EQUATION SHEETS

Transformation to discrete systems:

Forward difference $s = \frac{z-1}{T}$

Backward difference $s = \frac{z-1}{Tz}$

Bilinear transformation $s = \frac{2}{T}\frac{z-1}{z+1}$

Final Value $y_{ss} = \lim_{z-1}\left[\left(1-z^{-1}\right)Y(z)\right]$

$$
\begin{aligned}
z &= e^{sT} \\
&= e^{-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}} \\
&= e^{-\zeta\omega_n T}(\cos\omega_n T\sqrt{1-\zeta^2} \pm j\sin\omega_n T\sqrt{1-\zeta^2}).
\end{aligned}
$$

Control Algorithm for Desired Input-Output Relationship

$$T(z) = \frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1+D(z)G(z)}$$

$$D(z) = \frac{1}{G(z)}\frac{T_d(z)}{1-T_d(z)}$$

Controller Design Based on Error-Input Relationship

$$\frac{e_n}{r_n} = \frac{1}{1+D(z)G(z)} = T_{de}(z)$$

$$D(z) = \frac{m_n}{e_n} = \frac{1}{G(z)}\ \frac{1-T_{de}(z)}{T_{de}(z)}$$

Command generation:

$R_i = A_i \cdot B_i$ or $R^*{}_i = A^*{}_i \cdot B^*{}_i$

$R_i = \left[r(t_{i-1}) \quad \dot{r}(t_{i-1}) \quad r(t_i) \quad \dot{r}(t_i)\right]$ and $B_i = \left[b(t_{i-1}) \quad \dot{b}(t_{i-1}) \quad b(t_i) \quad \dot{b}(t_i)\right]$

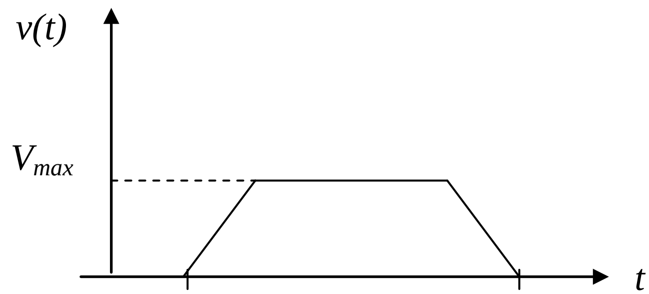
$t = t_{i-1} + (t_i - t_{i-1})u_i$

Position $\mathbf{r}(t) = \mathbf{A}_i \cdot \mathbf{b}(t)$

Velocity $\dot{\mathbf{r}}(t) = \mathbf{A}_i \cdot \dot{\mathbf{b}}(t)$

<u>Three splines with constant acceleration</u>

$$\Delta t_1 = t_1 - t_i = \frac{V_{\max}}{A_{\max}}$$



$$\Delta t_2 = \frac{\Delta x_2}{V_{\max}} = \frac{\Delta x - 2\Delta x_1}{V_{\max}}$$

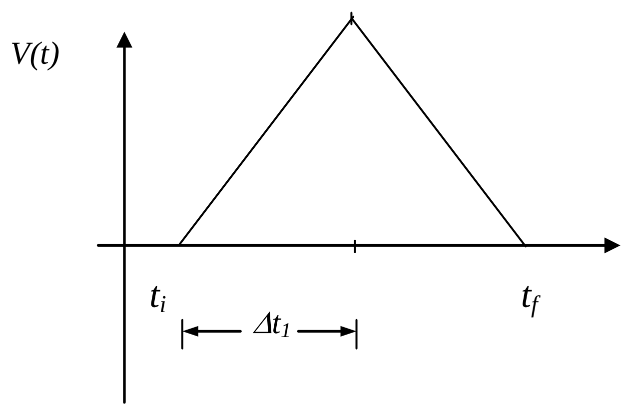
$$= \frac{\Delta x}{V_{\max}} - \Delta t_1$$

$$\Delta t_3 = \Delta t_1$$

For $\Delta x < \left(2\Delta x_1 = \frac{V_{\max}{}^2}{A_{\max}} \right)$

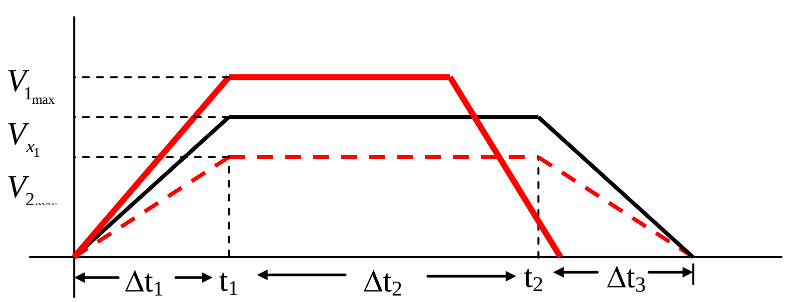
$$\frac{\Delta x}{2} = \frac{1}{2}(\Delta t_1)^2 A_{\max}$$

