Homework #4

1. (a)

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+2y(t)=u(t) \qquad \text{T=0.1 sec}$$

$$\frac{d^2y(t)}{dt^2}=\frac{1}{T^2}\left[y(k)\text{-}2y(k\text{-}1)+y(k\text{-}2)\right]$$

$$\frac{dy(t)}{dt}=\frac{1}{T}\left[y(k)-y(k\text{-}1)\right]$$

$$\frac{1}{T^2}\left[y(k)-2y(k\text{-}1)+y(k\text{-}2)\right]+\frac{2}{T}\left[y(k)\text{-}y(k\text{-}1)\right]+2y(k)=u(k\text{-}1)$$

$$\left[\frac{1}{T^2}+\frac{2}{T}+2\right]y(k)=\left(\frac{2}{T^2}+\frac{2}{T}\right)y(k\text{-}1)\text{-}\frac{1}{T^2}y(k\text{-}2)+u(k\text{-}1)$$

$$122y(k)=220y(k\text{-}1)\text{-}100y(k\text{-}2)+u(k\text{-}1)$$

$$y(k)=1.803y(k\text{-}1)\text{-}0.820y(k\text{-}2)+0.82\text{x}10^{-2}u(k\text{-}1)$$

(b)

Use definition 6

$$\frac{1}{2}\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+y(t)=\frac{1}{2}u(t)$$

$$\omega_n^2=2 \qquad \omega_n=1.414$$

$$\frac{2\zeta}{\omega_n}=1 \qquad \therefore \zeta=0.707$$

$$\zeta\omega_n=1$$

$$\omega_d=\omega_n\sqrt{1-\zeta}^2$$

$$=1.414\sqrt{1-0.707^2}=1$$

$$a_1=2e^{-0.707\times 1.414\times 0.1}\cos(0.1)$$

$$=1.8$$

$$a_2=-e^{-2\times 0.707\times 1.414\times 0.1}=-0.819$$

$$e^{-\zeta\omega_n T}=e^{-0.707\times 1.414\times 0.1}=0.905$$

$$b_1=1-\frac{1}{1}\times 0.905\times\sin(0.1)-0.905\cos(0.1)$$

$$=0.0092$$

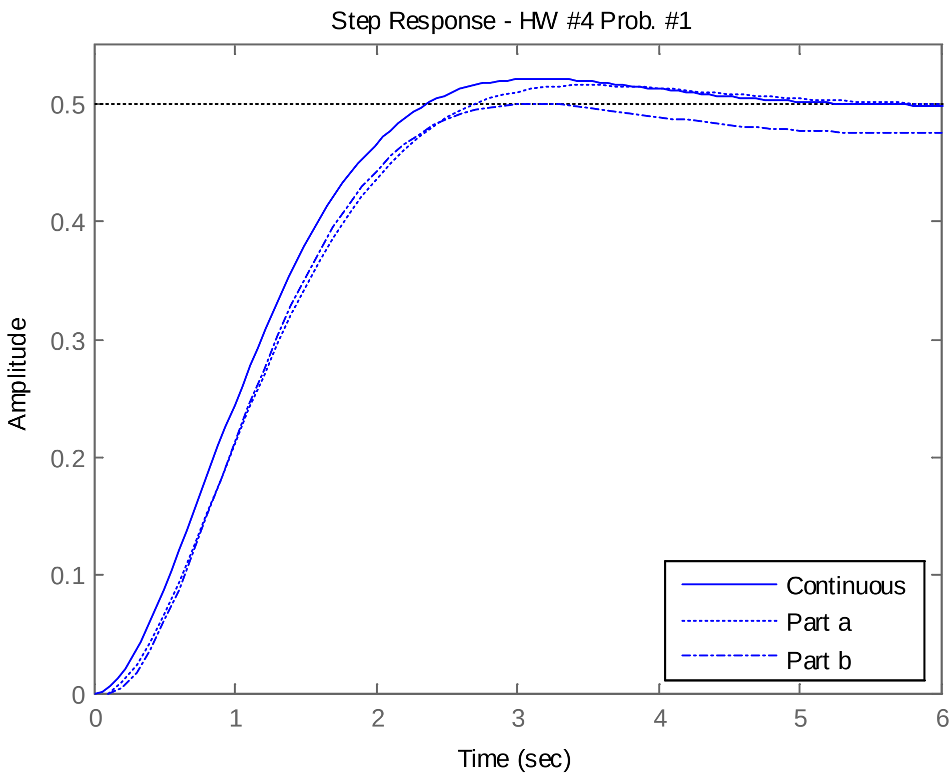
$$b_2=0.905\left(0.905+\frac{1}{1}\sin(0.1)-\cos(0.1)\right)$$

$$=0.0089$$

$$y(k)-1.8y(k-1)+0.819y(k-2)=\frac{1}{2}\left[0.0092u(k-1)+0.0089u(k-2)\right]$$

$$y(k)=1.8y(k-1)-0.819y(k-2)+0.00464(k-1)+0.00445u(k-2)$$

Comparison of responses (Simulations)



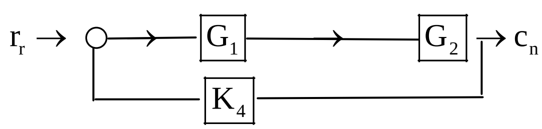
2.

