Unification of field emission and space charge limited emission with collisions

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Electron emission plays a vital role in device design for systems with pressures ranging from vacuum to atmospheric pressure. Nonuniform pressure in vacuum devices and gap sizes below microscale for electronics near atmospheric pressure necessitate further theoretical characterization of the transition between electron emission phenomena. This letter incorporates collisions into analytical equations describing the transition from the Fowler-Nordheim (FN) equation for field emission to space-charge limited emission (SCLE). We recover the Child-Langmuir (CL) law for vacuum, SCLE at high mobility \( l \), and the Mott-Gurney (MG) law for collisional SCLE at low \( l \). The exact solutions follow asymptotic solutions for FN at low voltage \( V \), before transitioning to MG at higher \( V \), and, ultimately, to CL independent of \( l \). We also define a never before seen “triple-point,” where the asymptotic solutions of all three electron emission regimes converge. Fixing \( V \), \( l \), or gap distance \( D \) uniquely specifies the other two parameters to achieve this triple point, which defines a regime where the electron emission mechanism is very sensitive to experimental conditions. The implications on device design are discussed.

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atmospheric pressure or SCLE for imperfect vacuum, this letter assesses the transitions between FN, MG, and CL by including electron mobility into the equation for electron motion.

We start with a one-dimensional, planar diode with the cathode at $x = 0$ and the anode at $x = D$, fixed at $V$ with respect to the cathode. The gap is filled with a neutral gas with electron mobility $\mu$, which varies inversely with pressure $p$. We assume electron emission from the cathode with negligible initial ($t = 0$) velocity and accelerated by the surface electric field $E_s = E(0)$, or $x(0) = 0, v(0) = 0, a(0) = eE_0/m$. Coupling Poisson’s equation with continuity, $J = env$, where $e$, $n$, and $v$ are the electron charge, number density, and velocity, respectively, yields

$$\frac{d^2\phi}{dx^2} = \frac{J}{\varepsilon_0 v},$$

where $\phi$ is the electric potential, $\varepsilon_0$ is the permittivity of free space, and $J = AE_0^2 \exp(-B/E_0)$ from the FN relation, where $A$ and $B$ are the semi-empirically derived constants. We write the electron force balance as

$$m \frac{dv}{dt} = e \frac{d\phi}{dx} - ev - \mu v,$$

where the second term on the right-hand-side represents friction.\(^{14}\) We nondimensionalize (1) and (2) by

$$\phi = \phi_0 \psi; \quad J = I_0 J_0; \quad x = x_0 \xi; \quad t = t_0 T; \quad \mu = \mu_0 \mu; \quad E = E_0 E; \quad v = v_0 \nu;$$

$$\phi_0 = \frac{e^2}{m \alpha_2}; \quad J_0 = AB^2; \quad x_0 = \frac{e_0}{m \alpha_2}; \quad t_0 = \frac{e_0}{\alpha_1}; \quad \mu_0 = \frac{e_0}{m \alpha_2}; \quad E_0 = B; \quad v_0 = \frac{x_0}{t_0},$$

where the bars denote the dimensionless parameters and subscript 0 designates the scaling terms. Recasting (1), (2), and the FN equation using (3) gives

$$\frac{d^2\psi}{dx^2} = \frac{\psi}{\xi},$$

$$\frac{d\psi}{dx} \frac{d\psi}{d\xi} = \frac{\psi}{\xi},$$

$$\frac{J}{E_0} e^{-1/\xi},$$

and

$$\frac{\psi}{E_0} e^{-1/\xi}. \quad (4)$$

Differentiating (5) with respect to $\xi$, transforming to the Llewellyn form by using $\psi \equiv d\xi/d\xi$ to change variables,\(^{17}\) and combining with (4) yields

$$J = \frac{d^2\psi}{dx^2} + \frac{1}{\xi} \frac{d\psi}{dx} \frac{d\psi}{d\xi}. \quad (7)$$

Solving (6) and (7) gives

$$\psi(\xi) = \frac{\psi}{\xi} \left[ (\xi - E_0) (e^{-1/\xi} - 1) + \xi \right], \quad (8)$$

and

$$\psi(\xi) = \frac{\psi}{\xi} \left[ (\xi - E_0) (-\xi e^{-1/\xi} - \xi + \xi^2) + \xi^2 / 2 \right]. \quad (9)$$

