

1. Bollinger 2.2

$$0.02 \frac{d^2\theta(t)}{dt^2} + \frac{d\theta(t)}{dt} = 10v(t)$$

$$x(t) = 5\theta(t)$$

$$0.02 \frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} = 50v(t)$$

(a)

$$\frac{d^2x(t)}{dt^2} = \frac{\frac{x_n - x_{n-1}}{T} - \frac{x_{n-1} - x_{n-2}}{T}}{T} = \frac{1}{T^2}(x_n - 2x_{n-1} + x_{n-2})$$

$$\frac{dx(t)}{dt} = \frac{1}{T}(x_n - x_{n-1})$$

$$0.02 \frac{1}{T^2}(x_n - 2x_{n-1} + x_{n-2}) + \frac{1}{T}(x_n - x_{n-1}) = 50v_n$$

$$200(x_n - 2x_{n-1} + x_{n-2}) + 100(x_n - x_{n-1}) = 50v_n$$

$$x_n = 1.667x_{n-1} - 0.667x_{n-2} + 0.167v_n$$

or using Table 2.1 in the book

$$x_n = 1.6065x_{n-1} - 0.6065x_{n-2} + 0.1065v_{n-1} + 0.09025v_{n-1}$$

(b) Laplace T.F.

$$\frac{X(s)}{V(s)} = \frac{50}{0.02s^2 + s}$$

i) Forward difference

$$\frac{X(z)}{V(z)} = \frac{50}{0.02 \left(\frac{1-z^{-1}}{0.01z^{-1}} \right)^2 + \frac{1-z^{-1}}{0.01z^{-1}}} = \frac{50(0.01)^2 z^{-2}}{0.02 - 0.03z^{-1} + 0.01z^{-2}}$$

$$\therefore x_k = 1.5x_{k-1} - 0.5x_{k-2} + 0.25v_{k-2}$$

ii) Backward difference

$$\frac{X(z)}{V(z)} = \frac{50}{0.02 \left(\frac{1-z^{-1}}{0.01} \right)^2 + \frac{1-z^{-1}}{0.01}} = \frac{50(0.01)^2}{0.03 - 0.05z^{-1} + 0.02z^{-2}}$$

$$\therefore x_k = 1.667x_{k-1} - 0.667x_{k-2} + 0.167v_k$$

iii) Bilinear

$$\frac{X(z)}{V(z)} = \frac{50}{0.02 \left(200 \frac{1-z^{-1}}{1+z^{-1}} \right)^2 + 200 \frac{1-z^{-1}}{1+z^{-1}}} = \frac{50(1+2z^{-1}+z^{-2})^2}{1000 - 1600z^{-1} + 600z^{-2}}$$

$$\therefore x_k = 1.6x_{k-1} - 0.6x_{k-2} + 0.05v_k + 0.1v_{k-1} + 0.05v_{k-2}$$

2.

(a) 2.29(a)

$$y_n - 1.6y_{n-1} + 1.13y_{n-2} = 0$$

$$(1 - 1.6z^{-1} + 1.13z^{-2})y_n = 0$$

C.E.

$$z^2 - 1.6z + 1.13 = 0$$

$$z = 0.8 \pm \sqrt{0.8^2 - 1.13}$$

$$= 0.8 \pm j0.7$$

$$|z| = \sqrt{0.8^2 + 0.7^2} = \sqrt{1.13} = 1.06 > 1$$

\therefore unstable

(b) 2.29(c)

$$y_{n+3} - 2y_{n+2} + 1.5y_{n+1} - 0.5y_n = 0$$

$$(1 - 2z^{-1} + 1.5z^{-2} - 0.5z^{-3})y_n = 0$$

C.E.

$$z^3 - 2z^2 + 1.5z - 0.5 = 0$$

$$z = 1, 0.5 \pm j0.5$$

\therefore marginally stable

(c) 2.29(d)

$$y_n = y_{n-1} + 1.05x_n - x_{n-1} = 0$$

C.E.

$$z - 1 = 0$$

\therefore marginally stable

3. Bollinger 3.10

a. Using long division

$$c_n = \frac{9}{(1-B)(1-0.1B)} S_n$$
$$c_n = (9 + 9.9B + 9.99B^2 + 9.999B^3 + \dots) S_n$$

b. Using the partial-fraction expansion

$$c_n = \frac{9}{(1-B)(1-0.1B)} S_n$$
$$c_n = \left[\frac{10}{(1-B)} + \frac{-1}{(1-0.1B)} \right] S_n$$
$$= 10(1 + B + B^2 + B^3 + \dots) S_n - (1 + 0.1B + 0.01B^2 + 0.001B^3 + \dots) S_n$$
$$= (9 + 9.9B + 9.99B^2 + 9.999B^3 + \dots) S_n$$

c. Using the final value theorem

$$c_\infty = \lim_{B \rightarrow 1} (1-B) \frac{9}{(1-B)(1-0.1B)} = 10$$

4.

