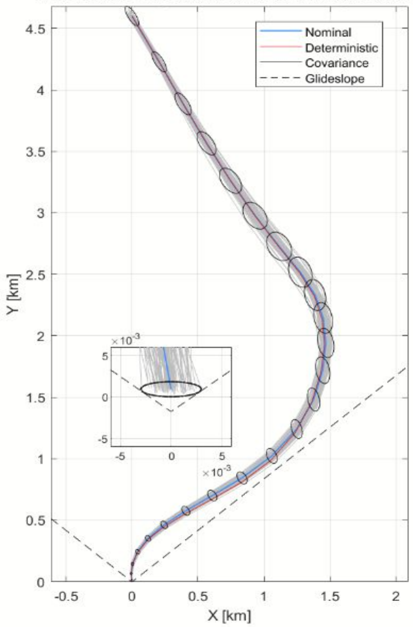
Stochastic Sequential Convex Programming for Rocket Landing

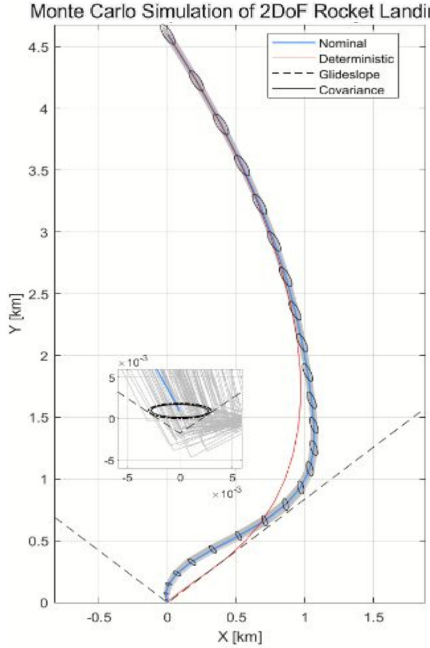
Travis Hastreiter, Atharva Awasthi

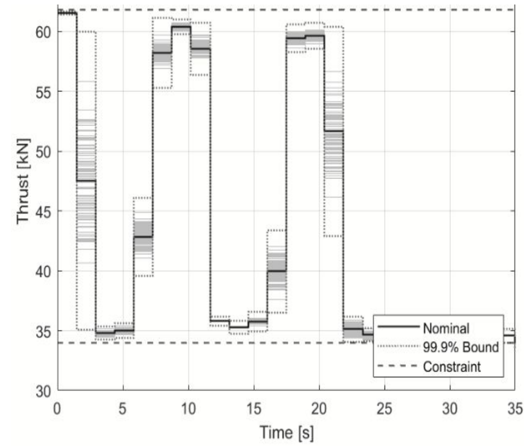
- Using Penalized Trust Region (PTR) for the SCP algorithm.
- Sources of Uncertainty: Initial state, Estimation error, Disturbances
- Optimized state and control covariance
- Adjusted the objective function and path constraints to account for worst case scenario under uncertainty.

Monte Carlo Simulation of 3DoF Rocket Landing

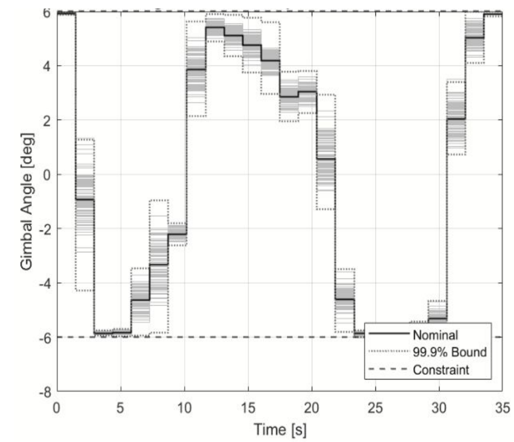


Monte Carlo Simulation of 2DoF Rocket Landing



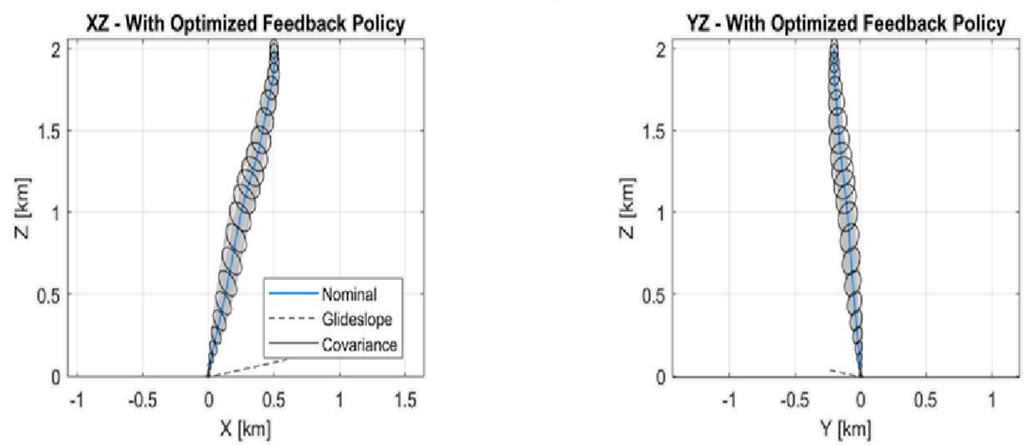


3DoF Stochastic Thrust vs Time



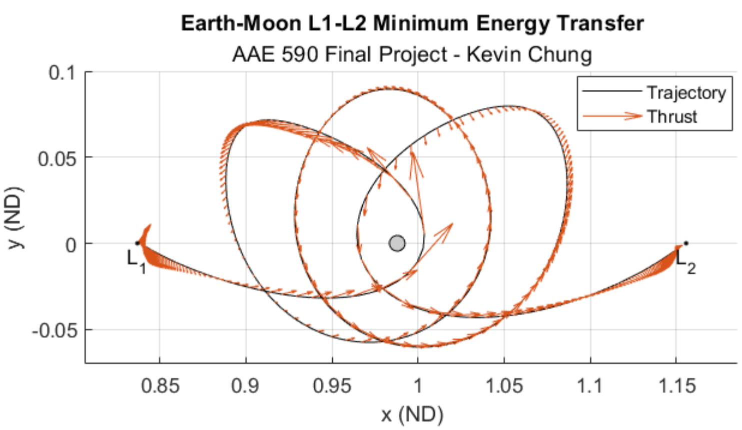
3DoF Stochastic Gimbal Angle vs Time

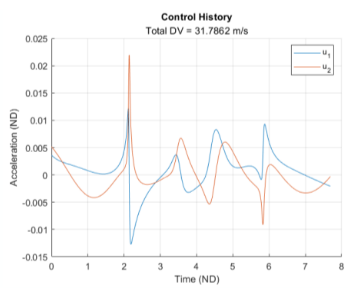
Monte Carlo Simulation of 6DoF Rocket Landing

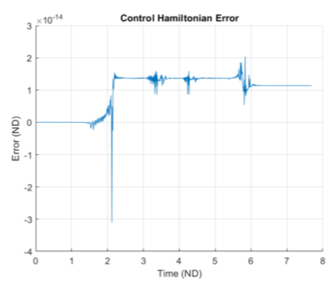


L1-L2 Minimum Energy Transfer in the Earth-Moon CR3BP

Kevin Chung







Objective

- Determine a minimum energy transfer from Earth-Moon L1 to L2 via Pontryagin-based indirect method

Approach

- Determine a reference trajectory via invariant manifolds, then solve for the optimal control to reach L2 via single shooting with a fixed transfer duration
- Using the fixed-time trajectory as a reference, allow the transfer duration to vary to further decrease energy cost

Results

- A low energy transfer is determined that costs 32 m/s of delta-V

Discussion

- A low-cost transfer from L1 to L2 can be determined via single shooting given a high-quality initial reference trajectory
- The approach is extremely sensitive to the initial guess; other methods are recommended to more robustly determine optimal trajectories

PURDUE UNIVERSITY
School of Aeronautics and Astronautics

Earth-Moon Cr3BP Orbit Transfer via Lyapunov Control

James Gilliam and Caleb Balzer

PURDUE UNIVERSITY | School of Aeronautics and Astronautics
COLLEGE OF ENGINEERING