We solve for the critical current density $J(\psi)$ at the transit time $T$ such that $\psi(\xi) = \xi$ and $\psi(\xi) = \xi$ for fixed $\xi$ and $\xi$. Rather than using an energy balance to find $\psi$, we integrate (5) with respect to $\xi$ and change variables to $\xi$ to obtain

$$\psi = \frac{\psi}{\xi} \left[ \frac{\psi}{\xi} \right]_0^T + \int_0^T \frac{d\psi(\xi)}{dx} \frac{d\psi(\xi)}{d\xi}, \quad (10)$$

where $\psi(\xi)$ given by (8). We numerically solve for $T$ by substituting $J(\psi)$ from (6) into (9) and evaluating at $\psi = \xi$. We then compute $\psi$ by substituting $T$, (6), and (8) into (10). With $\xi$ and $\xi$ fixed, this gives $J$ and $\xi$ parametrically in terms of $\xi$.

While an exact solution requires solving (9) numerically, an asymptotic analysis can elucidate the transitions between FN, MG, and CL. For high mobility (low pressure), expanding the exponential in (6) and (9) eliminates the collisional terms, yielding $\psi(\xi) \approx (\xi T^2) / (2 + ET)$ and $\psi(\xi) \approx (\xi T^2) / (6 + (ET)^2 / 2)$. The second term of (10) can be neglected, giving $\psi = (\xi T^2) / 2$ or $\psi(\xi) = (\xi T^2)^{1/2}$. Setting the approximations of $\psi(\xi)$ equal and solving for $\xi$ yields $\xi = \exp(1/\psi)/E$, where $\xi = -1 + [1 + (2\xi)^{1/2}] (\xi^{1/2})^{1/2}$. Incorporating $\xi$ into the simplified expression $\psi(\xi)$ above gives the transcendental function

$$\xi(\xi + 3) = 6 \xi \exp(-2/\xi),$$

which has the asymptotic limits of

$$J_{CL} = \left( \frac{4 \sqrt{2} / 9}{\psi} \right)^{3/2} \frac{\psi}{\psi^2},$$

for the CL limit at large $\psi$, and

$$J_{FN} = \left( \frac{\psi^2}{\psi^2} \right) e^{-\psi/\psi} \frac{\psi}{\psi^2},$$

for the FN limit at small $\psi$.\(^{17}\)

At low mobility (high pressure), $\exp(-1/\psi) \approx 0$, and (8) and (9) become $\psi(\xi) \approx \psi E + \xi T$ and $\psi(\xi) \approx \psi E + \xi T^2 / 2$, respectively. At $\psi(\xi) = \xi$, $T = \xi E / J$ with $\xi = -1 + [1 + (2\xi^2) / (\psi E^2)]^{1/2}$. Using these relationships to explicitly solve (10) gives

$$\psi = \left( \frac{\psi^2}{\psi^2} \right) (1 + 2 \xi + \xi^2) + \left( \frac{\psi^2}{\psi^2} \right) (\xi + \xi^2 + \xi^3 / 3).$$

Applying (6) and the definition of $\xi$ to (13) and neglecting higher order terms of $\psi$ yields

$$J_{MG} = \left( \frac{9 \psi^2}{\psi^2} \right), \quad (14)$$

which is MG.\(^{23}\) In the low current limit when space charge becomes negligible, $J \ll \psi E / 2D$. A binomial expansion yields $\chi \approx (\psi E / 2D)^2 - (\psi E / 2D)^2 / 8$. Solving (13) in this limit and neglecting higher order terms give $\psi = E D$, which we combine with (6) to obtain $J = \left( \frac{\psi^2}{\psi^2} \right) e^{-\psi/\psi}$, which is the same FN limit derived in (12) and Ref. 17 for vacuum.

We derive relationships for the transition between any two electron emission regimes by setting the respective asymptotic $J$ equal. Setting $J_{FN} = J_{MG}$ yields
and electron emission directly transitions from Fowler-Nordheim (FN) to Child-Langmuir (CL), as in vacuum. At the triple point, Mott-Gurney (MG) becomes important. Mathematically, (19) shows that the MG regime disappears (FN directly transitions to CL) when $D \leq V \ln \left(\frac{9\sqrt{V}}{4\sqrt{2}}\right)$, as

$J_{MG} = J_{CL}$, which we define as a “triple-point” that simultaneously satisfies all three electron emission laws.

