Zero Phase Error Tracking Algorithm for Digital Control

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A digital feedforward control algorithm for tracking desired time varying signals is presented. The feedforward controller cancels all the closed-loop poles and cancellable closed-loop zeros. For uncancellable zeros, which include zeros outside the unit circle, the feedforward controller cancels the phase shift induced by them. The phase cancellation assures that the frequency response between the desired output and actual output exhibits zero phase shift for all the frequencies. The algorithm is particularly suited to the general motion control problems including robotic arms and positioning tables. A typical motion control problem is used to show the effectiveness of the proposed feedforward controller.

I Introduction

In servo control, two fundamental problems are the point to point control problem and the tracking (path following) control problem. The point to point control problem is concerned with moving the control object from one point to another. Therefore, the transient path to the final point is often not important in point to point control. In tracking control, the control object must be moved along the desired trajectory; e.g., automated arc welding and painting using a robot arm (Dorf, 1983).

This paper is concerned with the tracking problem for single-input, single-output systems. Although generation of the desired trajectory is an important problem in tracking (Brady et al., 1983), the desired trajectory is assumed to be given in this paper. Given the desired trajectory, the optimal linear tracking approach (Anderson and Moore, 1971), preview control approach (Tomizuka and Whitney, 1975; Tomizuka et al., 1984), and independent tracking and regulation approach (Landau and Lozano, 1981) are all known to be effective for the purpose of tracking. See also Snyder (1985) for other approaches.

In the optimal linear tracking and preview control approaches, the tracking error is traded off against the controlling effort. Therefore, tracking with zero error is not attempted and the output normally follows a smoothed desired trajectory, which may or may not be desirable depending upon the application; for example, if the desired trajectory is obtained by real time measurement that is contaminated by noise, the smoothing function may be appropriate. In the independent tracking and regulation approach, the reference input to the feedback loop is computed so that the plant output follows the desired trajectory with zero tracking error. In this design method, the feedback loop is designed for regulation purposes, and the feedforward controller is placed for tracking purposes. The method is based on pole/zero cancellation. When the controlled plant possesses uncancellable zeros, e.g., unstable zeros, the plant output cannot follow arbitrary desired trajectories. This problem arises in any design approach for tracking controllers based on pole/zero cancellation.

In this paper, it is proposed to design the feedforward controller by combinations of pole/zero cancellation and phase cancellation. The latter is applied to uncancellable zeros. The phase cancellation assures that the frequency response between the desired output and actual output exhibits zero phase shift for all the frequencies.

In the next section, a perfect tracking controller (PTC), which computes the controlling input for tracking with zero error, is considered. Limitations of the perfect tracking controller are discussed, and the zero phase error tracking control (ZPETC) is proposed to overcome the limitations. In Section 3, PTC and ZPETC are compared for a typical motion control example. Conclusions are given in Section 4.

II Tracking Controller Design Based on Pole/Zero Cancellation and Phase Cancellation

To achieve superior tracking, the feedforward controller is required in addition to the feedback controller. It should be noted that the major function of feedback controllers is regulation against disturbance inputs.

The feedback controller and feedforward controller can be designed simultaneously or separately. In the optimal linear tracking approach, preview control approach and independent tracking and regulation approach, the feedback controller and feedforward controller are designed simultaneously. In this paper, we assume that the feedback controller already exists.

2.1 Perfect Tracking Controller. Let us suppose that the closed-loop (discrete time) transfer function, which includes the controlled plant and feedback controller, is expressed as

$$G_{\text{closed}}(z^{-1})=\frac{z^{-d}B_c(z^{-1})}{A_c(z^{-1})} \qquad (1)$$

where z^{-d} represents a d-step delay normally caused by the delay in the plant and

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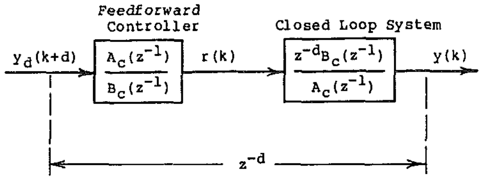