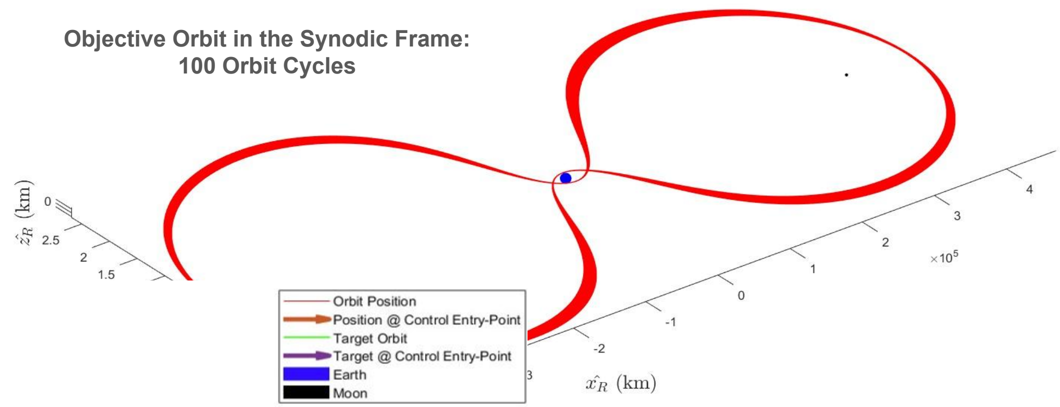
Project Objective

Cislunar space travel is an increasingly prominent topic, especially with projects such as Artemis and Gateway gaining momentum for utilizing the Moon as a launching point for future deep space missions. Translunar ejection is expensive though, and much equipment, such as exercise, living facilities, and landers, do not need to be sent up repeatedly. The goal of this research is to transfer into and maintain a synchronized figure-8 orbit around both the Earth and Moon to facilitate a cislunar highway.

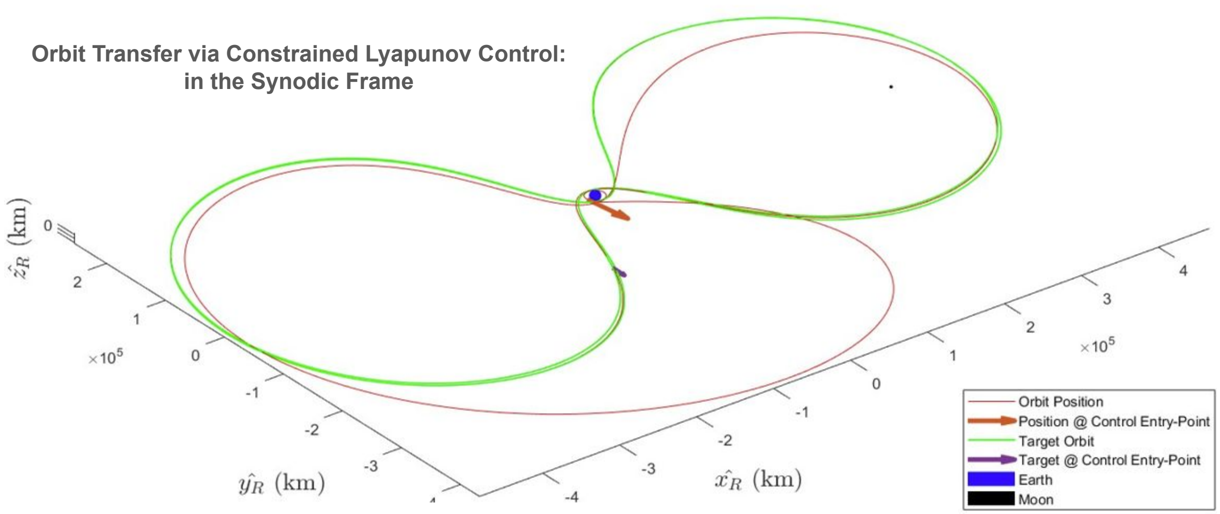
Approach

This is done by modeling the Cr3BP dynamics as a perturbed two-body problem. A stable Lyapunov controller is then derived with critical damping and applied to transfer a spacecraft from LEO to the desired figure-8 orbit. A basic Monte Carlo analysis is employed to find the optimal time in the orbit propagation to enter the transfer for fuel conservation.

**Objective Orbit in the Synodic Frame:
100 Orbit Cycles**



**Orbit Transfer via Constrained Lyapunov Control:
in the Synodic Frame**



Discussion

Results show the transfer to be feasible and with an optimal entry to the transfer, potentially practical for transporting equipment and other cargo. Whether the Lyapunov transfer can be performed quickly enough to be practical for transporting people remains an open question for future research.

AAE 590: *Applied Control is Astronautics*
Final Project - Spring 2025

Formation Flying for Interferometry Near Sun–Earth L2 Via Convex Optimization

AAE 59000 Applied Control In Astronautics – Paulo Ramirez, Kaylee Spencer

Objectives:

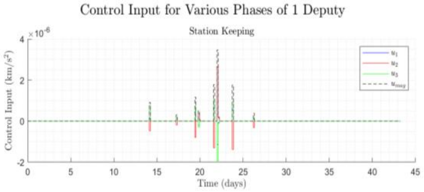
- Level 1 – Single-Satellite Station Keeping and Stable Halo Orbit Correction
- Level 2 – Multi-Agent Interferometry Formation
- Level 3 – Formation Reconfiguration Control Maneuver and LEO to Halo Orbital Transfer

Results:

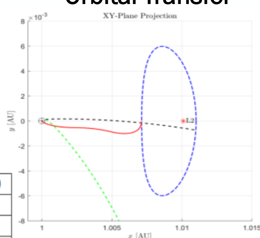
- Station keeping maintains 9 spacecraft in interferometry formation without a single structure
- 2 maneuver types adjust formation radius and plane angle

Maneuver Type	Δv per Satellite (m/s)
Station-Keeping	0.15
Radial Increase δr to $2\delta r$	6.35
Radial Decrease $2\delta r$ to $0.5\delta r$	11.02
Radial Increase $0.5\delta r$ to δr	7.03
Formation Plane Change 0^o to 45^o	10.19
Formation Plane Change 45^o to -45^o	11.80

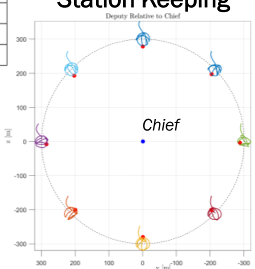
Control Input for Various Phases of 1 Deputy



Orbital Transfer



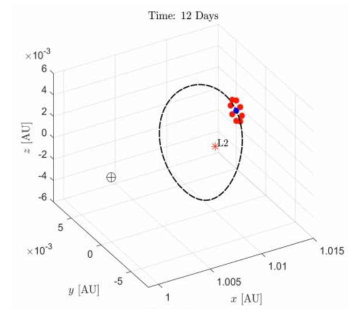
Station Keeping

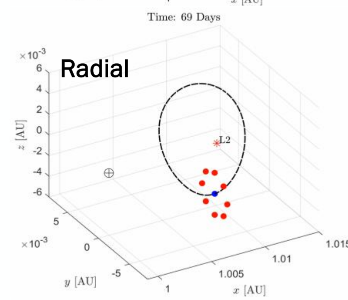


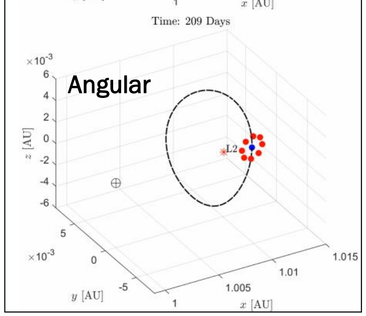
Discussion:

- Space telescope made from formation of chief and deputy satellites can adjust focal point to observe deep space
- Attitude integration could allow for greater degrees of control and more realistic analysis of resulting telescope capabilities

Formation Reconfiguration:







Methodologies Used

Nonlinear CR3BP Equations:

$$\ddot{x} - 2\dot{y} = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3},$$
$$\ddot{y} + 2\dot{x} = y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3},$$
$$\ddot{z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3},$$
$$\mu = \frac{m_2}{m_1 + m_2}$$
$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}, \quad r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$$