(a)

$$G_c(s) = K_p + \frac{K_I}{s} = K \frac{(s+a)}{s}$$

$$G(s) = G_c(s) \cdot G_p(s) = \frac{K(s+a)}{s} \frac{1}{(0.5s+1)} = \frac{2K(s+a)}{s(s+2)}$$

C.E.:

$$s^2 + 2(K+1)s + 2Ka$$

$$\% \text{ overshoot} = 100 \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 5$$

$$\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = -2.996$$

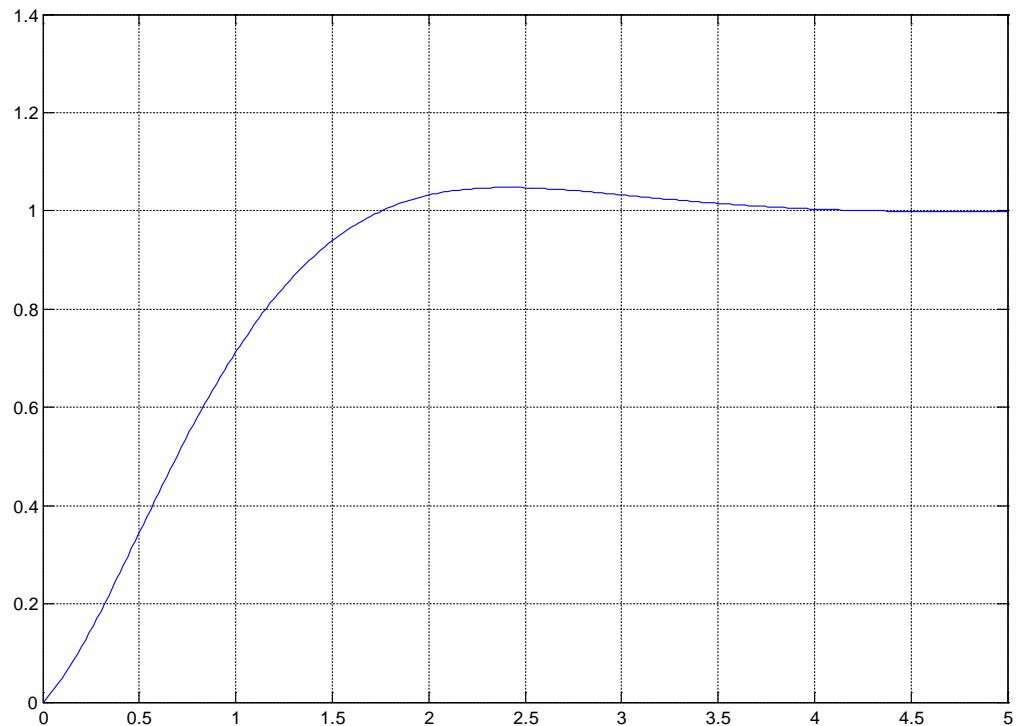
$$2.0984\zeta^2 = 1 \Rightarrow \zeta \geq 0.69 \text{ we choose } \zeta = 0.7$$

$$\frac{4}{\zeta\omega_n} \leq 6 \Rightarrow \zeta\omega_n \geq 0.667 \text{ we choose } \zeta\omega_n = 1.2 \Rightarrow \omega_n = 1.714$$

$$\therefore s^2 + 2(K+1)s + 2Ka = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2.4s + 2.94$$

$$\therefore K = 0.2 \quad a = 7.35$$

(b)



(c) T=0.1 seconds for this problem.

$$G_p(z) = (1 - z^{-1}) Z \left[\frac{2}{s(s+2)} \right] \quad (e^{-aT} = e^{-2 \times 0.1} = 0.8187)$$

$$= (1 - z^{-1}) \frac{z(0.1813)}{(z-1)(z-0.8187)} = \frac{0.1813}{z-0.8187} = \frac{0.1813z^{-1}}{1-0.8187z^{-1}}$$

$$D(z) = K_p + \frac{K_I T z}{z-1} = 2 + \frac{0.147 z}{z-1} = \frac{2.147 - 2z^{-1}}{1-z^{-1}} = \frac{2.147(1-0.9315z^{-1})}{1-z^{-1}}$$

$$G(z) = G_p(z) D(z) = \frac{0.3893(1-0.9315z^{-1})z^{-1}}{(1-z^{-1})(1-0.8187z^{-1})}$$

$$T(z) = \frac{G(z)}{1+G(z)} = \frac{0.3893z^{-1} - 0.3626z^{-2}}{(1-1.8187z^{-1} + 0.8187z^{-2}) + 0.3893z^{-1} - 0.3626z^{-2}} = \frac{0.3893z^{-1} - 0.3626z^{-2}}{1-1.4294z^{-1} + 0.4561z^{-2}}$$

(d)

$$\therefore y(k) = 1.4294y(k-1) - 0.4561y(k-2) + 0.3893r(k-1) - 0.3626r(k-2)$$

$$y(1) = 1.4294y(0) - 0.4561y(-1) + 0.3893r(0) - 0.3626r(-1) = \boxed{0.3893}$$

$$\begin{aligned} y(2) &= 1.4294y(1) - 0.4561y(0) + 0.3893r(1) - 0.3626r(0) \\ &= 1.4294 \times 0.3893 - 0 + 0.3893 - 0.3626 = \boxed{0.5832} \end{aligned}$$

$$\begin{aligned} y(3) &= 1.4294y(2) - 0.4561y(1) + 0.3893r(2) - 0.3626r(1) \\ &= 1.4294 \times 0.5832 - 0.4561 \times 0.3893 + 0.3893 - 0.3626 = \boxed{0.6828} \end{aligned}$$