Fig. 1 Feedforward controller for perfect tracking

$$B_c(z^{-1}) = b_{c0} + b_{c1}z^{-1} + \ldots + b_{cm}z^{-m},\ b_{c0} \neq 0$$
$$A_c(z^{-1}) = 1 + a_{c1}z^{-1} + \ldots + a_{cn}z^{-n}$$

The input and output of the closed-loop transfer function are the reference input and actual output, respectively, i.e.,

$$y(k) = G_{\text{closed}}(z^{-1})r(k) \tag{2}$$

We consider a feedforward tracking controller which provides the reference input in the form,

$$r(k) = \frac{A_c(z^{-1})}{B_c(z^{-1})} y_d(k+d) \tag{3}$$

where $y_d(k+d)$ is the d-step ahead desired output. The feedforward controller utilizes the future desired output in order to compensate for the d-step delay in the closed-loop transfer function.

This tracking controller provides $y(k) = y_d(k)$ (perfect tracking) as long as all the initial conditions are zero and is called the perfect tracking controller (PTC). PTC is depicted in Fig. 1. PTC cancels all the closed loop poles and zeros so that the overall transfer function from $y_d(k)$ to $y(k)$ is unity. Since the closed-loop zeros, i.e., the characteristic roots of $B_c(z^{-1})$, are the poles of PTC, they must be inside the unit circle in the z-plane for PTC to be implementable. If any closed-loop zero is outside or on the unit circle, the output of PTC, $r(k)$, explodes or oscillates. Even if all the zeros are inside the unit circle, some zeros, e.g., those on the negative real axis and close to -1, may make $r(k)$ highly oscillatory. As is clear from these observations, locations of closed-loop zeros are of fundamental importance in the design of a feedforward controller based on pole/zero cancellation.

2.2 Uncancellable Closed Loop Zeros. We factorize $B_c(z^{-1})$ into two parts and write

$$B_c(z^{-1}) = B_c^a(z^{-1})B_c^u(z^{-1}) \tag{4}$$

where B_c^a includes zeros of the closed-loop system which are acceptable as poles of the compensator (3), and $B_c^u(z^{-1})$ includes those unacceptable as compensator poles. Notice that all zeros of the closed-loop system (1) outside of the unit circle are included in $B_c^u(z^{-1})$.

The tracking controller which cancels all the closed-loop poles and zeros from $B_c^a(z^{-1})$ is

$$r(k) = \frac{A_c(z^{-1})}{B_c^a(z^{-1})B_c^u(1)} y_d^*(k+d) \tag{5}$$

where $y_d^*(k)$ is related to the desired trajectory $y_d(k)$ as will be discussed later and $B_c^u(1)$ in the denominator is to scale the steady state gain of the tracking controller to the reciprocal of that of the closed-loop transfer function given by equation (1).

For zero initial state, equations (2), (4) and (5) yield

$$y(k) = z^{-d}[B_c^u(z^{-1})/B_c^u(1)]y_d^*(k+d)$$
$$= [B_c^u(z^{-1})/B_c^u(1)]y_d^*(k) \tag{6}$$

Equation (6) implies that if the desired trajectory can be expressed as

$$y_d(k) = [B_c^u(z^{-1})/B_c^u(1)]y_d^*(k) \tag{7}$$

*Strictly speaking, equation (2) should be written as $Y(z) = G_{\text{closed}}(z^{-1})R(z)$ where $Y(z)$ and $R(z)$ are the z-transforms of $y(k)$ and $r(k)$. In this paper, all the signals are expressed in the time domain.

tracking is perfect. In other words, equation (7) imposes a condition on the desired trajectory for perfect tracking. In practice, however, $y_d(k)$ may not be necessarily expressed in the form of equation (7). Given arbitrary $y_d(k)$, equation (7) cannot be utilized backward for determining $y_d^*(k)$: i.e., any attempt to determine $y_d^*(k)$ from (7) is equivalent to use of the feedforward compensator (3). The designer faces the same problem in other schemes based on pole/zero cancellation such as the independent regulation and tracking approach (Landau et al., 1983).