Figure 2 shows alternative ways to depict the transition between electron emission mechanisms by plotting $V$ as a function of $D$ for $\pi = 7 \times 10^3$ and as a function of $\pi$ for $D = 10^3$. For small $D$ or large $\pi$, electron emission is driven by either FN or CL, as for vacuum. At the triple point, the solution bifurcates, making for asymptotic solution for the transitions between FN to MG, (15), and MG to CL, (16), relevant. Thus, for a given $\pi$ and $D$, one transitions from FN to MG to CL as $V$ increases. Figure 2 shows that the MG regime disappears when electrons lose only a small fraction of their energy in collisions with the neutral gas atoms at high $V$ or collisions become infrequent at low $D$. Thus, regardless of $P$ the diode behaves like vacuum for sufficiently small $D$ or large $V$.

Mathematically, (19) shows that the MG regime disappears (FN directly transitions to CL) when $D \leq V \ln \left(\frac{9\sqrt{V}}{4\sqrt{2}}\right)$, as
derived previously for vacuum.\textsuperscript{57} In other words, each pressure (inverse mobility\textsuperscript{58}) has a region of $\nabla$ and $D$ for which the diode exhibits vacuum–like behavior. Writing this relationship as an equality gives the triple point as

$$D = \nabla \ln \left( \frac{9 \sqrt{\nabla}}{4 \sqrt{2}} \right).$$  \hspace{1cm} (21)

Since (21) comes from setting the asymptotic solutions equal, the triple point does not exactly correspond to the exact solution of (9). While the triple point may not correspond to an exact solution, it does indicate a regime where the dominant electron emission mechanisms would be highly sensitive to small perturbations in the controllable parameters $D$, $\nabla$, and $\pi$ and by external factors, such as resistance.\textsuperscript{46}

Since the triple point occurs when $J_{MG} = J_{FN}$ and $J_{MG} = J_{CL}$, we add these equations, $2J_{MG} = J_{FN} + J_{CL}$, to obtain

$$\pi = D \left( \frac{16 \sqrt{2}}{81 \sqrt{\nabla}} + \frac{4}{9} \sqrt{\nabla} \right).$$  \hspace{1cm} (22)

which defines the triple point in terms of $D$, $\nabla$, and $\pi$. Combining (21) and (22) gives the triple point as a function of only $\pi$ and $\nabla$ as

$$\pi = \left( 16 \sqrt{2} / 81 \right) \ln \left( \frac{9 \sqrt{\nabla}}{4 \sqrt{2}} \right).$$  \hspace{1cm} (23)

Equations (17), (21), and (23) describe the triple point; fixing $D$, $\nabla$, or $\pi$ specifies the other two parameters at the triple point. Figure 3 shows this property of the triple point, plotting $\nabla$ as a function of $\pi$ and $D$.

The triple point has practical applications as a design tool. Consider a 250 nm gap filled with either nitrogen or argon. Using (3) with $A = (1.4/\varphi) \times 10^{-6} \cdot 4.36/\varphi$, $B = 6.49 \times 10^3 \varphi\varphi^{-1}$, and $\varphi = 2$ eV\textsuperscript{10} gives the scaling terms in (3) as $\varphi_0 = 2.66 \times 10^{-5}$ V, $x_0 = 1.45 \times 10^{-11}$ m, and $\mu_0 = 1.18 \times 10^{-7}$ m\textsuperscript{2}/(Vs). This gives $D = 1.73 \times 10^5$. Solving (21) and (23) yields $\nabla = 1.95 \times 10^2$ (V = 159 V) and $\pi = 2.19 \times 10^5$ ( $\pi = 2.86 \times 10^{-5}$ m\textsuperscript{2}/(Vs)), respectively. Since we can approach the triple point asymptotically from the field emission regime, $E = V/D$, giving $E = 2.08 \times 10^8$ V/m. Defining $v_\varphi = \mu_0 E$ gives the electron drift velocity for this system as $v_\varphi = 5.35 \times 10^6$ m/s. For argon, $\mu_A = 1.148 \times 10^9 / (T_e \varphi P_{o0})$, where $T_e$ is the electron temperature in K, $\mu_s$ is the argon electron mobility in m\textsuperscript{2}/(Vs), and $P_{o0}$ is the reduced pressure of argon in Torr.\textsuperscript{45} Obtaining $T_e$ from $mv_\varphi^2/2 = k_b T_e$, where $k_b$ is the Boltzmann constant, gives $P_{o0} = 243$ Torr. For nitrogen, $v_\varphi = 3.3 \times 10^6 \sqrt{E/P}$, where $v_\varphi$ and $E$ are in cgs units and P is in Torr.\textsuperscript{47} giving $P = 791$ Torr. Thus, the triple point yields similar physically realizable conditions for these gases.

