CHAPTER 4
DYNAMICS OF A SYSTEM OF PARTICLES

- **We consider a system consisting of *n* particles**
- **One can treat individual particles, as before; i.e.,one can draw FBD for each particle, define a coordinate system and obtain an expression of the absolute acceleration for the particle. One can then use Newton's second law and proceed to get *n* second-order coupled ODEs.**
- **Focus here is on overall motion of the system- also a precursor to rigid body dynamics.**

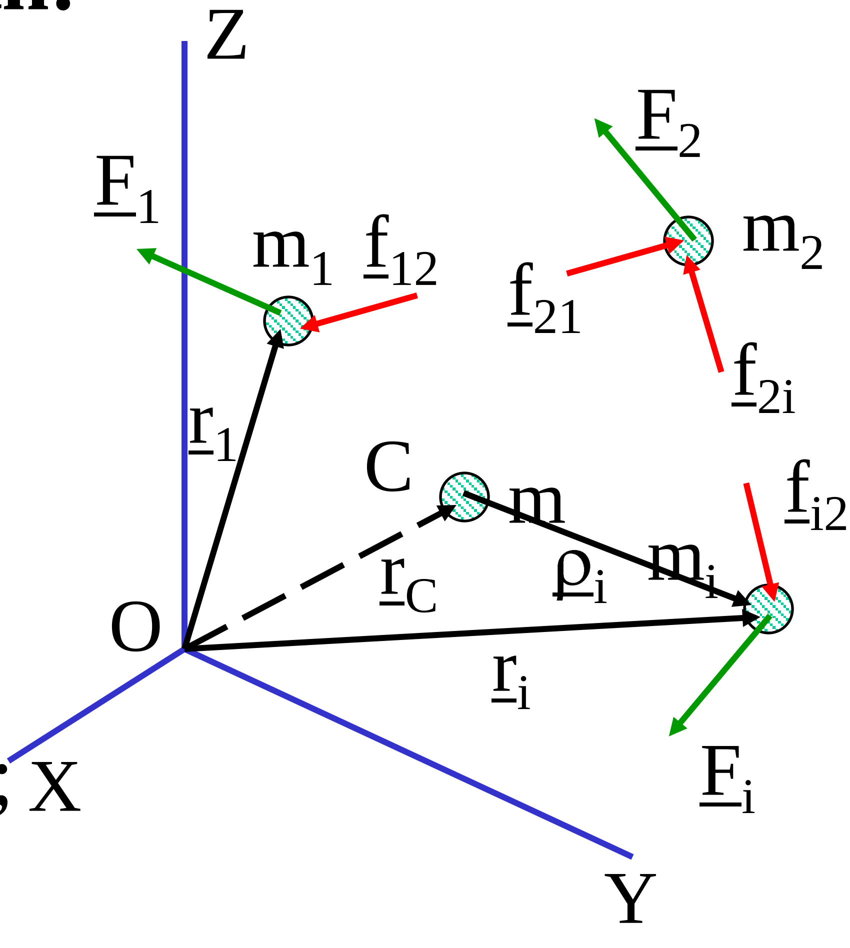
4.1 Equations of Motion:

Consider a system with:

- ***n* particles**
- **masses -** m_i
- **positions -** $\underline{r}_i$

There are two types of forces acting:

- **External forces -** $\underline{F}_i$;
- **Internal forces -** $\underline{f}_{ij}$



$\underline{f}_{ij}$ - **force on the** i^{th} **particle due to its interaction with the** j^{th} **particle**

- **Newton's 3rd law** $\rightarrow \underline{f}_{ij} = -\underline{f}_{ji}$

  (internal forces are equal and opposite)

Also $\underline{f}_{ij} = 0 \text{ when } i = j, i.e. \ \underline{f}_{ii} = 0$

- **Newton's 2nd law for** i^{th} **particle:**

$$\boxed{m_i \ddot{\underline{r}}_i = \underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij}}, \quad i = 1, 2, 3, \ldots, n$$

Now, for 3-dimensional motions, the position of each particle (in Cartesian coordinates) is:

$$\underline{r}_i = x_i \underline{i} + y_i \underline{j} + z_i \underline{k}, \quad i = 1, 2, 3, \ldots, n$$

Thus, each equation in Newton's second law has 3 scalar second-order ordinary diff. equations. $\rightarrow$ **3*n* scalar second-order o.d.e.'s for the system**

In order to solve for the motion, one needs to know:

- **external forces** $\underline{F}_i$ **on each of the particles**
- **nature of internal forces** $\underline{f}_{ij}$

e.g., Newton's law of gravitation:

$$\underline{f}_{ij} = G \frac{m_i m_j}{\left|\underline{r}_j - \underline{r}_i\right|^2} \frac{(\underline{r}_j - \underline{r}_i)}{\left|\underline{r}_j - \underline{r}_i\right|}$$

$$or, \quad \underline{f}_{ij} = -G m_i m_j (\underline{r}_i - \underline{r}_j) / \left|\underline{r}_j - \underline{r}_i\right|^3$$

We also need:

- **initial conditions:** $\underline{r}_i(0), \dot{\underline{r}}_i(0), i = 1, 2, \ldots\ldots, n$

The general solutions to these nonlinear ODEs are unknown; they are difficult to solve except for in some very simple cases and small *n*.

Suppose we would like to get overall motion of the system, not those of individual particles.

Adding the *n* equations:

$$\sum_{i=1}^{n} m_i \underline{\ddot{r}}_i = \sum_{i=1}^{n} \underline{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{f}_{ij}$$

Now, $\sum_{i=1}^{n}\sum_{j=1}^{n} \underline{f}_{ij} = 0$ **(net interaction force is zero)**

- $m = \sum_{i=1}^{n} m_i$ **- total mass**
- $m\underline{r}_C(t) = \sum_{i=1}^{n} m_i \underline{r}_i(t)$ **- defines center of mass; note that it is a function of time since the particles move.**

Thus, addition of Eqns. $\rightarrow \quad \sum_{i=1}^{n} \underline{F}_i = \sum_{i=1}^{n} m_i \ddot{\underline{r}}_i = m\ddot{\underline{r}}_C$

- Let $\underline{F} \equiv \sum_{i=1}^{n} \underline{F}_i$ - **total external force**
- $\boxed{\underline{F} \equiv \sum_{i=1}^{n} \underline{F}_i = m\ddot{\underline{r}}_C}$ **Equation of motion for the center of mass**

$\rightarrow$ **Internal forces do not affect the motion of the center of mass.**

