Part II: Lagrange Multiplier Method & Karush-Kuhn-Tucker (KKT) Conditions

KKT Conditions

General Non-Linear Constrained Minimum:
Min: f[x]
Constrained by: h[x] = 0 (m equality constraints)
g[x] ≤ 0 (k inequality constraints)

Introduce slack variables s_i for the inequality contraints: g_i [x] + s_i^2 == 0
and construct the monster Lagrangian:

$$L[x,\lambda,\mu] = f[x] + \lambda\, h[x] + \sum \mu_i (g_i[x] + s_i^2)$$

Recall the geometry of the Lagrange multiplier conditions: The gradient of the objective function must be orthogonal to the tangent plane of the (active) constraints. That is the projection of the gradient of f onto the space of directions tangent to the constraint "surface" is zero. The KKT conditions are analogous conditions in the case of constraints.

The KKT conditions are the following:

1) Gradient of the Lagrangian = 0
2) Constraints: h[x] = 0 (m equality constraints) & g[x] ≤ 0 (k inequality constraints)
3) Complementary Slackness (for the s_i variables) μ.s == 0
4) Feasibility for the inequality constraints: $s_i^2 \geq 0$
5) Sign condition on the inequality multipliers: $\mu \geq 0$

One final requirement for KKT to work is that the gradient of f at a feasible point must be a linear combination of the gradients for the equality constraints and the gradients of the active constraints: this is often called regularity of a feasible point.

At a feasible point for the constraints, the active constraints are those components of g with g_i [x] = 0 (if the value of the constraining function is < 0, that constraint is said to be inactive).

The solution of a set of KKT equations proceeds by cases according to which inequality constraints are Active & Inactive.

Example: Chong Zak Example 20.2

Consider a circuit with a 20V battery and two resistances in series: R and 10 ohm.
We will investigate the maximization of the power in each of the resistors separately

■ Subproblem One: Max in resistor R

Maximize power absorbed by resistance R
Equivalently: minimized: - Power absorbed by R= - i^2 R = $-400\,R/(10+R)$^2 subject to -R ≤ 0.

The lagrangian is L[R,λ] =

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L[R_, μ_] := -400 R/(10 + R)^2 + μ (-R)
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D[L[R, μ], R]
D[L[R, μ], μ]
```

$$\frac{800\,R}{(10+R)^3}-\frac{400}{(10+R)^2}-\mu$$

```
-R
```

KKT also gives us the complementary slackness: μ.R = 0 and the sign condition for the inequality constraints: $\mu \geq 0$.
But, if $\mu > 0$, then R==0 which is gives no power absorbed by R.
If the constraint is inactive we must solve

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Solve[(D[L[R, μ], R] /. {μ → 0}) == 0, R]
```

```
{{R → 10}}
```

This gives the maximum

■ Subproblem Two: Maximization of power in 10 ohm resistor

Find the value of R such that maximal power is delivered to the 10 ohm resistor, i.e:
Min: - :\frac{4000}{(10+R)^2}: subject to -R ≤ 0

```
L[R_, μ_] := -4000/(10 + R)^2 + μ (-R)
```

Again, KKT gives us a complementary slackness condition: μ.R = 0 and the sign condition for the inequality constraints: $\mu \geq 0$.
But, if $\mu = 0$, we must solve

```
Solve[(D[L[R, μ], R] /. {μ → 0}) == 0, R]
```

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{}
```

This first case has no solutions. However, R = 0 is possible in the case $\mu > 0$

KKT conditions for a Linear Program: (Franklin: p. 197)

▪ Lagrangian

Min: c^T .x subject to the linear constraints:
A.x == b, x ≥ 0,

the associated Lagrangian is:

L[x,λ] = c^T .x + Transpose[λ] (A.x - b) + μ.(-x+$s^{[2]}$)

The KKT conditions yield:
c^T - Transpose[λ].A + Transpose[μ] == 0
A.x - b ==0
x ≥ 0,
2 Transpose[μ].s == 0
μ≥ 0

Eliminating μ, we obtain the following: c^T - Transpose[λ].A ≤ 0, (c^T - Transpose[λ].A).x= - Transpose[μ]).($s^{[2]}$) = 0 yielding
c^T .x = Transpose[λ].A.x= Transpose[λ].b which is the **equilibrium condition** in mild disquise!

Example: Pedregal Example 3.10, p. 82

A certain electrical networks is designed to supply power x_i thru 3 channels.

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p[x_, y_, z_] = z + (x^2 + y^2 + z^2 / 10) / 2;
```