2.19 (b)

$$\frac{c_n}{r_n} = \frac{\frac{k_c}{1\text{-}B}\bullet K_p}{1+\frac{k_c}{1\text{-}B}\bullet K_p\bullet K_2}\bullet K_1$$

$$= \frac{K_1 k_c K_p}{\left(1+K_2 k_c K_p\right)\text{-}B}$$

2.19 (d)



$$G_1(B)\frac{LTB}{\left(1+LTBK_2\right)-B} = \frac{LTB}{1+\left(LTK_2\text{-}1\right)B}$$

$$G_2(B) = K_1\frac{TB}{J\left(1\text{-}B\right)+K_3TB}$$

$$= K_1\frac{TB}{J+\left(K_3T\text{-}J\right)B}$$

$$T(B) = \frac{c_n}{r_r} = \frac{\left(LTB\right)\bullet\left(K_1TB\right)}{\left[1+\left(LTK_2\text{-}1\right)B\right]\left[J+\left(K_3T\text{-}J\right)B\right]+\left(LTB\right)\left(K_1TB\right)K_4}$$

$$= \frac{K_1LT^2B^2}{J+\left(JLTK_2\text{-}J+K_3T\text{-}J\right)B+\left[\left(LTK_2\text{-}1\right)\left(K_3T\text{-}J\right)+K_1K_4LT^2\right]B^2}$$

$$= \frac{K_1LT^2B^2}{J+\left(JLTK_2+K_3T\text{-}2J\right)B+\left[\left(LTK_2\text{-}1\right)\left(K_3T\text{-}J\right)+K_1K_4LT^2\right]B^2}$$

3. Bollinger 3.16

a.

$$C_n = \frac{4(1\text{-}\delta)(1\text{-}0.5B)B}{(1\text{-}\delta B)(1\text{-}B) + 2(1\text{-}\delta)B(1\text{-}0.5B)} S_n$$

where $\delta = e^{-1} = 0.368$

$$\frac{m_n}{\ell_n} = \frac{1-0.5B}{1-B}$$

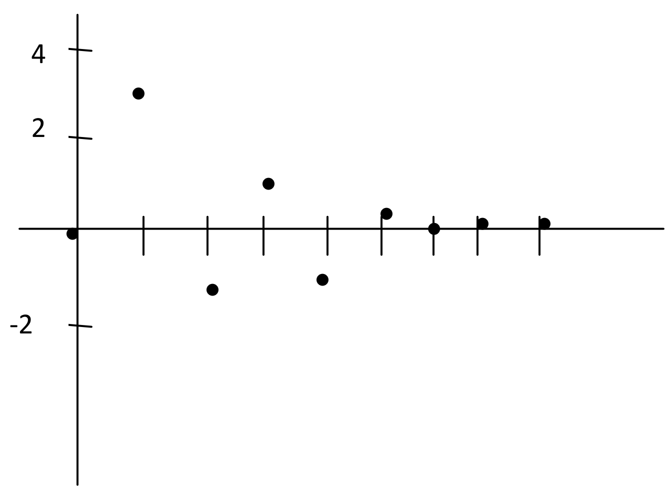
$$r_n = 2S_n$$

$$\therefore \frac{c_n}{r_n} = \frac{2(1-\delta)(1-0.5B)B}{(1-\delta B)(1-B) + 2(1-\delta)B(1\text{-}0.5B)}$$

$$= \frac{1.264(1-0.5B)B}{(1-0.368B)(1-B) + 2(0.632)B(1\text{-}0.5B)}$$

$$= \frac{1.264B(1-0.5B)}{1-0.104B-0.264B^2} = \frac{1.264B-0.632B^2}{1-0.104B-0.264B^2}$$

b.



$$c_n = \frac{2.518B(1-0.5B)}{1-0.104B-0.264B^2} = 2.518B - B^2 + 0.561B^3 - 0.206B^4 + \cdots$$

c. from part (a)

$$\frac{c_n}{r_n}=\frac{\dfrac{2(1-\delta)(1-0.5\mathrm{B})\mathrm{B}}{(1-\delta\mathrm{B})(1-\mathrm{B})}}{1+\dfrac{2(1-\delta)\mathrm{B}(1-0.5\mathrm{B})}{(1-\delta\mathrm{B})(1-\mathrm{B})}}$$

$$\therefore \frac{\mathrm{c_n}}{\mathrm{e_n}}=\frac{2(1-\delta)(1-0.5\mathrm{B})}{(1-\delta\mathrm{B})(1-\mathrm{B})}$$

$$\mathrm{G_{p(B)}}=\frac{\mathrm{c_n}}{\mathrm{m_n}}=\frac{\mathrm{c_n}}{\mathrm{e_n}}\bullet\frac{\mathrm{e_n}}{\mathrm{m_n}}=\frac{2(1-\delta)(1-0.5\mathrm{B})\mathrm{B}}{(1-\delta\mathrm{B})(1-\mathrm{B})}\bullet\frac{(1-\mathrm{B})}{1-0.5\mathrm{B}}$$

$$=\frac{2(1-\delta)\mathrm{B}}{1-\delta\mathrm{B}}=\frac{2(1-\delta)z^{-1}}{1-\delta z^{-1}}$$