Multi-Agent Structure:

$$x \in \mathbb{R}^6 \rightarrow x_d \in \mathbb{R}^{6\times D}$$

- Where D is number of deputies

Convex Optimization:

Objective Function:

$$\min_{u_k} \sum_{k=0}^{N-1} \sum_{i=1}^{D} \|u_{d,i,k}\|_2 \Delta t$$

Constraints:

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k + \mathbf{c}_k, \quad k = 0, 1, \ldots, N-1$$
$$\mathbf{x}_0 = \mathbf{x}_{\text{init}}$$
$$\mathbf{x}_N = \mathbf{x}_{\text{final}}$$
$$\|\mathbf{u}_k\|_2 \le u_{\max}, \quad \forall k$$
$$\|\mathbf{x}_{\text{deputy}}(k) - \mathbf{x}_{\text{ref}}(k)\|_2 \le \epsilon, \quad \forall k$$

Halo Orbit Correction:

- Enforce **symmetry** (across yz-plane).
- Enforce **periodicity** (matching x, z, $\dot{y}$).

$$\Psi = \begin{bmatrix} x(T/2) - x(0) \\ z(T/2) - z(0) \\ \dot{y}(T/2) + \dot{y}(0) \end{bmatrix} \in \mathbb{R}^3$$

Earth-to-Halo Transfer Correction:

- Target **final position** close to **target point**.
- Constrain **right ascension (RA)** and **declination (DEC)** angles.
- Enforce a **cone constraint**.

$$\Psi = \begin{bmatrix} \|\mathbf{r}_f - \mathbf{r}_{\text{target}}\| - R_{\text{target}} \\ \text{RA}_f - \text{RA}_{\text{target}} \\ \text{DEC}_f - \text{DEC}_{\text{target}} \end{bmatrix} \in \mathbb{R}^3$$

Linearization About Reference Trajectory:

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\mathbf{u} + c(t)$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x_{\text{ref}}} \quad B = \begin{bmatrix} 0_{3\times3} \\ I_{3\times3} \end{bmatrix} \quad c = f(x_{\text{ref}}) - Ax_{\text{ref}}$$

$$\frac{d}{dt}\begin{bmatrix} \delta\mathbf{x} \\ \Phi \\ \Psi_B \\ \Psi_c \end{bmatrix} = \begin{bmatrix} A\delta\mathbf{x} + c \\ A\Phi \\ \Phi^{-1}B \\ \Phi^{-1}c \end{bmatrix}$$

$$\mathbf{A}_d(k) = \Phi(t_{k+1})$$
$$\mathbf{B}_d(k) = \Phi(t_{k+1})\Psi_B(t_{k+1})$$
$$\mathbf{c}_d(k) = \Phi(t_{k+1})\Psi_c(t_{k+1})$$
$$\Downarrow$$
$$\mathbf{x}_{k+1} = A_k \mathbf{x}_k + B_k \mathbf{u}_k + c_k$$

Multi-Objective Indirect Optimal Control for Low-Thrust Cislunar Return Trajectories

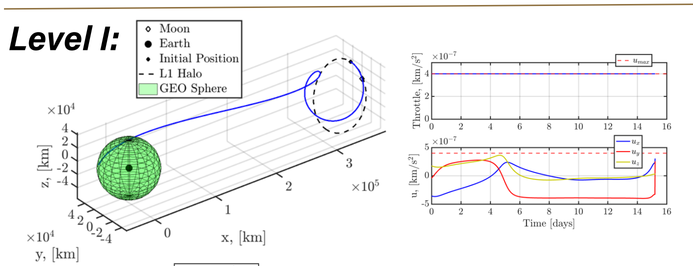
Mikayla Gallagher

Objective:
Simultaneously optimize spacecraft control for prescribed weighting between the objectives of minimizing time-of-flight and minimizing fuel-usage in the cislunar regime to develop a trade-space of optimal low-thrust trajectories from an L1 Halo Orbit to the GEO-sphere

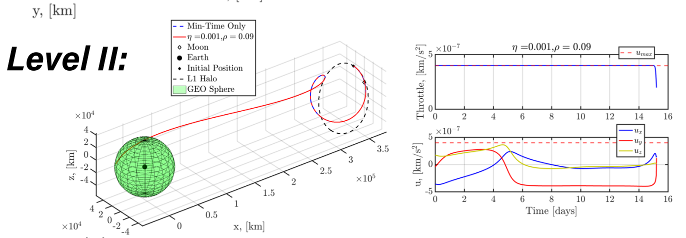
Approach:
Level I: Obtain an optimal minimum time low-thrust transfer from an L1 Halo (after 1 m/s step-off) to anywhere on GEO-sphere with free final velocity using Pontryagin-based Indirect Optimal Control methods
Level II: Develop multi-objective framework for the same transfer conditions and use minimum time solution to obtain a weighted solution for a multi-objective optimization very close to a pure minimum time problem
Level III: Derive and implement the analytical gradient to overcome numerical sensitivities to obtain a family of trajectories weighted from pure minimum time to pure minimum fuel and perform sharpness continuation at each weight

Results:

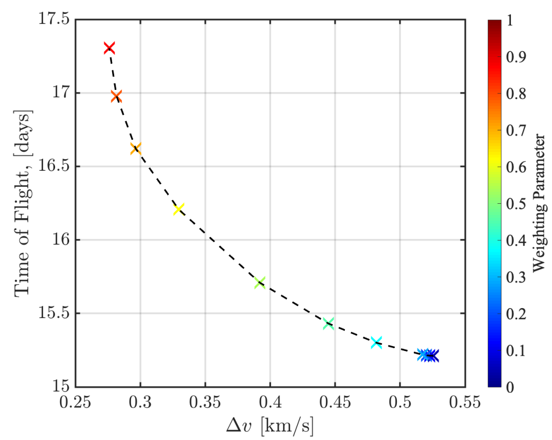
Level I:

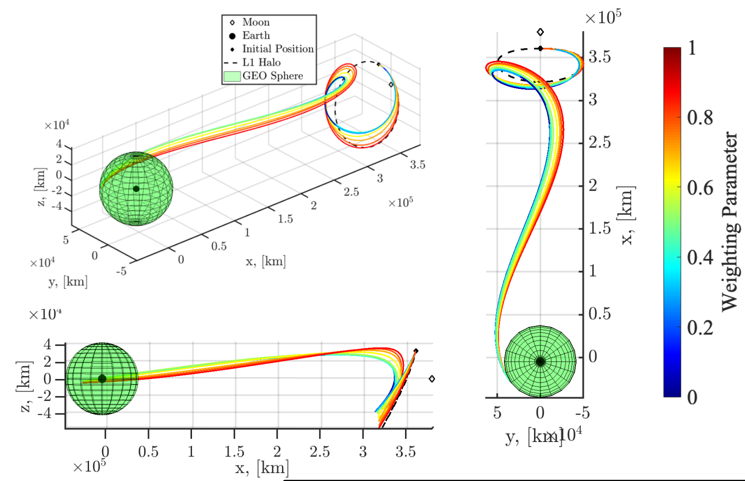


Level II:



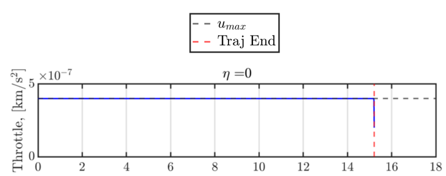
Level III:

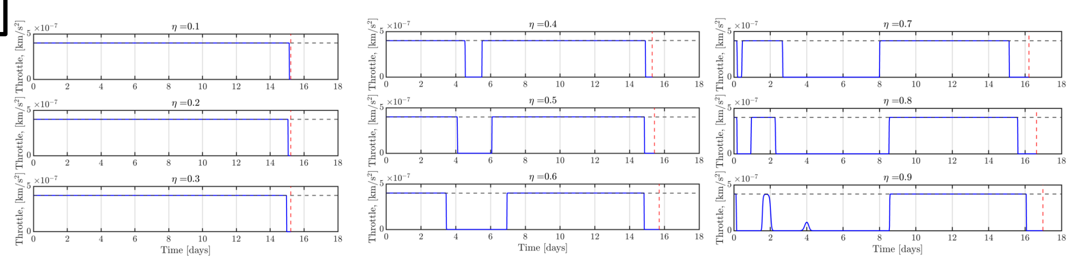


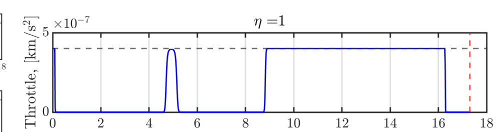


Pure Min Time TOF = 15.2121 Days

Pure Min Fuel TOF = 17.3045 Days







Minimum Time Objective ⟷ **Minimum Fuel Objective**

PURDUE UNIVERSITY
School of Aeronautics and Astronautics

Discussion:
The Level I objective of obtaining a minimum time optimal trajectory was achieved using small random initial guesses for the initial costates and Lagrange multiplier and random guesses for the final time close to a ballistic trajectory solution. The Level II objective was achieved by implementing the minimum time only solution in the multi-objective framework with a weighting parameter equal to 0.001, however numerical sensitivities prevented sufficient smoothing continuation or continuation to other values of the weighting parameter. This was achieved in Level III by deriving and implementing the analytical gradient. These results produced a family of optimal solutions for weighting parameters [0,1] with intuitive trajectories where minimum fuel leaning solutions led to longer flight paths. The control profiles also demonstrated longer coasting periods as the weighting parameter increased towards the minimum fuel objective.

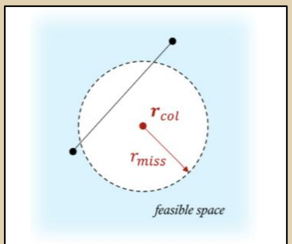
COLLISION AVOIDANCE IN THE CR3BP USING SEQUENTIAL CONVEX OPTIMIZATION

Objective

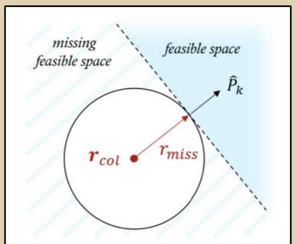
Implement convex optimization to model collision avoidance scenarios in the CR3BP with impulsive and low-thrust control (reference orbit: 9:2 NRHO).

Methods

- Approximate non-convex avoidance sphere via series of convex, half-plane constraints defined using reference diverted trajectory



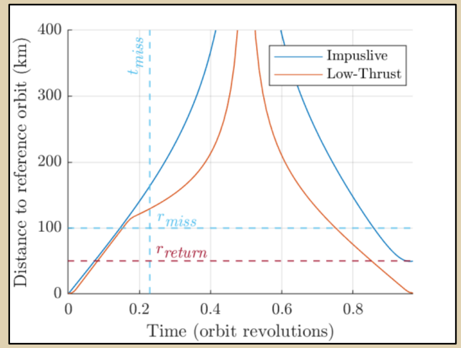
(a) Original, non-convex avoidance constraint.



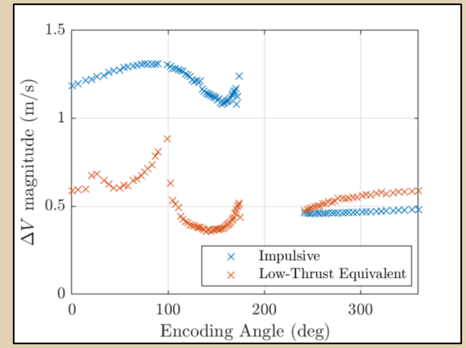
(b) Convexified avoidance constraint with loss.

- Sequential optimization: update reference and plane constraints after each iteration
  - Impulsive: find consistent ΔV_0 solution
  - Low-thrust: include slack variables and trust regions to improve convergence

Results



Sample diverted trajectory distances to NRHO for impulsive and low-thrust



Collision avoidance costs across NRHO (assuming 300 s I_{sp} for low-thrust equivalent)

Takeaways

- Optimization produces low cost diverted trajectories
- Convexification reduces solution space but still produces good solutions
- Low-thrust allows trajectories to return exactly to baseline orbit and often results in propellant savings
- Sequential convex programming is challenging to converge, especially for the low thrust problem
- Optimization still requires dynamical information to initialize the reference trajectories

PURDUE AERONAUTICS & ASTRONAUTICS

Low Thrust Earth-Neptune Trajectory with Gravity Assist

Objective

Investigate optimal minimum fuel low thrust trajectories from Earth to Neptune using a Jupiter gravity assist and compare to results to a similar impulsive trajectory

Methodology

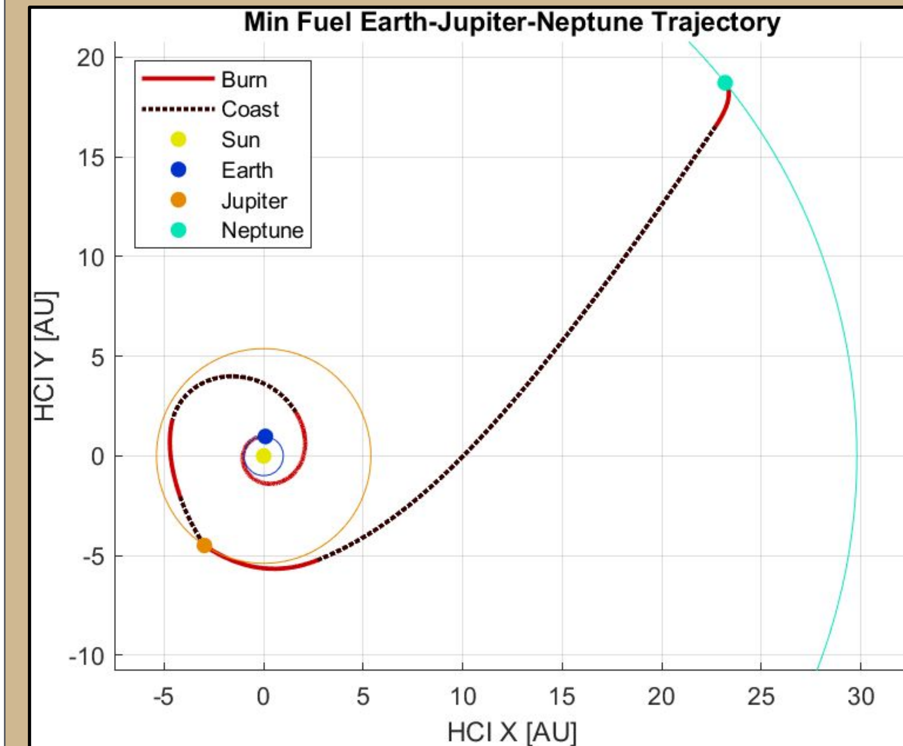
Pontryagin Shooting Method Optimization: Repeatedly propagate state and costate dynamics, using a numerical root finding function to find an initial costate which satisfies transversality conditions.

Homotopic Approach: Start with the solution to an easier to converge problem (min-energy), and incrementally change the objective function to transition from min-energy to min-fuel while ensuring convergence

Discussion

- While a low thrust trajectory requires move delta-v, it has a lower propellent mass fraction (m_{prop}/m_0 = 0.73) than a comparable chemically propelled impulsive trajectory (m_{prop}/m_0 = 0.94).
- Shooting method convergence is very sensitive to initial conditions and costate guess, which makes finding minimum fuel trajectories difficult. Adding mass as a state variable increases this difficulty.
- Trajectory could be further optimized by freeing the flyby date and velocity and optimizing both segments as a single trajectory

Results



Total delta-v:	38.64 km/s
Time of Flight:	19 years
Propellent Mass Fraction:	0.731

Hannah Kadlec & Ishaan Rao

Convex Gateway Rendezvous Under Cislunar Dynamics

Chase Loeb, Sarah DeVito

School of Aeronautics and Astronautics

Objective: Rendezvous with Gateway along a fuel-efficient low-thrust trajectory originating from Earth

Pontryagin-based Minimum Fuel Trajectory

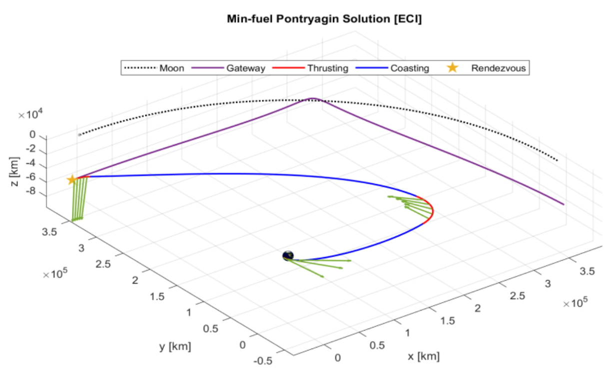
An initial fuel-efficient trajectory can be generated using a shooting function approach, which seeks to minimize $||u||_2$ under nonlinear CR3BP dynamics. The spacecraft is assumed to be launched into a highly eccentric orbit about the Earth before performing its own maneuvers with a low thrust engine.