Contraint: h[x,y,z]=x+y+z== 5, x,y,z ≥0
Note that this gives a "slanted" triangles with vertices on the axes 5 units from the origin.

```
h[x_, y_, z_] = x + y + z
```

The KKT conditions give:
1) ∇f + λ ∇h + μ ∇g = {x,y,1+z/10} + λ {1,1,1} + {μ_1 ,μ_2 ,μ_3 } =={0,0,0}
2) Constraint: h==5
3) $\mu_1\, x = 0$, $\mu_2\, y = 0$, $\mu_3\, z = 0$,
Checking for active constraints will divide in the consideration of 8 cases for the three inequality constraints.

KKT: Necessary Conditions for Quad. Program

■ Review: Quadratic Programs

The general quadratic program proposes to minimize an objective function of the form: Min: x.Q.x/2 + p.x subject to the linear constraints:

A.x == b, x ≥ 0

Note that we may assume that Q is a symmetric matrix (and that Q is the Hessian of the objective function.) This is a non-linear program problem, for the objective function is a quadratic function (if Q is non-zero.)

■ Wolfe's Reduction to LP

Given the Quadratic Program as above, the associated Linear Program is:

A.x == b
Q.x + Transpose[A].u - v == -p
x ≥ 0, v≥ 0, x.v == 0

Note that the vector u is not necessarily positive. The only additional feature is the **exclusion rule**: x.v == 0, this requirement states that the i^{th} components of x and v can not both be positive simultaneously.

Finally, note that the existence of a solution of the associated LP is only necessary for the existence of a solution to the QP and is sufficient in case Q is positive semi-definite.

In practice we solve the LP program:

A.x==b
Q.x + Transpose[A].u -v + D.z == -p
x ≥ 0l, u free, v ≥ 0, z ≥ 0
Min: {1, . . .,1}.z

Initial Feasible Solution: x, u=0, v=0, z where x is a basic feasible solution of A.x ==b, x ≥ 0, D is a diagonal matrix with entries ± 1 to correct the signs of z and z is a chosen such that Q.x + D.z == - p, z ≥ 0.

■ KKT conditions for a Quadratic Program

■ Lagrangian

Given: x.Q.x/2 + p.x subject to the linear constraints:

A.x == b, x ≥ 0,

the associated Lagrangian is:

L[x,λ] = x.Q.x/2 + p.x + λ (A.x - b) + μ.x

The KKT conditions yield:

Q.x + p - Transpose[A].λ + μ == 0

Transpose[λ].(A.x - b) ==0

Transpose[μ].x == 0

A.x == b, x ≥ 0,

Exercises

I. Write down the KKT conditions for the problem:
Min f[x] = $- x_1^3 + x_2^2 - 2 x_1 x_3^2$ subject to the constraints:
$2 x_1 + x_2^2 + x_3 - 5 == 0$
$5 x_1^2 - x_2^2 - x_3 \geq 2$
$x_i \geq 0$ for i = 1,2,3.

Verify that the KKT conditions are satisfied at (1,0,3).

II. Write down the KKT conditions for the problem:
Min f[x] = $x_1^2 + x_2^2 + x_3^2$ subject to the constraints:
$-x_1 + x_2 - x_3 \geq -10$
$x_1 + x_2 + 4 x_3 \geq 20$
Find all the solutions.

III. Here's an exercise from Grieg p. 142):
Minimize: f[x] = (x_1 -1)^2 + (x_2 -2)^2+ (x_3 -3)^2+ (x_4 -4)^2

subject to: $- x_1 - x_2 - x_3 - x_4 + 5 \geq 0$
$- 3 x_1 - 3 x_2 - 2 x_3 - x_4 + 10 \geq 0$
x ≥ 0

Verify the KT conditions and then expand f to obtains a Quadratic Program and compare the KT conditions with Wolfe's.

Solve the problem using Wolfe's method .

Bonus: Solve the problem using a projected gradient methods (mimic affine scaling) and a projected Newton's method. Compare the results.