4.2 Work and Kinetic Energy

- **The motion of individual particle is defined by**

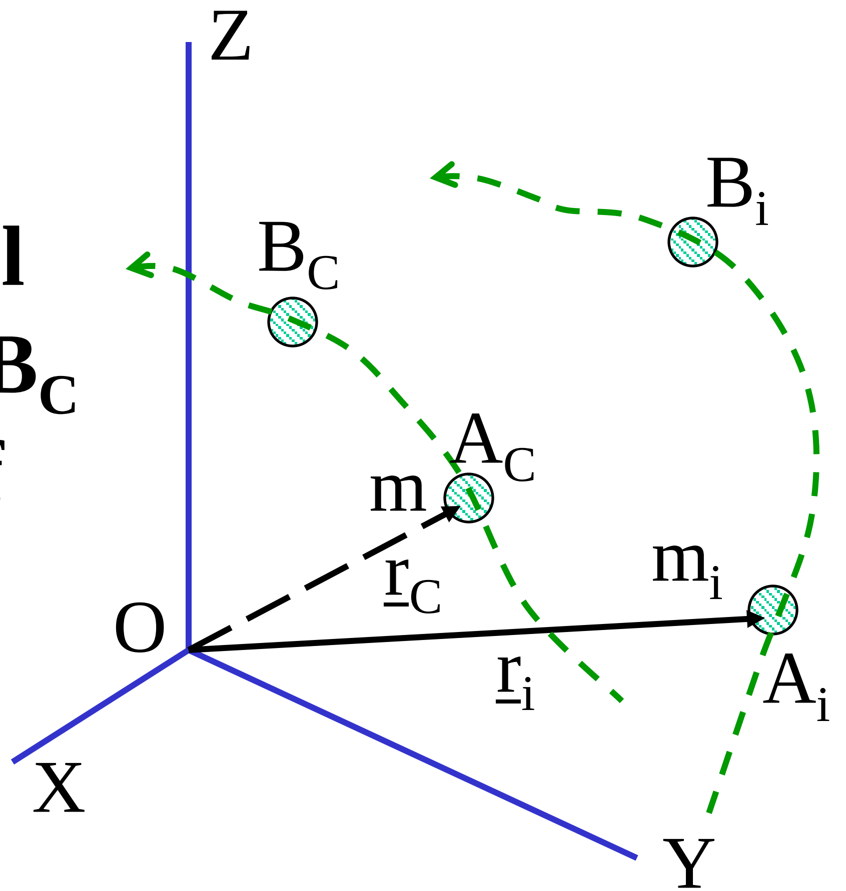
$$m_i \ddot{\underline{r}}_i = \underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij}, \quad i = 1, 2, 3, \ldots, n$$

- **The motion of center of mass C is defined by**

$$\underline{F} \equiv \sum_{i=1}^{n} \underline{F}_i = m \ddot{\underline{r}}_C$$

where the total mass is $m = \sum_{i=1}^{n} m_i$

Consider a motion of the system. The initial state is A, and the final state is B. Let A_C and B_C denote the positions of the CM.



- **Now, for the CM**

$$\underline{F} \equiv m\ddot{\underline{r}}_C$$

$$\rightarrow \int_{A_C}^{B_C} \underline{F} \bullet d\underline{r}_C = \int_{A_C}^{B_C} m\ddot{\underline{r}}_C \bullet d\underline{r}_C = (mv_C^2/2)\Big|_{A_C}^{B_C}$$

- **work-energy statement for the CM**

Note that $\int_{A_C}^{B_C} \underline{F} \bullet d\underline{r}_C$ **is only the work done by external forces, and it is related to the change in** **translational kinetic energy associated with the CM**

- **Let** $W_i \equiv$ **work done on the** i^{th} **particle by all the forces acting on it in moving from** A_i to B_i

$$W_i = \int_{A_i}^{B_i} (\underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij}) \bullet d\underline{r}_i$$

Now: $\underline{r}_i = \underline{r}_C + \underline{\rho}_i$

where $\underline{\rho}_i$ **- position of** i^{th} **particle relative to the CM of the system**

- **Total work done=sum of the work done on all particles:** $W = \sum_{i=1}^{n} W_i$

$$\rightarrow W = \sum_{i=1}^{n} \int_{A_i}^{B_i} (\underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij}) \bullet (d\underline{r}_C + d\underline{\rho}_i)$$

$$Now,\ \sum_{i=1}^{n} \int_{A_i}^{B_i} (\underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij}) \bullet d\underline{r}_C = \int_{A_C}^{B_C} (\sum_{i=1}^{n} \underline{F}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{f}_{ij}) \bullet d\underline{r}_C$$

$$\rightarrow W = \int_{A_C}^{B_C} \underline{F} \bullet d\underline{r}_C + \sum_{i=1}^{n} \int_{A_i}^{B_i} (\underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij}) \bullet d\underline{\rho}_i$$

work done by total ext. forces through the displ. of the CM

summation of the work done on all the particles through their displacements relative to the CM

- **For each particle, the work done is:**

$$W_i = \frac{1}{2} m_i \underline{\dot{r}}_i \bullet \underline{\dot{r}}_i \Big|_{A_i}^{B_i} = \frac{1}{2} m_i (\underline{\dot{r}}_C + \underline{\dot{\rho}}_i) \bullet (\underline{\dot{r}}_C + \underline{\dot{\rho}}_i) \Big|_{A_i}^{B_i}$$

$$\rightarrow W = \sum_{i=1}^{n} W_i = m\dot{\underline{r}}_C \bullet \dot{\underline{r}}_C \Big|_{A_C}^{B_C} / 2 + \dot{\underline{r}}_C \bullet \sum_{i=1}^{n} m_i \dot{\underline{\rho}}_i \Big|_{A_i}^{B_i}$$

$$+ \sum_{i=1}^{n} m_i \dot{\underline{\rho}}_i \bullet \dot{\underline{\rho}}_i \Big|_{A_i}^{B_i} / 2$$

$$Now, \quad \sum_{i=1}^{n} m_i \underline{\rho}_i = 0 \rightarrow \sum_{i=1}^{n} m_i \dot{\underline{\rho}}_i = 0$$

$$\rightarrow \boxed{W = m\dot{\underline{r}}_C \bullet \dot{\underline{r}}_C \Big|_{A_C}^{B_C} / 2 + \sum_{i=1}^{n} m_i \dot{\underline{\rho}}_i \bullet \dot{\underline{\rho}}_i \Big|_{A_i}^{B_i} / 2}$$

$\equiv \mathrm{T_B - T_A}$ - **the sum of increase/change in KE of the system.**

$$\boxed{W_{A \to B} = T_B - T_A}$$

work-energy principle for the system of particles

T = K.E. at any instant

$$= mv_C^2/2 + \sum_{i=1}^{n} m_i \dot{\underline{\rho}}_i \bullet \dot{\underline{\rho}}_i/2$$

Recalling the work-energy principle for the CM:

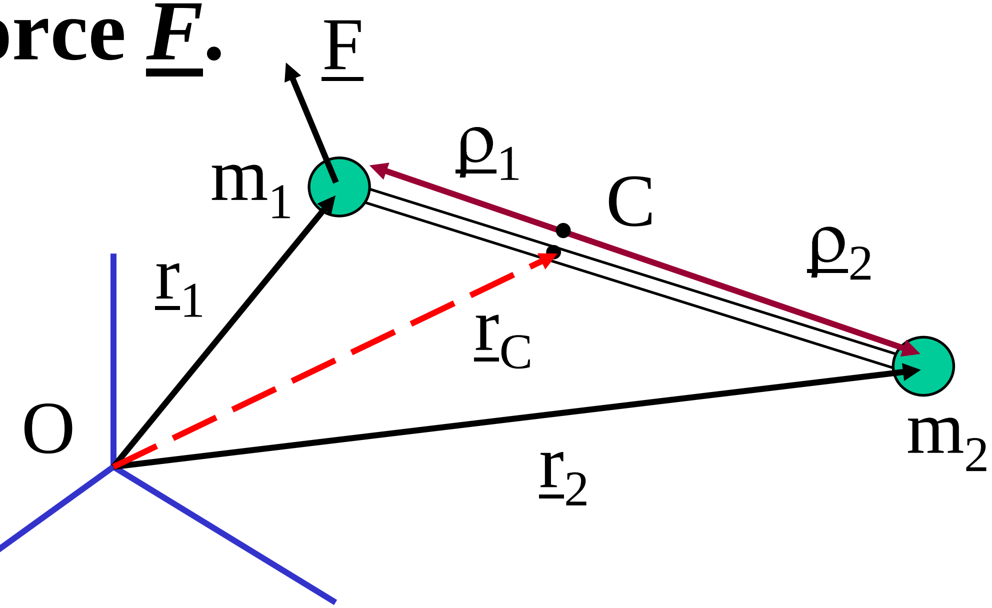
$$\rightarrow \sum_{i=1}^{n} \int_{A_i}^{B_i} (\underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij}) \bullet d\underline{\rho}_i = \sum_{i=1}^{n} m_i \dot{\underline{\rho}}_i \bullet \dot{\underline{\rho}}_i \Big|_{A_i}^{B_i} /2$$