If equation (7) cannot be utilized for generating $y_d^*(k)$ from $y_d(k)$, a reasonable compromise would be to set $y_d^*(k) = y_d(k)$ and to use equation (5) as a tracking controller at the expense of delay between the desired trajectory and actual trajectory. The delay is as given in equation (6). A better method for generating $y_d^*(k)$ from $y_d(k)$ is developed below.

2.3 Zero-Phase Error Tracking Controller. Tracking error caused by the tracking controller (5) can be studied in the frequency domain. Consider a sinusoidal $y_d^*(k)$ given by

$$y_d^*(k) = \text{a}\sin(\omega Tk),\quad 0 \leq \omega T \leq \pi$$

where T is the sampling period for digital control. The sinusoidal steady state plant output is written as

$$y(k) = \text{a}\, M \sin(\omega Tk + \phi)$$

where by noting equation (6) the gain M and phase shift ϕ are determined from the frequency transfer function

$$B_c^u(e^{-j\omega T})/B_c^u(1) \tag{8}$$

i.e., M and ϕ are, respectively, given by

$$M = |B_c^u(e^{-j\omega T})/B_c^u(1)| \text{ and } \phi = \measuredangle B_c^u(e^{-j\omega T})/B_c^u(1) \tag{9}$$

Writing

$$B_c^u(z^{-1}) = b_{c0}^u + b_{c1}^u z^{-1} + \ldots + b_{cs}^u z^{-s} \tag{10}$$

the frequency transfer function (8) becomes

$$B_c^u(e^{-j\omega T})/B_c^u(1) = \text{Re}(\omega) - j\text{Im}(\omega) \tag{11}$$

where the real and imaginary parts are, respectively, given by

$$\text{Re}(\omega) = [b_{c0}^u + b_{c1}^u\cos(\omega T) + \ldots$$
$$+ b_{cs}^u\cos(s\omega T)]/(b_{c0}^u + b_{c1}^u + \ldots + b_{cs}^u)$$
$$\text{Im}(\omega) = [b_{c1}^u\sin(\omega T) + \ldots$$
$$+ b_{cs}^u\sin(s\omega T)]/(b_{c0}^u + b_{c1}^u + \ldots + b_{cs}^u) \tag{12}$$

By using $\text{Re}(\omega)$ and $\text{Im}(\omega)$ given by equation (12), the frequency response (9) can be expressed as

$$M = \sqrt{\text{Re}^2(\omega) + \text{Im}^2(\omega)} \text{ and } \phi = \tan^{-1}[\text{Im}(\omega)/\text{Re}(\omega)] \tag{13}$$

Notice that from the frequency response, both the gain and phase errors are evident. In many practical situations, the desired trajectory is smooth or is composed of low frequency components. Equations (12) and (13) suggest that both the gain and phase errors should be small in the low frequency region (i.e., $\cos(l\omega T) \simeq 1$ and $\sin(l\omega T) \simeq 0$). However, from a tracking viewpoint, even a small amount of phase error may severely degrade the tracking performance since the phase shift of ϕ implies a time delay of ϕ/ω.

The feedforward compensator for zero phase error tracking is obtained by utilizing the following property: for

$$[B_c^u(z^{-1})/B_c^u(1)][B_c^u(z)/B_c^u(1)] \tag{14}$$

where

$$B_c^u(z) = b_{c0}^u + b_{c1}^u z + \ldots + b_{cs}^u z^s \tag{15}$$

the frequency transfer function is given by

$$[B_c^u(e^{-j\omega T})/B_c^u(1)][B_c^u(e^{j\omega T})/B_c^u(1)]$$
$$= [\text{Re}(\omega) - j\text{Im}(\omega)][\text{Re}(\omega) + j\text{Im}(\omega)]$$
$$= \text{Re}(\omega)^2 + \text{Im}(\omega)^2 \tag{16}$$