Corresponding continuous model.

$$\tau\frac{\mathrm{dc(t)}}{\mathrm{dt}}+\mathrm{c(t)}=2\mathrm{m(t)}$$

$$\tau=1$$

$$\text{where } \delta=\mathrm{e}^{-T/t}=\mathrm{e}^{-1/1}=\mathrm{e}^{-1}$$

Controller.

$$\mathrm{m_n}-\mathrm{m_{n-1}}=\mathrm{e_n}-0.5\mathrm{e_{n-1}}$$

$$\text{or}\quad\frac{1-0.5\mathrm{B}}{1-\mathrm{B}}$$

PI controller:

$$\mathrm{k_p}+\frac{\mathrm{k_I T}}{1-\mathrm{B}}=\frac{(\mathrm{k_p}+\mathrm{k_I}T)-\mathrm{k_p B}}{1-\mathrm{B}}$$

$$\mathrm{T}=1\qquad \mathrm{k_p}+\mathrm{k_I}=1$$

$$\mathrm{k_p}=0.5$$

$$\therefore \mathrm{k_I}=0.5$$

4.

a.

$$D(z) = K_p + \frac{K_I T}{1 - z^{-1}} = \frac{\left(K_p + K_I T\right) - K_p z^{-1}}{1 - z^{-1}}$$

$$\frac{M(z)}{\mathrm{E}(z)} = \frac{\left(K_p + K_I T\right) - K_p z^{-1}}{1 - z^{-1}}$$

$$m(k) = m(k-1) + \left(K_p + K_I T\right)e(k) - K_p e(k-1)$$

b.

$$G_p(z) = \left(1 - z^{-1}\right) Z\left[\frac{2}{s(s+2)}\right] \qquad \left(e^{-aT} = e^{-100\times 0.02} = 0.135\right)$$

$$= \left(1 - z^{-1}\right)\frac{z(0.865)}{(z-1)(z-0.135)} = \frac{0.865}{z - 0.135} = \frac{0.865z^{-1}}{1 - 0.135z^{-1}}$$

$$D(z) = \frac{(1+0.04) - z^{-1}}{1 - z^{-1}} = \frac{1.04 - z^{-1}}{1 - z^{-1}}$$

$$G(z) = G_D(z) D(z) = \frac{0.865\left(1.04 - z^{-1}\right)z^{-1}}{\left(1 - z^{-1}\right)\left(1 - 0.135z^{-1}\right)}$$

$$T(z) = \frac{G(z)}{1 + G(z)} = \frac{0.9z^{-1} - 0.865z^{-2}}{\left(1.135z^{-1} + 0.135z^{-2}\right) + 0.9z^{-1} - 0865z^{-2}} = \frac{0.9z^{-1} - 0.865z^{-2}}{1 - 0.235z^{-1} - 0.73z^{-2}}$$

$$\therefore \mathrm{y}(\mathrm{k}) = 0.235\mathrm{y}(\mathrm{k}\text{-}1) + 0.73\mathrm{y}(\mathrm{k}\text{-}2) + 0.9\mathrm{r}(\mathrm{k}\text{-}1) - 0.865\mathrm{r}(\mathrm{k}\text{-}2)$$

$$\mathrm{y}(1) = 0.235\mathrm{y}(0) + 0.73\mathrm{y}(-1) + 0.9\mathrm{r}(0) - 0.865\mathrm{r}(-1) = \boxed{0.9}$$

$$\mathrm{y}(2) = 0.235\mathrm{y}(1) + 0.73\mathrm{y}(0) + 0.9\mathrm{r}(1) - 0.865\mathrm{r}(0)$$

$$= 0.235\mathrm{x}0.9 + 0 + 0.9 - 0.865 = \boxed{0.2465}$$

$$\mathrm{y}(3) = 0.235\mathrm{y}(2) + 0.73\mathrm{y}(1) + 0.9\mathrm{r}(2) - 0.865\mathrm{r}(1)$$

$$= 0.235\mathrm{x}0.2465 + 0.73x0.9 + 0.9 - 0.865 = \boxed{0.75}$$

c

$$
\begin{aligned}
y_{ss} &= \lim_{z\to 1}\left(1-z^{-1}\right)\frac{0.9z^{-1}-0.865z^{-2}}{1-0.235z^{-1}-0.73z^{-2}}\bullet\frac{1}{1-z^{-1}} \\
&= \frac{0.9-0.865}{1-0.235-0.73} = 1
\end{aligned}
$$