Work done by all forces (external as well as internal) in relative motion ≡ KE for relative motion

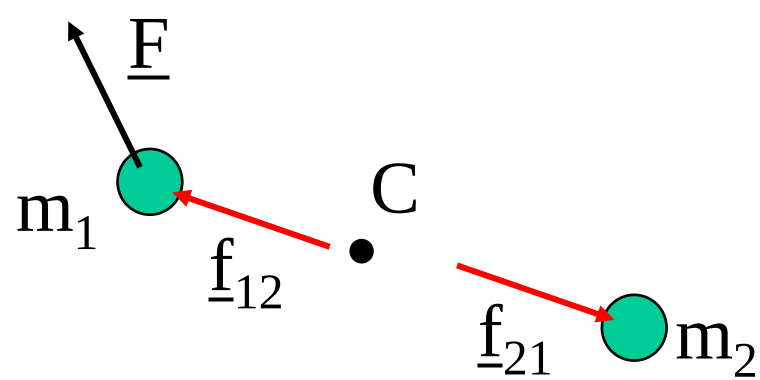
Important: In general, internal forces $\underline{f}_{ij}$ do work in any motion of the system. Sometimes, net work (that on the whole system) may be zero even though there is work done on individual particles.

Ex: Consider the force in a spring connecting two moving bodies - there is net work done by the spring force - evaluated by potential function ϕ_{sp} .

Ex 1: **Consider two particles connected by a massless rigid (inextensible) rod, and acted upon by a force $\underline{F}$.**



<u>FBDs</u> for individual particles are:



Note: $\underline{f}_{12} = -\ \underline{f}_{21}$

- **Work done in relative motion by internal forces:**

$$dW = \underline{f}_{12} \bullet d\underline{\rho}_1 + \underline{f}_{21} \bullet d\underline{\rho}_2 = \underline{f}_{12} \bullet (d\underline{\rho}_1 - d\underline{\rho}_2)$$

- **constraint** $\left|\underline{r}_{12}\right|^2 = (\underline{\rho}_2 - \underline{\rho}_1) \bullet (\underline{\rho}_2 - \underline{\rho}_1) = l^2$

- **Differentiate:**

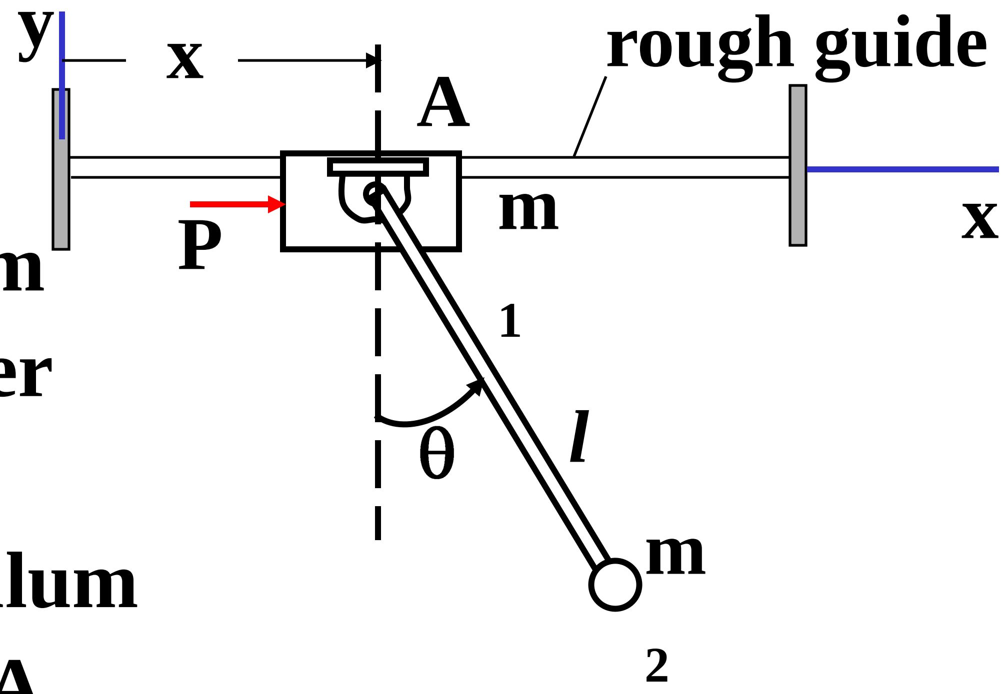
$$d(\left|\underline{r}_{12}\right|^2) = (\underline{\rho}_2 - \underline{\rho}_1) \bullet (d\underline{\rho}_2 - d\underline{\rho}_1) = 0$$

Now:

$$\underline{f}_{12} = \left|\underline{f}_{12}\right| (\underline{\rho}_1 - \underline{\rho}_2) / \left|\underline{\rho}_1 - \underline{\rho}_2\right|$$

Thus, $\boxed{\underline{f}_{12} \bullet (d\underline{\rho}_1 - d\underline{\rho}_2) = 0}$

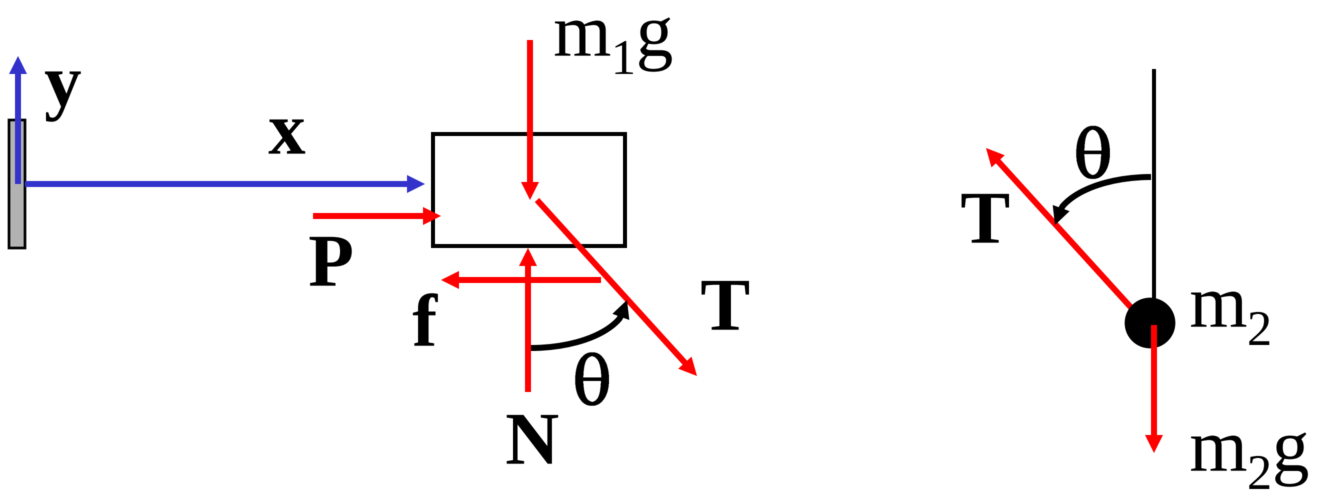
Ex 2:



Consider the system shown here. A slider moves on a rough guide, and a pendulum is attached to it at A.