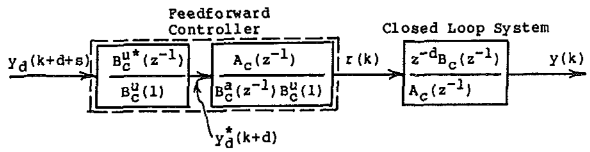

Fig. 2 Feedforward controller for zero phase error tracking

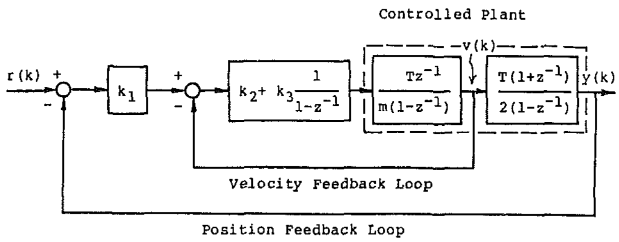

Fig. 3 Positioning system with proportional plus integral velocity controller and proportional position controller

where Re and Im are given by equation (12). Equation (16) does not have an imaginary part, which implies that the phase shift introduced by the transfer function (14) is zero for all frequencies. In the low frequency region, the frequency response gain is close to 1. Based on equations (6) and (14), we generate y_d^* from $y_d(k)$ by

$$y_d^*(k) = [B_c^u(z)/B_c^u(1)]y_d(k) \tag{17}$$

or

$$y_d^*(k) = [b_{c0}^u y_d(k) + b_{c1}^u y_d(k+1) + \ldots + b_{cs}^u y_d(k+s)]/B_c^u(1) \tag{18}$$

Then, from equations (5) and (18), the feedforward tracking controller is

$$r(k) = \frac{A_c(z^{-1})B_c^{u*}(z^{-1})}{B_c^a(z^{-1})[B_c^u(1)]^2} y_d(k+d+s) \tag{19}$$

where

$$B_c^{u*}(z^{-1}) = b_{cs}^u + b_{c(s-1)}^u z^{-1} + \ldots + b_{c0}^u z^{-s} \tag{20}$$

By using this feedforward controller, the overall transfer function from $y_d(k)$ to $y(k)$ becomes as given by equation (14): i.e., the phase shift between $y_d(k)$ and $y(k)$ is zero for all frequencies. Equation (19) is called the zero phase error tracking controller (ZPETC). Notice that the input to ZPETC is $y_d(k+d+s)$, the $(d+s)$-step ahead desired output.

If $y_d(k)$ is a sampled sinusoidal signal and the zero gain error is required also, $r(k)$ given by equation (19) can be scaled by $1/\{\mathrm{Re}(\omega)^2 + \mathrm{Im}(\omega)^2\}$ where $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$ are given by equation (12). ZPETC is depicted in Fig. 2.

When the zero phase error tracking controller is used, the overall transfer function from the desired output, $y_d(k)$, to the actual output, $y(k)$, is given by equation (14). From equation (14), the actual output becomes a moving average of the desired output. This relation can be utilized in error analysis: e.g., predicting the maximum error given a desired trajectory.

III Application to Motion Control

As an example, we consider a motion control problem where the controlled plant is a pure double integrator: i.e.,

$$G(s) = 1/(ms^2) \tag{21}$$

where the input is the force (torque) and the output is the position (angular position). When this plant is preceded by a zero order hold and sampler, the discrete time transfer function for digital control is

$$G(z) = \frac{1}{m} \frac{Tz^{-1}}{1-z^{-1}} \frac{T(1+z^{-1})}{2(1-z^{-1})} \tag{22}$$