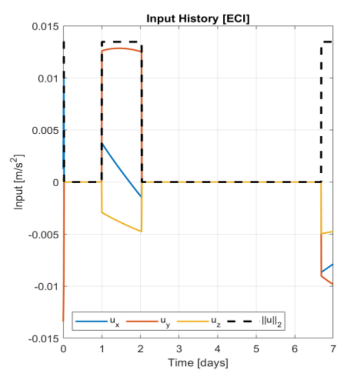
$$x_0 := \{h_{\text{alt}} = 500 \text{ km},\ \text{e} = 0.90,\ \omega = 180 \text{ deg},\ \nu = 0 \text{ deg},\ \text{i} = 0 \text{ deg}\}$$

$$u_{\max} = 0.0135\ \frac{\text{m}}{\text{s}^2}$$

$$\begin{aligned} \text{minimize} \quad & \int_{t_0}^{t_f} ||\vec{u}(t)||_2 \\ \text{subject to} \quad & \dot{x}(t) = f_{\text{CR3BP}}(\vec{x}(t)) + B_{\text{CR3BP}}\vec{u}(t) \\ & ||\vec{u}(t)||_2 \le u_{\max} \\ & \vec{x}(t_0) = x_0 \qquad \vec{x}(t_f) = x_{\text{Gateway}} \end{aligned}$$

Pontryagin-based optimal control is sensitive to the initial costate guess, it is possible for the solution to fall into many different local optima. Therefore, care must be taken to provide a "good" initial guess such that the solution is reasonable.





The Pontryagin solution results in a bang-bang control history, which is expected for a min-fuel objective. While this trajectory is ideal, its practical usefulness is limited when considering control system uncertainty and launch vehicle inaccuracy. Generating a new solution mid-journey may not be feasible, because the Pontryagin solution is extremely sensitive to initial conditions.

Corrective Maneuvers When Considering Model Uncertainty

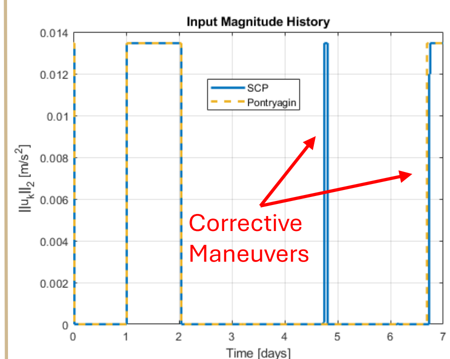
When adding uncertainty to the spacecraft's initial state and trajectory propagation via Brownian motion, several solutions that simply implement the Pontryagin control history do not successfully rendezvous with Gateway. Sequential Convex Programming is implemented via SCvx [1] to re-compute a min-fuel trajectory using the previous trajectory as a reference $\bar{x}$.

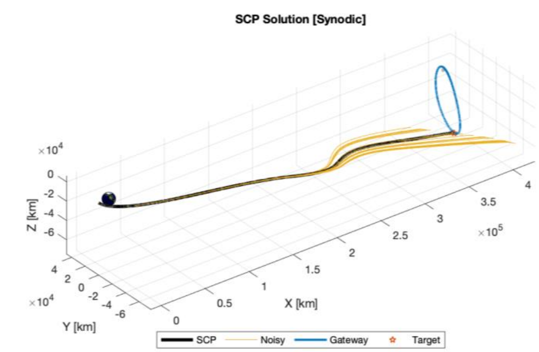
CR3BP dynamics are non-convex, the dynamics must be linearized and discretized around the reference trajectory [2]

$$\dot{x} = A(t)x + Bu + \{f(\bar{x}) - A(t)\bar{x}\}$$

$$x_{k+1} = A_k x_k + B_k u_k + F_k$$

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{N} ||\vec{u}_k||_2 \cdot \Delta t + w_\zeta \sum_{k=1}^{N} \max(0,\ \zeta_k) + w_\xi|\xi| \\ \text{subject to} \quad & \vec{x}_{k+1} = A_k\vec{x}_k + B_k\vec{u}_k + F_k \\ & ||\vec{u}_k||_2 - u_{\max} \le \zeta_k \\ & \vec{x}(t_f) - x_{\text{Gateway}} = \xi \qquad \vec{x}(t_0) = x_0 \\ & ||\bar{x} - x^*||_\infty \le \delta \text{ (variable trust region)} \end{aligned}$$



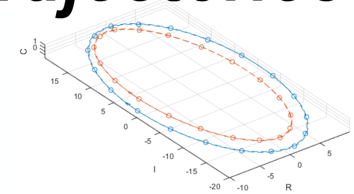


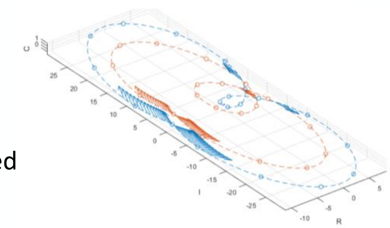
The trajectory re-calculation points are three predetermined events after the second planned maneuver, when the spacecraft is likely to significantly depart from the reference trajectory. The SCvx routine computes a new trajectory which may include corrective maneuvers. Despite many noisy trajectories missing Gateway, the corrected trajectory will successfully reach the goal state.

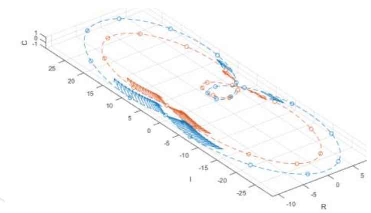
[1] Sequential Convex Programming for Nonlinear Trajectory Optimization, Kenshiro Oguri, Purdue University School of Aeronautics and Astronautics, [2] Low-Thrust Trajectory Design with Successive Convex Optimization for Libration Point Orbits, Yuki Kayama, Kathleen C. Howell, Mai Bando, and Shinji Hokamoto, Journal of Guidance, Control, and Dynamics 2022

Convex Optimization of Coupled Multi-Spacecraft Trajectories

AAE 590ACA – Mark Hurt

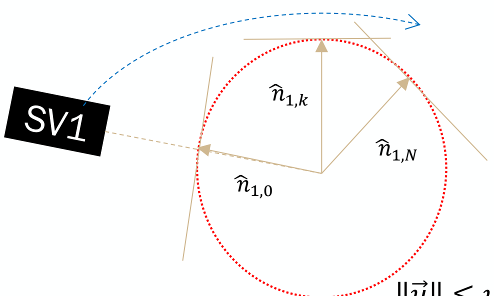






Motivation

- Increasing desire in industry to move to many inexpensive small-sats available throughout many orbits, rather than expensive, dedicated vehicles for Space Situational Awareness (SSA)
  - Redundancy through multiplicity
  - Inspection missions are improved with multiple cameras and viewpoints
  - Difficult to coordinate constellation orbit dynamics.



$$\hat{n}_{i,k}^T \vec{r}_i \ge r_{\min}$$

$$\|\vec{u}\| \le u_{\max}$$

$$\|\vec{r}_i\| \le r_{\max}$$

$$\|\vec{r}_i - \vec{r}_j\| \le \alpha_{\max}$$

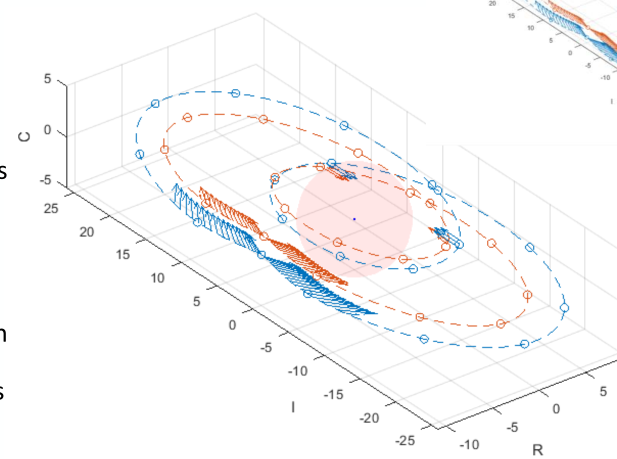
PURDUE UNIVERSITY | School of Aeronautics and Astronautics