- m_1 , m_2 **connected by a massless rigid link.**
- **Coulomb friction between** m_1 **and the horizontal guide. Force P acts on the block A.**

The FBDs are:



The positions of the two particles can now be defined: $\underline{r}_1(t) = x(t)\,\underline{i}$

$$\underline{r}_2(t) = \{x(t) + l\sin\theta\}\,\underline{i} - l\cos\theta\,\underline{j}$$

The equations of motion for the individual particles are:

$$\underline{\underline{m_1}}: \quad m_1\ddot{x}\underline{i} = (P - f + T\sin\theta)\underline{i} + (N - m_1 g - T\cos\theta)\underline{j}$$

$$\text{where} \quad f = \mu N(\text{sgn}(\dot{x}))$$

$$\underline{\underline{m_2}}: \quad m_2\{(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta)\underline{i}$$

$$+ (l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta)\underline{j}\} = -T\sin\theta\underline{i}$$

$$+ (T\cos\theta - m_2 g)\underline{j}$$

- **Try to write the equation of motion for the CM of the system.**

4.3 Conservation of Mechanical Energy

- **Suppose that the External forces are conservative, that is,** $\underline{F} \equiv \sum_{i=1}^{n} \underline{F}_i$ **are conservative.**

$\rightarrow \; E_C = T_C + V_C$ **for the CM of the system**

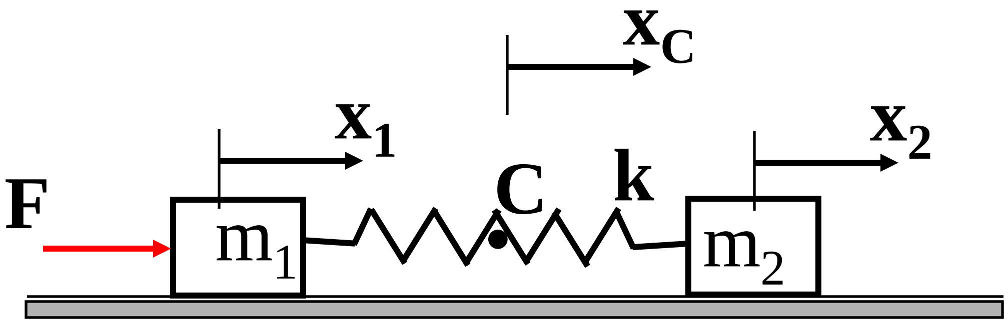
→Total energy conserved for motion of the CM

- **Suppose that Internal forces also conservative:**

→ E = T + V

Total energy conserved for the whole system.

Ex. 3 (4.2): Consider the system shown.



- m_1 and m_2 **connected by a massless spring.**
- **A constant force F applied to** m_1 **at t = 0.**
- **<u>No friction between the floor and the blocks</u>**

Find: $x_1(t)$; when masses are equal : $m_1 = m_2 = \bar{m}$

IC (t = 0), $x_1 = x_2 = \dot{x}_1 = \dot{x}_2 = 0$; **spring unstretched**

Motion of the CM:

$$\underline{r}_C = (\sum m_i \underline{r}_i)/m \quad \rightarrow \quad \boxed{x_C = (x_1 + x_2)/2}$$

Newton's Second law :

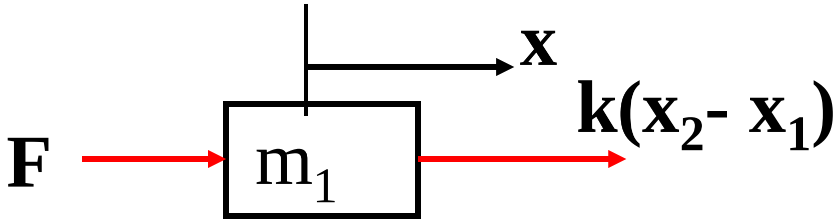
$$\sum F_x = F = m\ddot{x}_C \quad (m = m_1 + m_2 = 2\overline{m})$$

$$\rightarrow \quad \ddot{x}_C = F/2\overline{m}; \quad \text{Init. Conds. are}: x_C(0) = \dot{x}_C = 0$$

$$\rightarrow \quad \boxed{\dot{x}_C = (F/2\overline{m})t}; \quad \boxed{x_C = (F/2\overline{m})t^2/2}$$

Motion of the block m_1:

FBD:



Newton's law for block m_1:

$$\sum F_x = F + k(x_2 - x_1) = \bar{m}\ddot{x}_1; \text{ ICs.}: x_C(0) = \dot{x}_C = 0$$

Also, note that $x_2 = 2x_C - x_1$

$$\rightarrow \quad F - 2k(x_1 - x_C) = \bar{m}\ddot{x}_1 \qquad (1)$$

Also: $\bar{m}\ddot{x}_C = F/2 \qquad (2)$

$$(1)-(2) \rightarrow \boxed{\bar{m}(\ddot{x}_1 - \ddot{x}_C) + 2k(x_1 - x_C) = F/2}$$

ICs: $[x_C(0) - x_1(0)] = [\dot{x}_1(0) - \dot{x}_C(0)] = 0$

Soln: $\boxed{(x_C - x_1) = F\{1 - \cos\sqrt{2k/\bar{m}}\}/4k}$

(Harmonic oscillation)

Aside (steps involved in the solution):

The eqn. is : $\bar{m}\ddot{y} + 2ky = F/2$ where $y = (x_1 - x_C)$

The solution is $y(t) = y_h(t) + y_p(t)$

$\underline{\underline{y_p}}$: $2ky_p = F/2 \rightarrow y_p = F/4k$

$\underline{\underline{y_h}}$: $y_h(t) = A\cos\omega_n t + B\sin\omega_n t,$ where $\omega_n = \sqrt{2k/\bar{m}}$

$y(0) = 0 \rightarrow A + F/4k = 0 \rightarrow A = -F/4k$

$\dot{y}(0) = 0 \rightarrow B\omega_n = 0 \rightarrow B = 0$

Soln: $\boxed{y(t) = F\{1 - \cos\omega_n t\}/4k}$

(Harmonic oscillation)

Or $x_1(t) = (F/4\bar{m})t^2 + F\{1 - \cos\sqrt{2k/\bar{m}}\,t\}/4k$

Energy considerations: (verification)

Recall that $\dot{x}_C = v_C = (F/2\bar{m})t$

$\rightarrow$ $\text{K.E. of CM} = T_C = (2\bar{m})v_C^2/2 = (F^2/4\bar{m})t^2$

$\text{Work done on CM} = W_C = Fx_C = F(F/4\bar{m})t^2$

(for a constant force)

$\rightarrow$ Work done on CM (W_C)=change in K.E. of CM

Now, consider for the whole system:

Total KE:

$$T = [(2\bar{m})\dot{x}_C^2 + \bar{m}\{(\dot{x}_1 - \dot{x}_C)^2 + (\dot{x}_2 - \dot{x}_C)^2\}]/2$$

or $T = (F^2/4\bar{m})t^2 + (F^2/8k)\sin^2(\sqrt{2k/\bar{m}}t)$

Potential Energy:

$$V = k(x_1 - x_2)^2/2 = 2k(x_1 - x_C)^2$$

(Work done by internal forces)

or $V = (F^2/8k)\{1 - \cos(\sqrt{2k/\bar{m}}t)\}^2$

$\rightarrow T + V = E = (F^2/4\bar{m})t^2 + (F^2/4k)\{1 - \cos(\sqrt{2k/\bar{m}}t)\}$