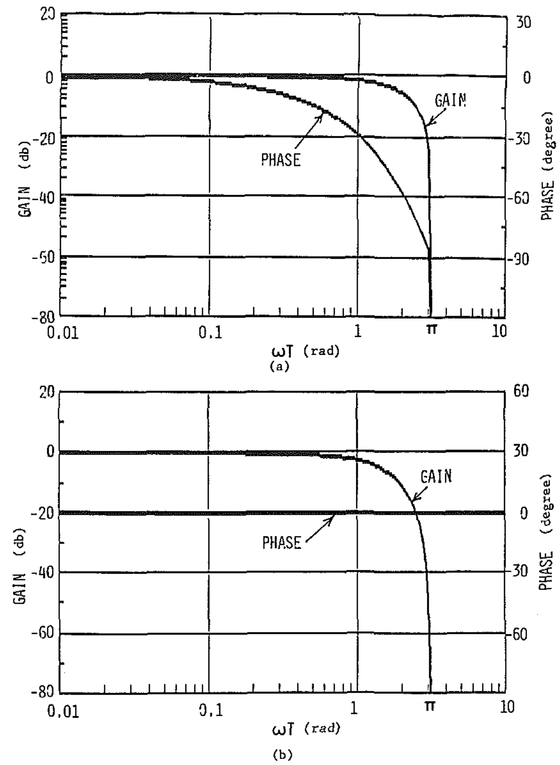

Fig. 4 Frequency response plots of (a) $(1+z^{-1})/2$ and (b) $(1+z^{-1})(1+z)/4$

where T is the sampling time. Figure 3 illustrates the relations among the various signals involved in this motion control problem.

Let us suppose that the overall feedback control scheme consists of a PI velocity loop controller and a P position loop controller (see Fig. 3). Then the overall closed loop transfer function from the position reference input $r(k)$ to the plant output $y(k)$ is

$$G_{\text{closed}}(z^{-1}) = \frac{z^{-1}B_c(z^{-1})}{A_c(z^{-1})}$$

$$= \frac{z^{-1}k_1T^2\{(k_2+k_3)-k_2z^{-1}\}(1+z^{-1})}{2(1-z^{-1})[m(1-z^{-1})^2 + T\{(k_2+k_3)-k_2z^{-1}\}z^{-1}] + z^{-1}k_1T^2\{(k_2+k_3)-k_2z^{-1}\}(1+z^{-1})} \tag{23}$$

Notice that the closed-loop transfer function (23) possesses two zeros: one, at $k_2/(k_2+k_3)$, is due to the velocity loop controller and is inside the unit circle for $k_2>0$ and $k_3>0$, and the other, at -1, due to the plant. Thus, *in this positioning problem*, $B_c^u(z^{-1}) = 1+z^{-1}$, and the zero phase error tracking controller is

$$r(k) = \frac{A_c(z^{-1})(1+z^{-1})}{4k_1T^2\{(k_2+k_3)-k_2z^{-1}\}} y_d(k+2) \tag{24}$$

$r(k)$ depends on the two step ahead value of the desired trajectory since in this problem $d=1$ and $s=1$. The transfer function from $y_d(k)$ to $y(k)$ becomes

$$[(1+z^{-1})(1+z)/4] \tag{25}$$

The frequency response of this transfer function is plotted in Fig. 4. Notice that that phase shift is zero for all frequencies and the low frequency gain remains close to 0 *db*. As a comparison, Fig. 4 includes the frequency response of

$$\frac{B_c^u(z^{-1})}{B_c^u(1)} = \frac{1+z^{-1}}{2} \tag{26}$$

which is the overall transfer function from $y_d(k)$ to $y(k)$ when

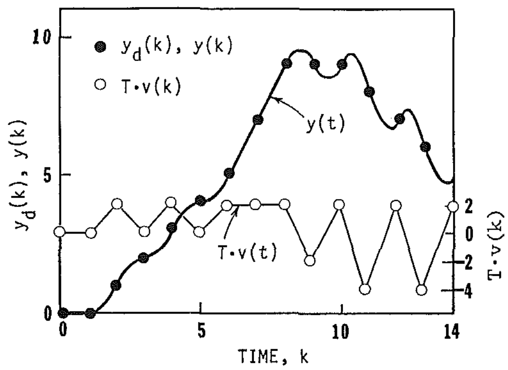

Fig. 5 Tracking performance of positioning system with perfect tracking controller

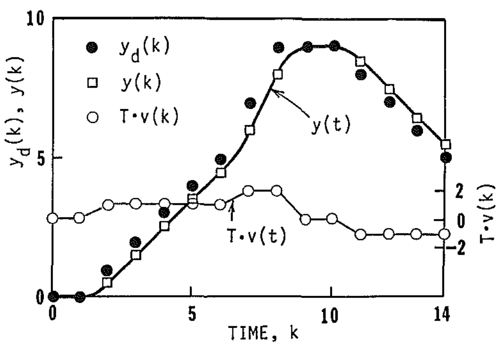

Fig. 6 Tracking performance of positioning system with tracking controller based on stable pole/zero cancellation

the tracking controller based on stable pole/zero cancellation, i.e., equation (5) with $y_d^*(k) = y_d(k)$, is used. In this case, the phase shift degrades the tracking performance significantly as will be shown below in simulation results.