Formulation

- Dynamics
  - Linearize the orbit about an assumed circular asteroid with Chlohessy-Wiltshire-Hills (CWH) Equations
  - Discretize the problem into N discrete-time steps.
  - Combine multiple spacecraft dynamics into a single problem, populating matrices in block-diagonal form
- Cost Function
  - Minimizing fuel use of each spacecraft at each time step
- Constraints
  - Thrusters - Maximum Acceleration Constraint
  - Camera - Maximum Distance Constraint
  - Crosslink - Maximum Distance between SVs
  - Collision - Minimum Distance Constraint (Nonconvex)
- Convexification
  - Define a hyperplane defined by a vector normal to the KOZ sphere, rotating to match the SV as it progresses around in an orbit.
  - Let $\hat{n}_{i,k}$ be the unit vector pointing towards the SV on the reference trajectory

Discussion

- Convex optimization is an increasingly efficient and robust method for trajectory optimization for arbitrarily large constellations
- Most constraints can be added directly, and those that can't can be implemented without significant inaccuracies



Touring Jupiter's Galilean Moons via Indirect Optimal Control

Mikayla Goulet AAE 590ACA

Objective

- Design a mission for a single spacecraft to visit all four Galilean moons (Io, Europa, Ganymede, and Callisto) with minimum-time transfers using indirect optimal control methods

Approach

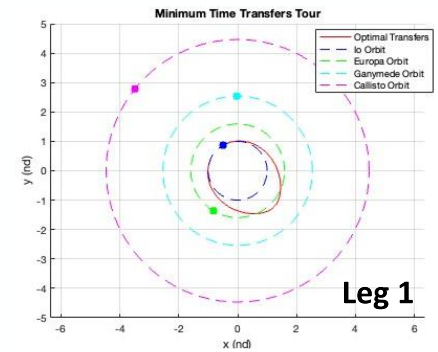
- Apply Pontryagin's Minimum Principle to optimize the control inputs and employ single point shooting methods to find the optimal initial costates
- Break the problem into four trajectories—parking orbit to moon 1, moon 1 to moon 2, moon 2 to moon 3, and moon 3 to moon 4—and piece together these individual legs to form one complete tour

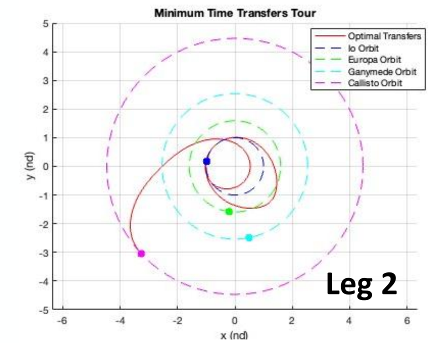
Results

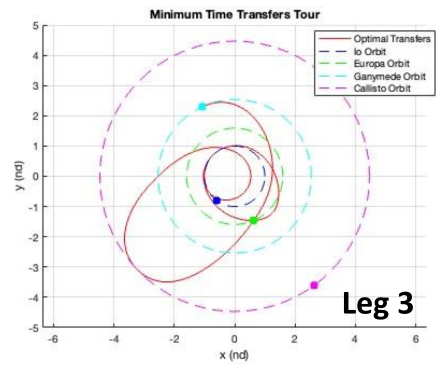
- Full trajectory showing all moon visits
- Plots of optimal control inputs over time and optimal costates over time

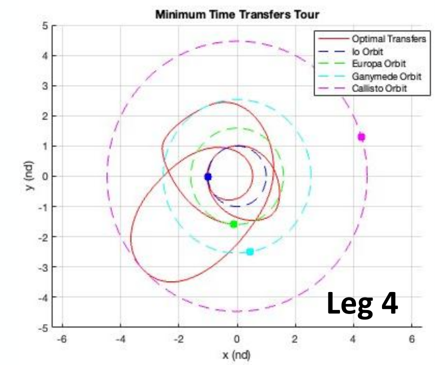
Key Takeaway

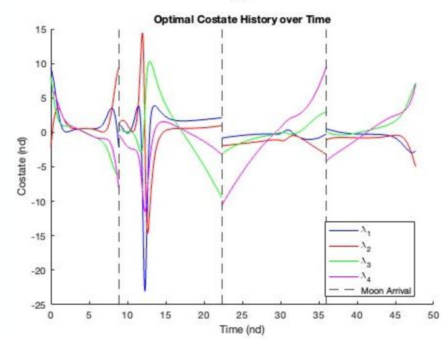
- It is possible to design a complete tour using this method, but it has its drawbacks as the problem is very sensitive to the estimate of the initial costates

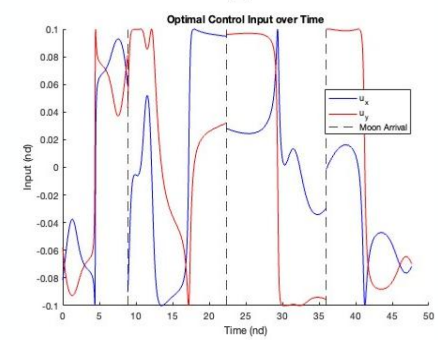








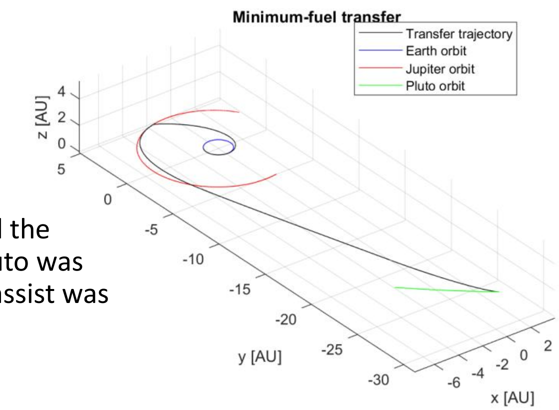


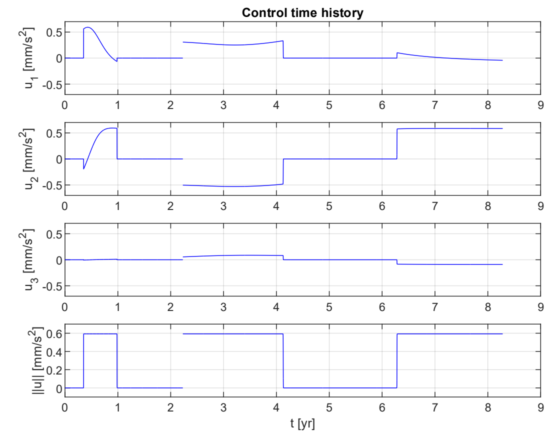


Optimal Non-Coplanar Low Thrust Transfer Trajectories from Earth to Pluto

Karel Hernández Bandrich

- **Objective**: to design trajectories for low-thrust spacecraft to reach Pluto considering its inclination and eccentricity, as well as SRP perturbations, incorporating a Jupiter gravity assist.
- **Approach**: Pontryagin's Minimum Principle was used to find the optimal trajectory. Ephemeris data for Earth, Jupiter, and Pluto was used to provide real initial and final conditions. The gravity assist was modeled using patched conics.