W = work done by the external force

$$Fx_1 = (F^2/4\bar{m})t^2 + (F^2/4k)\{1-\cos(\sqrt{2k/\bar{m}}\,t)\}$$

$$\rightarrow\ W = T + V - (0+0)$$

Work done by all forces (external and internal)
= (final total energy) − (initial total energy)

4.4 Linear Impulse and Momentum

Let, $\hat{\underline{F}} \equiv \int_{t_1}^{t_2} \underline{F}(\tau)d\tau$ **- lin. impulse of external forces**

Considering Newton's laws for motion of CM:

$$\hat{\underline{F}} \equiv \int_{t_1}^{t_2} \underline{F}(\tau)d\tau = \int_{t_1}^{t_2} m\ddot{\underline{r}}_C(\tau)d\tau = m(\underline{v}_{C2} - \underline{v}_{C1})$$

Let $\underline{p}(t) = \sum_{i=1}^{n} m_i \underline{v}_i(t) = m\underline{v}_C$ **- total linear momentum of the system at a given instant**

Then

$$\boxed{\hat{\underline{F}} = m(\underline{v}_{C2} - \underline{v}_{C1}) = \underline{p}(t_2) - \underline{p}(t_1)}$$

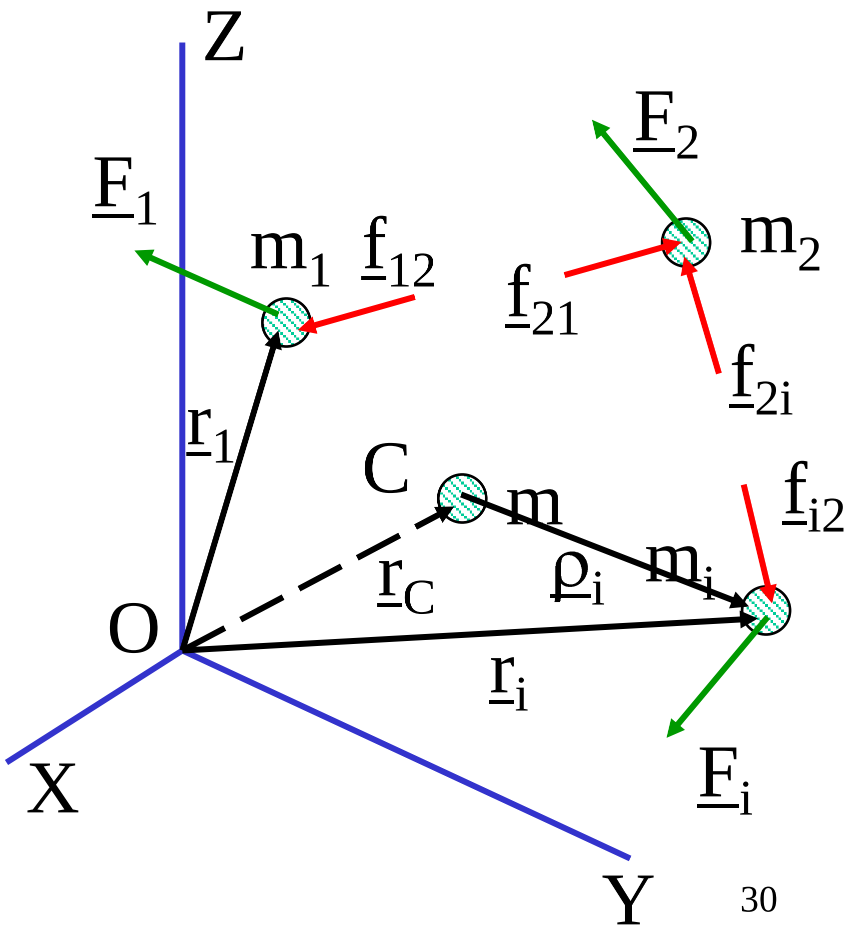
4.5 Angular Momentum:

The key point to consider here is the point about which the moment can be taken.

- **Moment about a fixed reference point:**

$$\underline{H}_{iO} = \underline{r}_i \times m_i \dot{\underline{r}}_i$$

(angular momentum of the ith particle about point O)



Total angular momentum of the system:

$$\underline{H}_O = \sum_{i=1}^{n} \underline{H}_{iO} = \sum_{i=1}^{n} \underline{r}_i \times m_i \dot{\underline{r}}_i$$

Rate of change of angular momentum :

$$\dot{\underline{H}}_O = \sum_{i=1}^{n} \dot{\underline{r}}_i \times m_i \dot{\underline{r}}_i + \sum_{i=1}^{n} \underline{r}_i \times m_i \ddot{\underline{r}}_i = \sum_{i=1}^{n} \underline{r}_i \times m_i \ddot{\underline{r}}_i$$

Now, using Newton' second law for a particle :

$$m_i \ddot{\underline{r}}_i = \underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij} \rightarrow \dot{\underline{H}}_O = \sum_{i=1}^{n} \underline{r}_i \times (\underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij})$$

or

$$\boxed{\dot{\underline{H}}_O = \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i = \underline{M}_O}$$

- **Reference point as the center of mass:**

Let $\underline{r}_i = \underline{r}_C + \underline{\rho}_i$

$$\rightarrow \underline{H}_O = \sum_{i=1}^{n} m_i (\underline{r}_C + \underline{\rho}_i) \times (\dot{\underline{r}}_C + \dot{\underline{\rho}}_i)$$

$$= m\underline{r}_C \times \dot{\underline{r}}_C + \underline{r}_C \times \sum_{i=1}^{n} m_i \dot{\underline{\rho}}_i + (\sum_{i=1}^{n} m_i \underline{\rho}_i) \times \dot{\underline{r}}_C$$

$$+ \sum_{i=1}^{n} m_i \underline{\rho}_i \times \dot{\underline{\rho}}_i$$

$$\rightarrow \boxed{\underline{H}_O = m\underline{r}_C \times \dot{\underline{r}}_C + \sum_{i=1}^{n} m_i \underline{\rho}_i \times \dot{\underline{\rho}}_i}$$

Thus, $$\boxed{\underline{H}_O = \underline{r}_C \times m\dot{\underline{r}}_C + \underline{H}_C}$$

where $\underline{H}_C = \sum_{i=1}^{n} \underline{\rho}_i \times m_i \dot{\underline{\rho}}_i$

(**Ang. momentum with respect to the CM, as viewed by a nonrotating observer moving with the CM**)

Now, differentiating:

$$\dot{\underline{H}}_O = \underbrace{\underline{r}_C \times m\ddot{\underline{r}}_C}_{\underline{r}_C \times \underline{F}} + \underbrace{\sum_{i=1}^{n} \underline{\rho}_i \times m_i \ddot{\underline{\rho}}_i}_{\sum_{i=1}^{n} \underline{\rho}_i \times \underline{F}_i}$$

$$\rightarrow \underline{M}_O = \underline{r}_C \times \underline{F} + \underbrace{\sum_{i=1}^{n} \underline{\rho}_i \times \underline{F}_i}_{\underline{M}_C} = \dot{\underline{H}}_C + \underline{r}_C \times m\ddot{\underline{r}}_C$$

Now $\underline{r}_C \times \underline{F} = \underline{r}_C \times m\ddot{\underline{r}}_C$ (for motion of CM)

$$\rightarrow \boxed{\underline{M}_C = \dot{\underline{H}}_C = \sum_{i=1}^{n} \underline{\rho}_i \times m_i \ddot{\underline{\rho}}_i}$$

