.16/R)^{-1}$. This implies that if we start at some large separations and decrease $R/a$, the series will diverge at $R/a \sim 2.16$! This is shown in Fig. 1 where both the original series and the accelerated series diverge for nearly touching spheres. From a rigorous viewpoint, it is not entirely clear at this point whether this is because of an inherent limitation of a method based on an expansion in $a/R$ (method of reflections) or because of some error in the analysis.

The results are well behaved for $R/a > 2.5$. Now the flow in question does not have high stresses anywhere in the limit of touching spheres, so we expect smooth behavior in the region $2.0 < R/a < 2.5$. A quadratic extrapolation of the
curve from the behavior in the range 2.5 to 3.0 is also plotted in Fig. 1. The extrapolated result produces an estimate for the drag in the touching case of 0.585, which is in good agreement with the experimental value of 0.574 obtained by Lasso and Weidman.  

Lengthy calculations are susceptible to computational errors and since this is the first exact treatment of this problem, meaningful comparisons with other results are not possible. The agreement of the extrapolated result with the experimental data does lend some support. Since the results presented above are so different from that encountered in the two-sphere analog, it should be mentioned here that the three-sphere drag was computed manually by the method of reflections without resorting to the addition theorems and recursion formulas. This second, independent calculation produced

\[
F = 8\pi \mu a \left[ \frac{1}{2} - \frac{3}{2} \left( \frac{a}{R} \right)^2 + \frac{9}{4} \left( \frac{a}{R} \right)^4 - \frac{35}{8} \left( \frac{a}{R} \right)^6 + \cdots \right] 
\]

before algebraic fatigue set in. This result agrees with the entries in Table I, thus supporting the overall result. The method of reflection produces simultaneously the results for other orientations of the uniform stream (i.e., sedimentation with the centers in a vertical plane), which we report here.  

(i) For a three-sphere symmetric cluster settling with the apex directed downwards, the drag on the leading sphere was found to be

\[
F = 8\pi \mu a \left[ \frac{1}{8} + \frac{45}{8} \left( \frac{a}{R} \right)^2 - \frac{691}{512} \left( \frac{a}{R} \right)^3 + \cdots \right] 
\]

(ii) For a three-sphere symmetric cluster settling with the apex directed horizontally, the drag on the leading sphere was found to be

\[
F = 8\pi \mu a \left[ \frac{15}{8} \left( \frac{a}{R} \right) - \frac{5447}{512} \left( \frac{a}{R} \right)^3 + \cdots \right].
\]

\[F = \frac{102}{4096} \frac{4096}{16384} \left( \frac{a}{R} \right)^4 - \frac{1085}{373} \left( \frac{a}{R} \right)^5 + \cdots \].

\[F = \frac{135}{4096} \frac{32}{8192} \left( \frac{a}{R} \right)^4 + \frac{689}{2192} \left( \frac{a}{R} \right)^5 + \cdots \].

\[F = \frac{1}{2} - \frac{3}{2} \left( \frac{a}{R} \right)^2 + \frac{9}{4} \left( \frac{a}{R} \right)^4 - \frac{35}{8} \left( \frac{a}{R} \right)^6 + \cdots \].

\[F = \frac{1}{8} + \frac{45}{8} \left( \frac{a}{R} \right)^2 - \frac{691}{512} \left( \frac{a}{R} \right)^3 + \cdots \]
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