System dynamics

$$\dot{\mathbf{x}} = \mathbf{f} = \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ -\frac{\mu}{r^3}\mathbf{r} + \mathbf{u} + \mathbf{a} \end{bmatrix}$$

Optimal control

$$\mathbf{u}^* = \Gamma^* \hat{\mathbf{u}}^*$$
$$\hat{\mathbf{u}} = \mathbf{p}/p \qquad \Gamma^* = \begin{cases} u_{max}, & if(p > 1) \\ 0, & if(p < 1) \end{cases}$$
$$\mathbf{p} = -\lambda_v$$

Transversality conditions

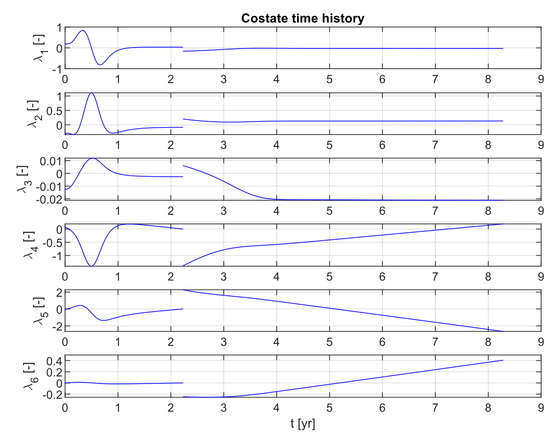
$$\psi = \begin{bmatrix} \mathbf{x}_f - \mathbf{x}_{arr,f} \\ -\lambda(t_f)^T \dot{\mathbf{x}}_{arr}(t_f) + H(t_f) \end{bmatrix}$$

Costate dynamics

$$\dot{\lambda}^T = -H_x^* = -\lambda^T \mathbf{f}_x^* = -\lambda^T A$$

$$A = \mathbf{f}_{x^*} = \begin{bmatrix} 0_3 & I_3 \\ \frac{\partial \mathbf{f}}{\partial \mathbf{r}} & 0_3 \end{bmatrix}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{r}} = \left(\frac{AG_0}{m} - \mu\right)\left(\frac{I_3}{r^3} - \frac{3\mathbf{r}\mathbf{r}^T}{r^5}\right)$$



- **Results and conclusions**: an optimal transfer trajectory was successfully found, although it is relatively expensive due to the alignment of the planets at the epoch. Further work includes refining design for the gravity assist and searching for better launch times to reduce Δv requirements.

PURDUE UNIVERSITY
School of Aeronautics and Astronautics

Lyapunov Controller for Nonlinear Relative Orbital Dynamics: A Polynomial Optimization Approach

By: Jose Daniel Hoyos

Objective

Design a Lyapunov-certified controller synthesis methodology in the relative orbital control scenario that can handle input saturation directly as well as other constraints.

Methodology

- Polynomial expansion of nonlinear terms.
- Search of $V(\cdot)$ and $u(\cdot)$ for the polynomial truncated dynamics.
- Use of 'grid search', running a bilinear solver (not convex) with several different parameters.
- Use of $V(\cdot)$ and $u(\cdot)$ to certify locally the original dynamics.

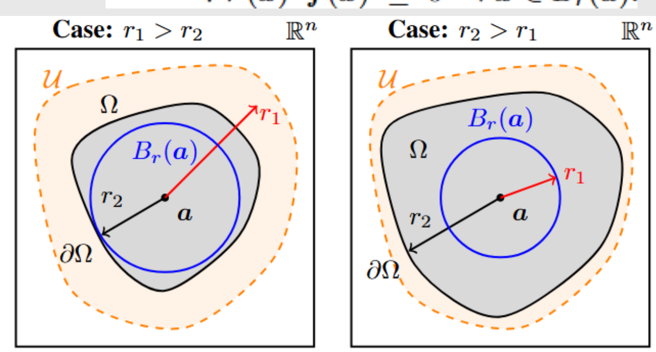
Results

- Analytical

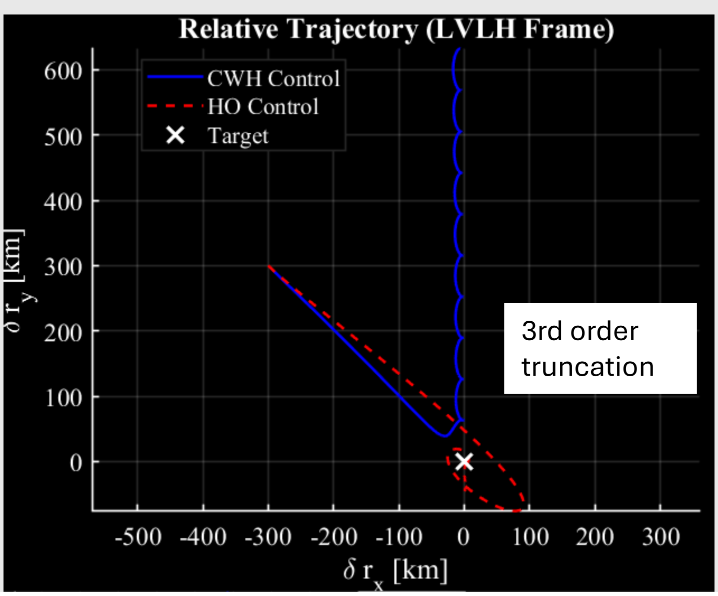
If $\nabla V(\boldsymbol{x})^\top \boldsymbol{f}_{(p)}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in \Omega$;

then there is an $r > 0$ such that

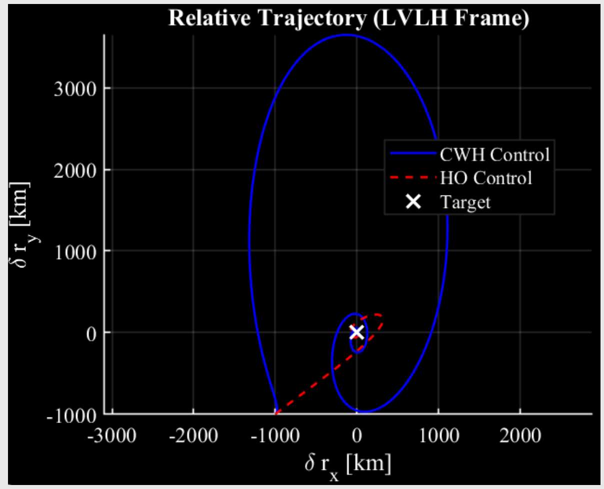
$$\nabla V(\boldsymbol{x})^\top \boldsymbol{f}(\boldsymbol{x}) \leq 0 \quad \forall \boldsymbol{x} \in B_r(\boldsymbol{a}).$$



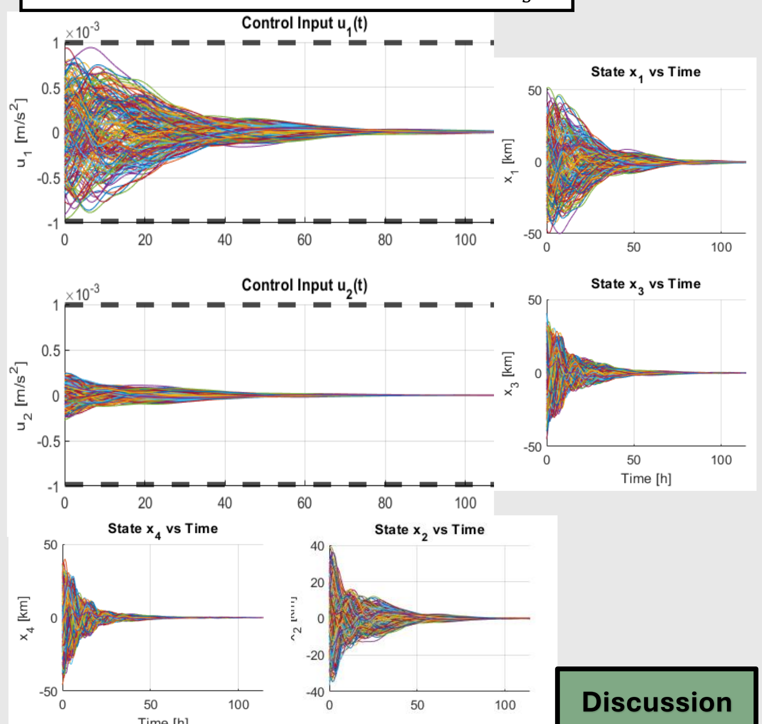
- Full dynamics propagation under control input history **(open-loop)** $u(\cdot)$ from Linearized equations + LQR vs Proposed method.



- Full dynamics propagation under control law **(closed-loop)** $u(\cdot)$ from Linearized equations + LQR vs Proposed method.