Reviewing: $\boxed{\underline{M}_O = \dot{\underline{H}}_O}$ (about fixed pointO)

$\boxed{\underline{M}_O = \dot{\underline{H}}_O}$ (about C, the CM)

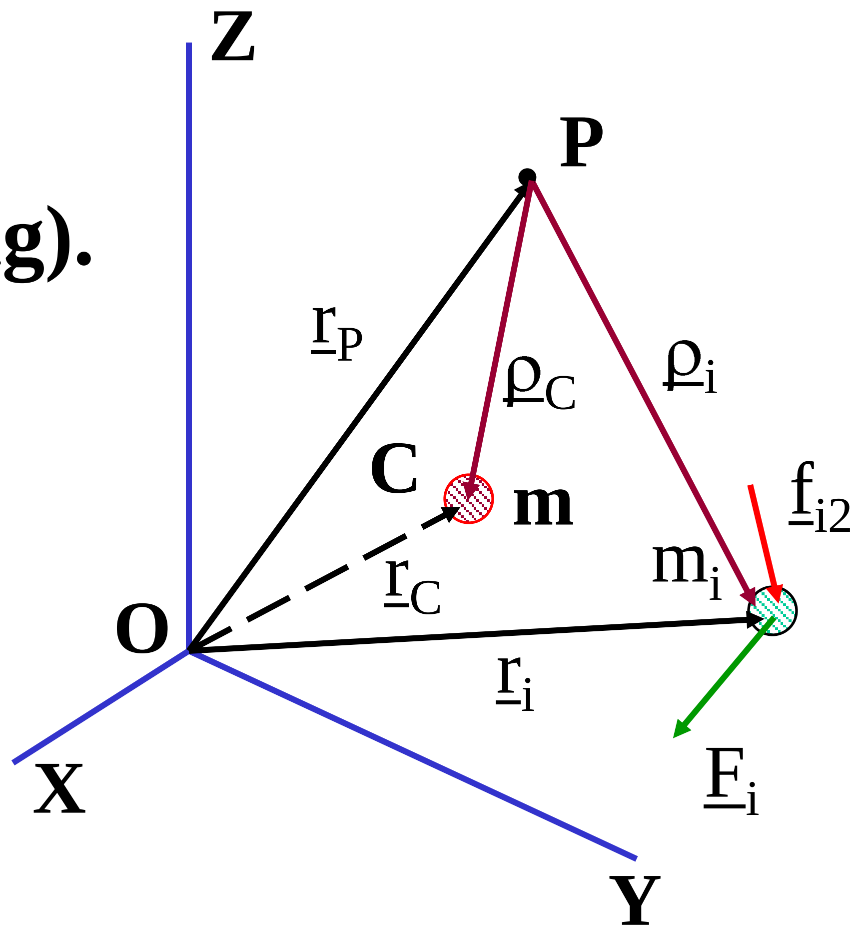
- **(very convenient for rigid bodies)**

• About an arbitrary reference point P:

Let P be an arbitrary point (could be moving).

Let $\underline{r}_i = \underline{r}_P + \underline{\rho}_i$

and $\underline{r}_C = \underline{r}_P + \underline{\rho}_C$



Then, one can show that

$$\underline{H}_P = \underline{\rho}_C \times m\dot{\underline{\rho}}_C + \underline{H}_C$$

- angular momentum about P

And $\underline{M}_P = \underline{\rho}_C \times m\ddot{\underline{r}}_P + \dot{\underline{H}}_P$

$\rightarrow$ **Choosing an arbitrary point for moments of forces results in an additional term in the moment equation.**

- **If P is a fixed point** $\rightarrow \underline{M}_P = \dot{\underline{H}}_P \quad (\ddot{\underline{r}}_P = 0)$
- **If P is the center of mass**

  $\rightarrow \underline{M}_P = \dot{\underline{H}}_P \quad (\underline{\rho}_C = 0)$
- **If P is such that** $\ddot{\underline{r}}_P$ **and** $\underline{\rho}_C$ **are parallel throughout the motion** $\rightarrow \underline{M}_P = \dot{\underline{H}}_P$

- **Computation of Kinetic energy using P as a reference point:**

The kinetic energy is: $T = \sum_{i=1}^{n} m_i \underline{\dot{r}}_i \bullet \underline{\dot{r}}_i / 2$

Now, $\underline{r}_i = \underline{r}_P + \underline{\rho}_i, \quad \underline{\dot{r}}_i = \underline{\dot{r}}_P + \underline{\dot{\rho}}_i \quad \rightarrow$

$$T = [m\left|\underline{\dot{r}}_P\right|^2 + \sum_{i=1}^{n} m_i \left|\underline{\dot{\rho}}_i\right|^2 + 2\underline{\dot{r}}_P \bullet m\underline{\dot{\rho}}_C]/2$$

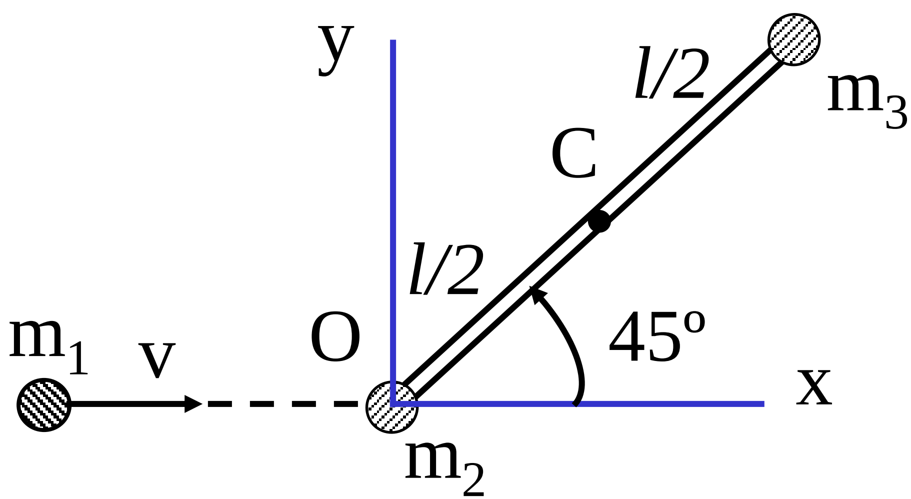
If $P = C$: $$T = [m\left|\underline{\dot{r}}_C\right|^2 + \sum_{i=1}^{n} m_i \left|\underline{\dot{\rho}}_i\right|^2]/2$$ (as before)

Ex. 4 (4.7):

Consider a particle traveling at a speed 'v' to the right. It strikes a stationary dumbbell (two particles connected by a massless rigid rod).