$$\therefore \quad \Delta t_1 = \left(\frac{\Delta x}{A_{\max}} \right)^{1/2}$$



<u>Multiple Axes</u>

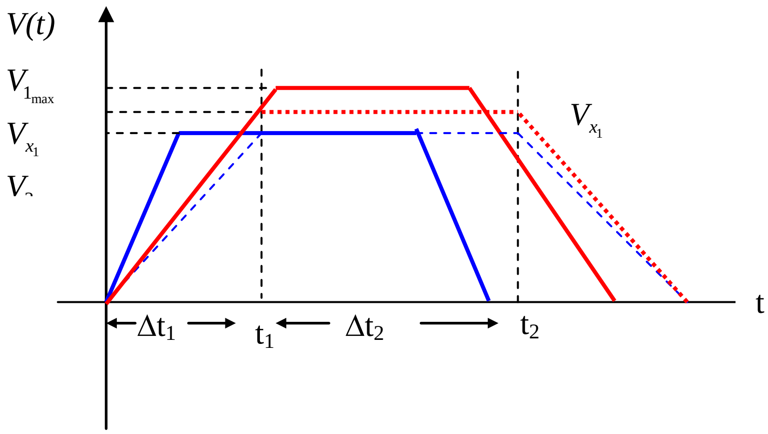
i). Case I. $V_{1_{\max}} > V_{2_{\max}}$ $A_{1_{\max}} > A_{2_{\max}}$



$$\Delta t_1 = \frac{V_{2_{\max}}}{A_{2_{\max}}} \qquad \frac{\Delta x_{11}}{\Delta x_{21}} = \frac{V_{x_1}}{V_{2_{\max}}}$$

$$V_{x_1} = V_{2_{\max}} \left(\frac{\Delta x_{11}}{\Delta x_{21}} \right) \qquad A_{x_1} = \frac{V_{x_1}}{\Delta t_1} = A_{2_{\max}} \left(\frac{\Delta x_{11}}{\Delta x_{21}} \right)$$

ii). Case II. $V_{1_{\max}} > V_{2_{\max}}$ $A_{2_{\max}} > A_{1_{\max}}$



$$V_{x_1} = \Delta t_1 \cdot A_{1_{\max}} \qquad V_{2_{\max}} = \Delta t_1 \cdot A_{x_2}$$

$$\frac{\Delta x_{11}}{\Delta x_{21}} = \frac{V_{x_1}}{V_{2_{\max}}} = \frac{A_{1_{\max}}}{A_{x_2}} \qquad \Delta t_1 = \frac{V_{x_1}}{A_{1_{\max}}} = \frac{\Delta x_{11}}{\Delta x_{21}} \cdot \frac{V_{2_{\max}}}{A_{1_{\max}}}$$

<u>DC Motor:</u>

$$V_A = R_A \cdot I_A + V_i + L_A \frac{dI_A}{dt}$$

$$V_i = K_i \phi \cdot n$$

$$T_m = J \frac{dn(t)}{dt} + T_L + bn(t)$$

$$P = V_A \cdot I_A$$

<u>AC Motor:</u>

$$T_m = \frac{pnV^2 sR_r}{\omega_p (R_r^2 + s^2 \omega_p^2 L_r^2)}$$

TABLE 8.1 Laplace Transforms and z-transforms of Simple Discrete Time Functions

$F(s)$ is the Laplace transform of $f(t)$, and $F(z)$ is the z-transform of $f(kT)$. *Note:* $f(t) = 0$ for $t = 0$.

Number	$\mathscr{F}(s)$	$f(kT)$	$F(z)$
1		$1,\ k = 0;\ 0,\ k \neq 0$	1
2		$1,\ k = k_o;\ 0,\ k \neq k_o$	z^{-k_o}
3	$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{1}{2!}(kT)^2$	$\frac{T^2}{2}\left[\frac{z(z+1)}{(z-1)^3}\right]$
6	$\frac{1}{s^4}$	$\frac{1}{3!}(kT)^3$	$\frac{T^3}{6}\left[\frac{z(z^2+4z+1)}{(z-1)^4}\right]$
7	$\frac{1}{s^m}$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}\right)$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}\right)$
8	$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
9	$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
10	$\frac{1}{(s+a)^3}$	$\frac{1}{2}(kT)^2e^{-akT}$	$\frac{T^2}{2}e^{-aT}z\frac{(z+e^{-aT})}{(z-e^{-aT})^3}$
11	$\frac{1}{(s+a)^m}$	$\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}\right)$	$\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}\right)$
12	$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
13	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(akT-1+e^{-akT})$	$\frac{z[(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})]}{a(z-1)^2(z-e^{-aT})}$
14	$\frac{b-a}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
15	$\frac{s}{(s+a)^2}$	$(1-akT)e^{-akT}$	$\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$
16	$\frac{a^2}{s(s+a)^2}$	$1-e^{-akT}(1+akT)$	$\frac{z[z(1-e^{-aT}-aTe^{-aT})+e^{-2aT}-e^{-aT}+aTe^{-aT}]}{(z-1)(z-e^{-aT})^2}$
17	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bkT}-ae^{-akT}$	$\frac{z[z(b-a)-(be^{-aT}-ae^{-bT})]}{(z-e^{-aT})(z-e^{-bT})}$
18	$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$
19	$\frac{s}{s^2+a^2}$	$\cos akT$	$\frac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$
20	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-akT}\cos bkT$	$\frac{z(z-e^{-aT}\cos bT)}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$
21	$\frac{b}{(s+a)^2+b^2}$	$e^{-akT}\sin bkT$	$\frac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$
22	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1-e^{-akT}\left(\cos bkT+\frac{a}{b}\sin bkT\right)$	$\frac{z(Az+B)}{(z-1)[z^2-2e^{-aT}(\cos bT)z+e^{-2aT}]}$

$A = 1 - e^{-aT}\cos bT - \frac{a}{b}e^{-aT}\sin bT$

$B = e^{-2aT} + \frac{a}{b}e^{-aT}\sin bT - e^{-aT}\cos bT$