- Constraint handling: $|u_i| \leq 0.001 \ \frac{m}{s^2}$



Discussion

Larger region of attraction when propagating closed-loop true dynamics compared to LQR.

Gets closer to desired state when propagating control history into the true dynamics compared to LQR (openloop).
The error in openloop is decreased by increasing the truncation polynomial order, but the optimization problem gets more expensive.
The method proved it can handle $u(\cdot)$ constraints directly (other Lyapunov methods can not).
The proposed method has a relative control effort compared to linearization + LQR.

Application of Koopman Operator to Optimal Control Problems in Astronautics

PURDUE UNIVERSITY
School of Aeronautics and Astronautics

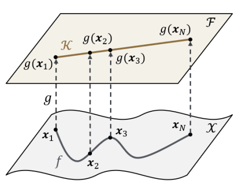
OVERVIEW

Koopman operator theory provides a linear representation of the nonlinear dynamics and state constraints. By investigating data-driven and Galerkin-based methods for Koopman operator approximation, we applied Koopman operator theory on the maximum radius orbit transfer problem and model predictive control for orbit transfer.

AAE-590-ACA
Zhi Xu

Introduction

➤ Linear Representation of Nonlinear Dynamics



Observable: $g(\boldsymbol{x}) \in \mathcal{F} \subseteq L^2$

Koopman Operator (KO):

$\mathcal{K}:\ g(\boldsymbol{x}) \mapsto g(\boldsymbol{x}^+)$ (*Linear*)

State: $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}^n$

System Dynamics:

$f:\ \boldsymbol{x} \mapsto \boldsymbol{x}^+$ (*Nonlinear*)

➤ Approximate KO on a Finite Subspace

System evolution with $\mathcal{K}: \mathcal{F} \to \mathcal{F}$ (*Infinite-dimensional*):

$$[\mathcal{K}g](\boldsymbol{x}) = g[f(\boldsymbol{x})] = g(\boldsymbol{x}^+)$$

Select a finite set of observables which span a finite subset $\mathcal{F}_c \subseteq \mathcal{F}$:

$$\Psi = [g_1, g_2, \ldots, g_m]^T, \qquad g: \mathbb{R}^n \to \mathbb{R}$$

System evolution with $K: \mathcal{F}_c \to \mathcal{F}_c$ (*finite-dimensional*):

$$[K\Psi](\boldsymbol{x}) = \Psi(\boldsymbol{x}^+) + \boldsymbol{r} \quad \to \quad K \cdot \boldsymbol{z} = \boldsymbol{z}^+ + \boldsymbol{r}$$

$\Psi: \mathbb{R}^n \to \mathbb{R}^m$: *Lifting function*
$\boldsymbol{z} = \Psi(\boldsymbol{x}) \in \mathbb{R}^m$: *Lifted state*
$K \in \mathbb{R}^{m \times m}$: *Approximated Koopman operator (matrix)*
$\boldsymbol{r}$: *Residual of approximation*

Method

➤ Computational Approaches for Approximated KO

- ***Regression on System Snapshots*** $\{\boldsymbol{x}_j, \boldsymbol{x}_j^+\}_{j=1}^J$

$$\min_K \frac{1}{2}\sum_{j=1}^J \|\Psi(\boldsymbol{x}_j^+) - K\Psi(\boldsymbol{x}_j)\|_2^2$$

Least square solution:

$$K = G^\dagger H, \qquad G = \frac{1}{J}\sum_j \Psi^\top(\boldsymbol{x}_j)\Psi(\boldsymbol{x}_j), \qquad H = \frac{1}{J}\sum_j \Psi^\top(\boldsymbol{x}_j)\Psi(\boldsymbol{x}_j^+)$$

- ***Projection using Galerkin Method***

$$\boldsymbol{r} = \Psi[f(\boldsymbol{x})] - K\Psi(\boldsymbol{x})$$

Galerkin orthogonal condition:

$$\langle \boldsymbol{r}, \Psi_j \rangle = \langle \Psi_i[f] - K\Psi_i, \Psi_j \rangle = 0$$
$$\to \ \langle K\Psi_i, \Psi_j \rangle = \langle \Psi_i[f], \Psi_j \rangle$$

With orthonormal basis ($\langle \Psi_i, \Psi_j \rangle = I$):

$$\to \ K = \langle \Psi_i[f], \Psi_j \rangle$$

➤ Optimal Control Problems with KO

- ***Controlled Dynamics***

For systems with control input $\boldsymbol{u}$, solve the approximated KO for augmented lifted state $\bar{\boldsymbol{z}} = [\boldsymbol{z}, \boldsymbol{u}]^\top, (\boldsymbol{z} = \Psi(\boldsymbol{x}))$:

$$\bar{\boldsymbol{z}}^+ = K\bar{\boldsymbol{z}} + \boldsymbol{r}, \ K = \begin{bmatrix} A & B \\ \ldots & \ldots \end{bmatrix}$$
$$\to \boldsymbol{z}^+ = A\boldsymbol{z} + B\boldsymbol{u} + \boldsymbol{r}$$

- ***Nonlinear State Constraints***

Nonlinear constraints on $\boldsymbol{x}$ can be transfer to linear constraints on $\boldsymbol{z}$ by projection:

$$h(\boldsymbol{x}) = 0 \ \to \ G \cdot \boldsymbol{z} = 0, G = \langle h, \Psi \rangle$$

E.g., with $\mathbf{z} = \Psi(x) = [1, x, x^2]^\top$, constraint:

$$x^2 + 2x = 0 \ \to \ [0\ 2\ 1] \cdot \boldsymbol{z} = 0$$

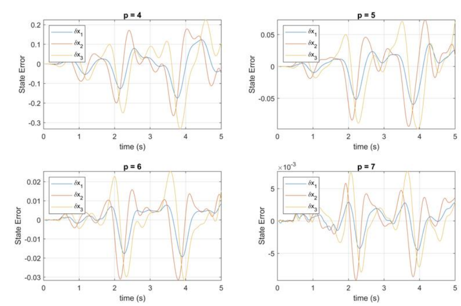
- ***Multivariate Basis Construction:***

$$\Psi(\mathbf{x}) = \Phi^{(1)}(x_1) \otimes \Phi^{(2)}(x_2) \otimes \cdots \otimes \Phi^{(n)}(x_n) = \bigotimes_{d=1}^{n} \Phi^{(d)}(x_d),$$

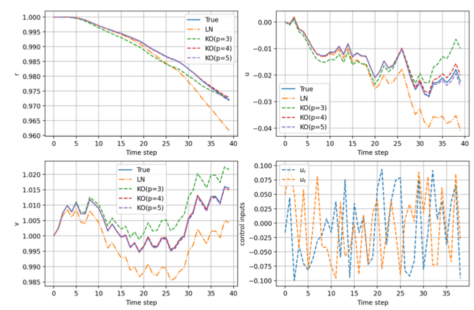
Monomials up to order p for univariate basis:

$$\Phi^{(d)}(x_d) = \{x^{i-1}\}_{i=1}^p$$

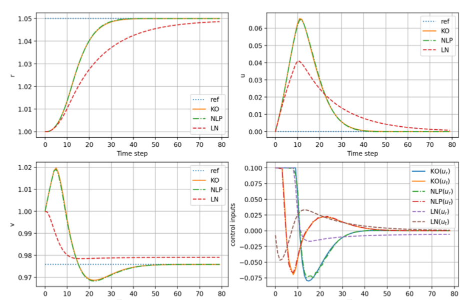
Simulation & Results



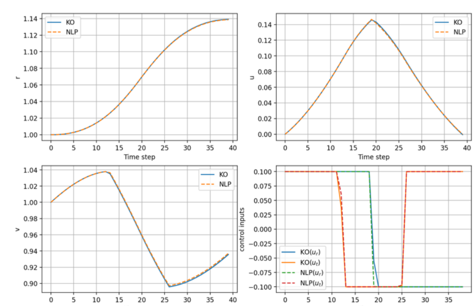
Approximation error with different selection of p



System propagation with random control input
(LN: Local linearization)



Solution comparison (NLP: Direct collocation with nonlinear programming solver, LN: Local linearization)



Solution comparison (NLP: Direct collocation with nonlinear programming solver)