The masses are:

$$m_1 = m_2 = m_3 = m$$

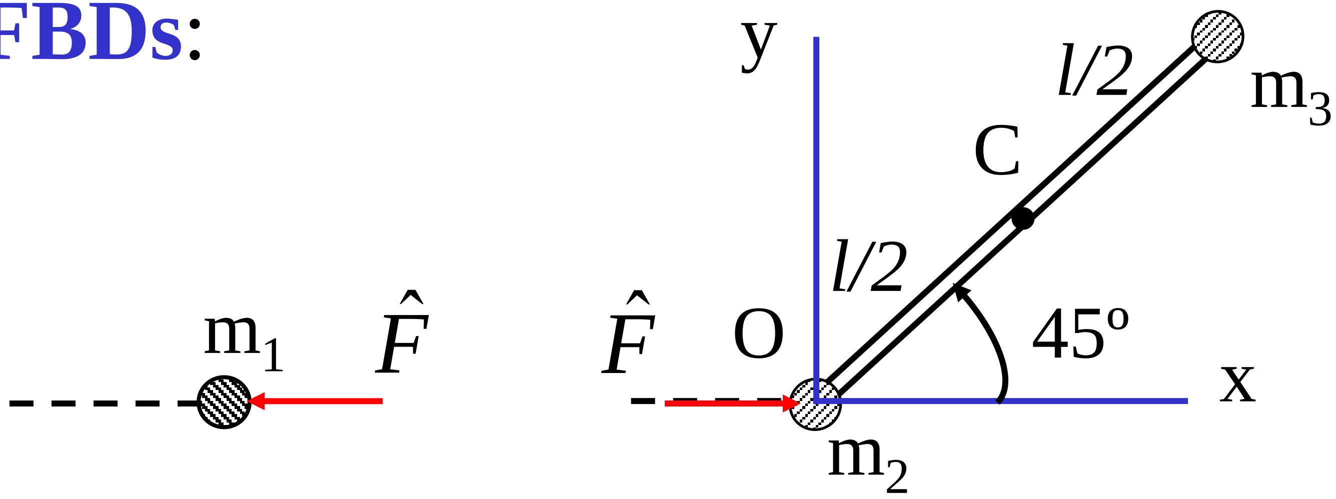


Assumption:

- **Perfectly elastic impact in** m_1 , m_2 **(*e*=1).**

Find: motion of the particles just after impact.

FBDs:



Observe that during impact:

- **Net force on the whole system = 0**

$\rightarrow$ **linear momentum conserved for the system**

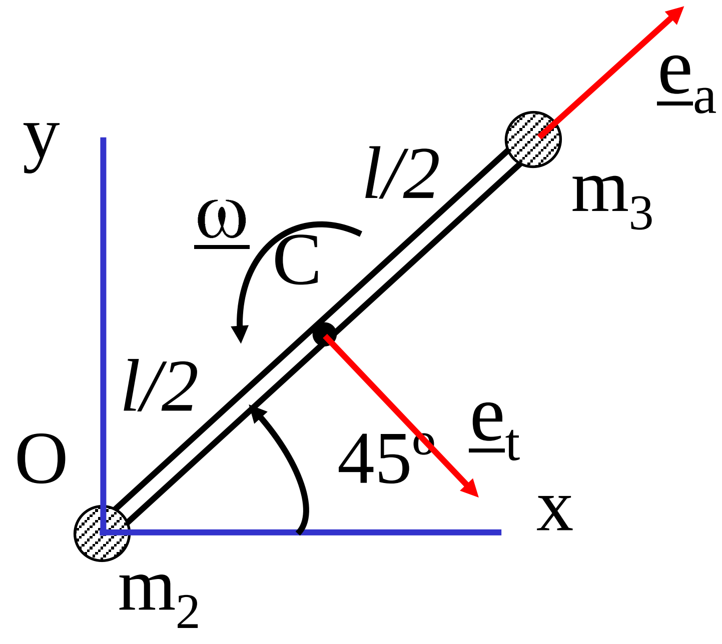
- **Resultant moment about O (a fixed point)= 0**

$\rightarrow$ **angular momentum about O conserved for the system**

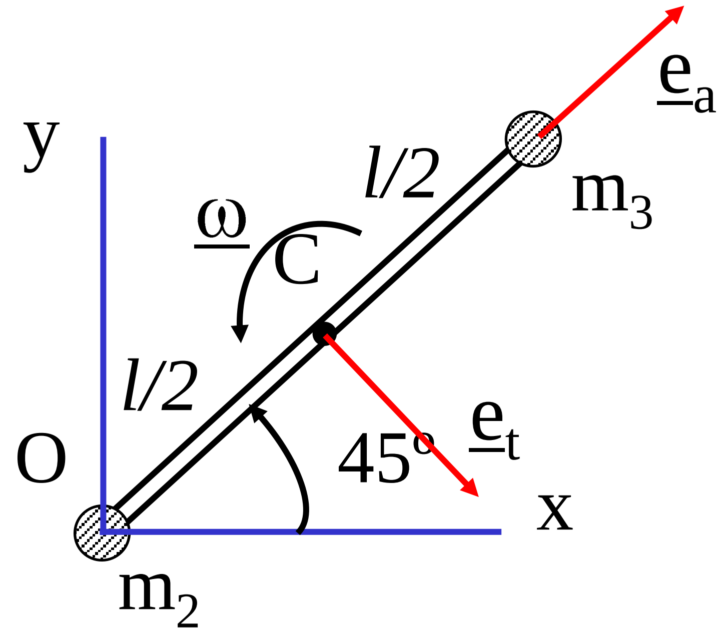
Set up of the problem:

Motion before impact: $\underline{v}_1 = v\underline{i}; \quad \underline{v}_2 = \underline{v}_3 = 0$

Motion after impact: It is convenient to think in terms of the motion of the CM, and rotational motion about CM. Use the triad $(\underline{e}_t, \underline{e}_a, \underline{e}_b)$ **to define the motion of the CM and the particles.**



Expressing velocities in terms of $(\underline{e}_t, \underline{e}_a, \underline{e}_b)$



$$\underline{v}_C = v_a \underline{e}_a + v_t \underline{e}_t$$

$$\underline{v}_2 = \underline{v}_C + \omega \underline{k} \times \underline{r}_{CO}$$

$$= \underline{v}_C + \omega \underline{k} \times (-l \underline{e}_a / 2)$$

$$= \underline{v}_C + (\omega l \underline{e}_t / 2)$$

$$\rightarrow \boxed{\underline{v}_2 = v_a \underline{e}_a + (v_t + \omega l / 2) \underline{e}_t}$$

Similarly, $\underline{v}_3 = \underline{v}_C + \omega \underline{k} \times \underline{r}_{C3} = \underline{v}_C - (\omega l / 2) \underline{e}_t$

$$\rightarrow \boxed{\underline{v}_3 = v_a \underline{e}_a + (v_t - \omega l / 2) \underline{e}_t}$$

linear momentum conserved for the system:

$$mv\underline{i} = mv_1\underline{i} + m\underline{v}_2 + m\underline{v}_3$$

$$or \quad v\underline{i} = v_1\underline{i} + 2v_a\underline{e}_a + 2v_t\underline{e}_t \qquad (1)$$

(a vector equation → 2 scalar equations)

angular momentum conserved for the system:

$$0 = \underline{r}_{O3} \times \underline{v}_3 = l\underline{e}_a \times [v_a\underline{e}_a + (v_t - \omega l/2)\underline{e}_t]$$

$$= -l(v_t - \omega l/2)\underline{k} \quad \rightarrow \quad \boxed{v_t = \omega l/2} \qquad (2)$$

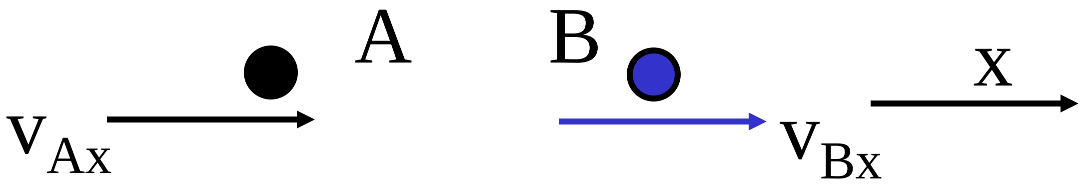
Note: The vectors $\underline{e}_t$ and $\underline{e}_a$ **can be expressed in terms of the unit vectors** $\underline{i}$ **and** $\underline{j}$ **as:**

$$\underline{e}_t = \underline{i}\cos 45° - \underline{j}\sin 45° = (\underline{i} - \underline{j})/\sqrt{2}$$