The triple point for a 250 nm diode filled with nitrogen occurs near atmospheric pressure. This is particularly relevant for microscale gas breakdown, where one transitions from Townsend avalanche to field emission for microscale gaps.\textsuperscript{3,4} Further reducing $D$ at atmospheric pressure will cause the transition from FE to SCLE. At sufficiently low $D$, collisions become unimportant and SCLE transitions from MG to CL. The above calculations indicate that reducing $D$ from approximately 1 $\mu$m to 250 nm for nitrogen causes electron emission to transition from FE to the triple point, suggesting that smaller gaps will result in vacuum–like electron emission even at atmospheric pressure.

Even microscale gaps at atmospheric pressure can exhibit transitions between FN, MG, and CL. For an atmospheric pressure nitrogen-filled diode with $D = 1 \mu$m ($D = 6.90 \times 10^8$), solving (15), (16), and (17) for the transitions from FN to MG, MG to CL, and CL to FN gives $\nabla = 6.74 \times 10^2$ (V = 1.79 kV), $\nabla = 3.10 \times 10^5$ (V = 8.24 kV), and $\nabla = 7.25 \times 10^5$ (V = 1.93 kV), respectively. Since the gap distance exceeds that of the triple point, FN transitions to MG at a lower $V$ expected from CL.\textsuperscript{17} This further demonstrates that even at high PD, emission behavior still follows CL for sufficiently high voltage.

In summary, this letter derives equations to represent the transition from FE to SCLE in vacuum (CL) and as a function of pressure (MG) by incorporating a frictional term in the force balance proportional to $P$. The exact numerical solution of the resulting equations agreed with asymptotic limits obtained from a matched asymptotic analysis for FE and CL at vacuum\textsuperscript{16} and MG at general $P$. Setting the asymptotic solutions for FN, CL, and MG equal yielded unique combinations of $D$, $V$, and $\mu$ ($\propto 1/P$) where all three emission mechanisms converge. Equating pairs of these asymptotic solutions creates phase diagrams defining the conditions for inter–mechanism transitions. The overall results show that regardless of pressure, electron emission is driven by CL at high $V$ and small $D$ and FN at low $V$. The phase diagrams also show that at large $D$ or low $\mu$, one may transition from FN to MG to CL with increasing $V$, further demonstrating...
that electron emission may follow vacuum scaling even at larger gaps or higher pressures. These results have implications at high pressure for predicting gas breakdown at the microscale and smaller gaps that are driven by electron emission and for vacuum devices containing non-vacuum regions. While the predicted triple point’s asymptotic nature means that it may not strictly exist, it does define a regime where electron emission may be easily perturbed between the three mechanisms. This letter assumed \( v(0) = 0 \); a nonzero initial electron velocity would likely shift the triple point to higher voltages since the electrons would spend less time in the gap. Furthermore, we hypothesize that similar triple points exist for every transition from any electron emission mechanism to SCLE. The exact mechanism—for instance, FE for non-planar or nanoscale diodes, thermionic emission, or photoemission—will modify the exact location of the triple point and possibly the slope of the transition from non-SCLE to SCLE behavior.

Further assessing the sensitivity of the electron mechanism to external perturbations, such as resistance\(^{11,12} \) or nonuniform electrodes,\(^ {13,14,15} \) will illuminate likely triple point ranges in future experiments. Future experimental measurements of current as a function of applied voltage for various gap distances and gases, extending previous experiments assessing the transition from FN to CL for nanogaps\(^ {1} \) may further refine the theory.

While this study has focused on leveraging the incorporation of collisions into SCLE from Ref. 44 with the unification of FE with SCLE at vacuum in Ref. 17, the transition from FE to MG to CL with reducing gap distance has clear implications on previous theoretical studies unifying PI and FE for microscale gas breakdown.\(^ {33,34} \) Although the current study is physically adjacent to these theories,\(^ {34,35} \) one can envision unifying this work with those and even further extending SCLE to the quantum scale;\(^ {36} \) however, a universal, unified theory is complicated by requiring a single set of scaling parameters. Much as Refs. 1, 17, and 44 unify the individual transitions between mechanisms, the current study could demonstrate the importance of collisions in the transition from FE to SCLE than a fully unified model would. Thus, it provides valuable information, particularly the presence of the “triple point,” and a starting point for developing a unified theory of gas breakdown.

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