Figure 5 shows the desired trajectory and actual velocity and trajectory when $y_d^*(k)$ is generated by equation (7): i.e., perfect tracking is attempted. Although the actual position matches the desired position at sampling instants, the tracking controller generates the mode, $(-1)^k$, and is not acceptable. The oscillatory mode disappears if we use the tracking controller based on stable pole/zero cancellation. However, the actual trajectory under this tracking controller lags the desired trajectory as shown in Fig. 6. Figure 7 shows the tracking performance with ZPETC. Although the actual position slightly misses the desired position when the desired trajectory makes a corner, the tracking performance is excellent both in terms of the tracking error and smoothness of velocity. In particular, ZPETC lets the plant output stay on any ramp trajectory after missing only the initial point. When it is important that the acutal output passes through a desired point at a specified velocity, this property can be utilized for generation of an appropriate trajectory pattern.

As stated at the end of Section II, the plant output under ZPETC becomes a moving average of the desired output sequence. In the present example, from equation (25),

$$y(k) = 0.25[y_d(k+1) + 2y_d(k) + y_d(k-1)] \tag{27}$$

Therefore, the tracking error, $e(k) = y_d(k) - y(k)$, becomes

$$e(k) = 0.25[-y_d(k+1) + 2y_d(k) - y_d(k-1)] \tag{28}$$

Equation (28) confirms that the tracking error is zero if $y_d(k)$ is either a step signal or a ramp signal. Equation (28) can be used for predicting the tracking error given a desired trajectory.

IV Conclusions

A simple tracking controller for digital control was

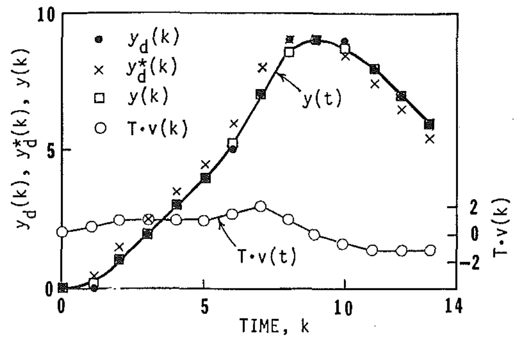

Fig. 7 Tracking performance of positioning system with zero phase error tracking controller

presented. The feedforward controller design assumed the existence of a feedback controller and was based on pole/zero cancellation and phase cancellation. It was shown that perfect tracking is not possible when the regulator loop possesses zeros outside or on the unit circle of the z-plane. For such cases, the feedforward controller (ZPETC) that assures zero phase error in the sense of frequency response was proposed. ZPETC utilizes the $(d+s)$-step ahead desired output, $y_d(k+d+s)$, where d is the number of delay steps in the closed loop transfer function and s is the number of closed loop zeros which are unacceptable for pole/zero cancellation. It should be noted that the optimal preview controller utilizes future desired outputs also and that the number of preview steps required for compensating the dynamic delay in the closed loop portion of the optimal system is normally much larger than $d+s$ (Tomizuka and Whitney, 1975).

ZPETC was examined in the detail for a typical motion control example. The tracking performance under ZPETC was shown to be excellent both in terms of the tracking error and smoothness of velocity. ZPETC has been successfully applied to the path following control for a robot arm (Kubo et al., 1986) and a machine tool control problem (Tomizuka et al., 1986).

Since ZPETC is based on pole/zero cancellation and phase cancellation, the tracking performance under ZPETC is sensitive to modelling errors and plant parameter variations. An adaptive ZPETC has been developed to ensure good tracking when the plant parameters are poorly known or vary during operation (Tsao and Tomizuka, 1986).

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