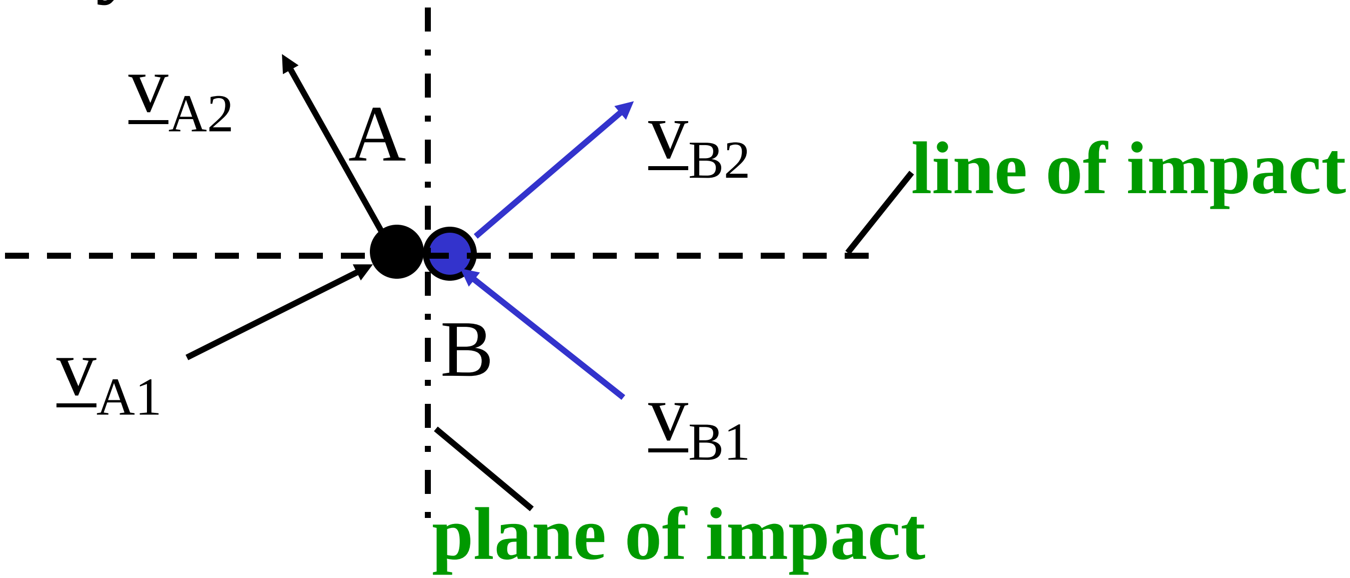
$$\underline{e}_a = \underline{i}\cos 45° + \underline{j}\sin 45° = (\underline{i} + \underline{j})/\sqrt{2}$$

→ **In equations (1) and (2),** v_1 , v_t , v_a , ω **are unknowns but there are only 3 equations. Thus one more relation is required.**

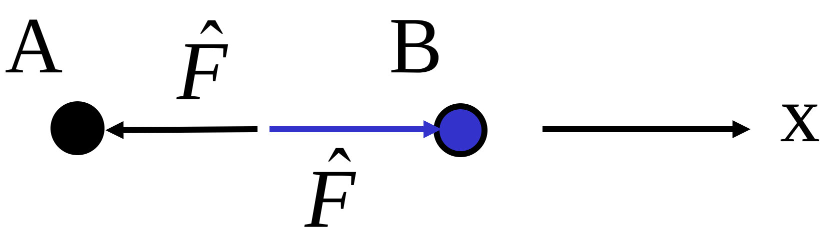
- **coefficient of restitution:**



Aside: central impact: Consider two particles A and B that collide with each other. The geometry and definitions of terms are:



FBDs:



No y-comp of force

Let $\underline{v}_A = v_{Ax}\underline{i} + v_{Ay}\underline{j}$ and $\underline{v}_B = v_{Bx}\underline{i} + v_{By}\underline{j}$ be velocities; $\underline{v}_{A1}, \underline{v}_{B1}$ before, and $\underline{v}_{A2}, \underline{v}_{B2}$ after. The coefficient of restitution is then defined as: the ration of the relative velocity after impact to the relative velocity before impact, for velocities along the line of impact:

$$e = -\frac{(v_{Bx2} - v_{Ax2})}{(v_{Bx1} - v_{Ax1})} = -\frac{(v_{Bx} - v_{Ax})_2}{(v_{Bx} - v_{Ax})_1}$$

- **For the system at hand, elastic impact: e = 1. Also,**

$$v_{Ax1} = v,\ v_{Ay1} = v_{Ay2} = 0,\ v_{Bx1} = 0,\ v_{Ax2} = v_1,$$

$$v_{Bx2} = (v_a + 2v_t)/\sqrt{2}, v_{By1} = v_{By2} = 0.\ \text{Thus}$$

$$e = 1 = -\frac{((v_a + 2v_t)/\sqrt{2} - v_1)}{(0 - v)}$$

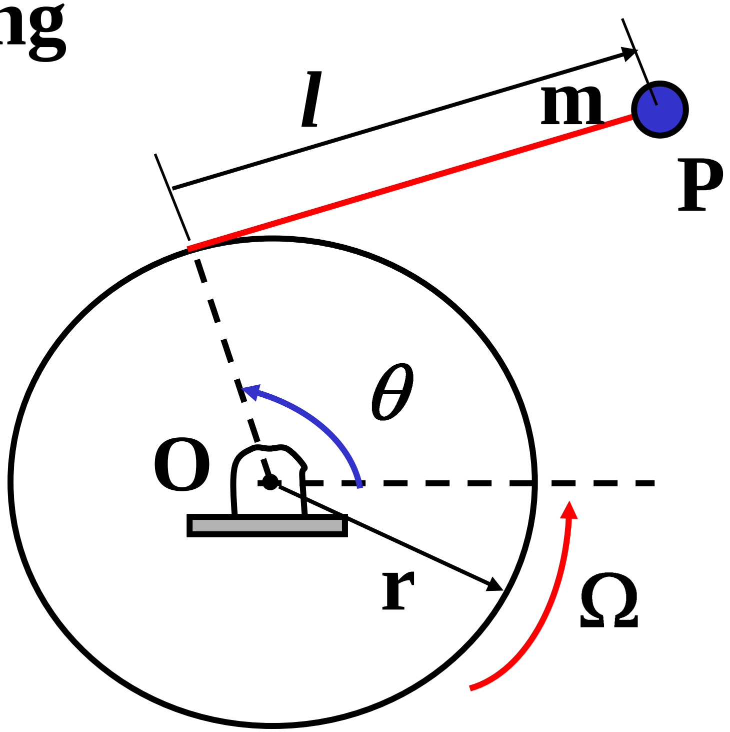
$$\rightarrow \quad v = v_a/\sqrt{2} + \sqrt{2}v_t - v_1 \qquad (3)$$

Solving (1), (2), and (3) $\rightarrow$

$$\boxed{v_a = (2\sqrt{2}/7)v,\ \ v_t = (2\sqrt{2}/7)v,\ v_1 = -v/7}$$

Ex. (Problem 3.19)

Consider a cylinder rotating at a constant rate Ω.



- **A thin, flexible and massless rope goes around the drum.**
- **There is no gravity, and the rope does not slip relative to the drum ;**

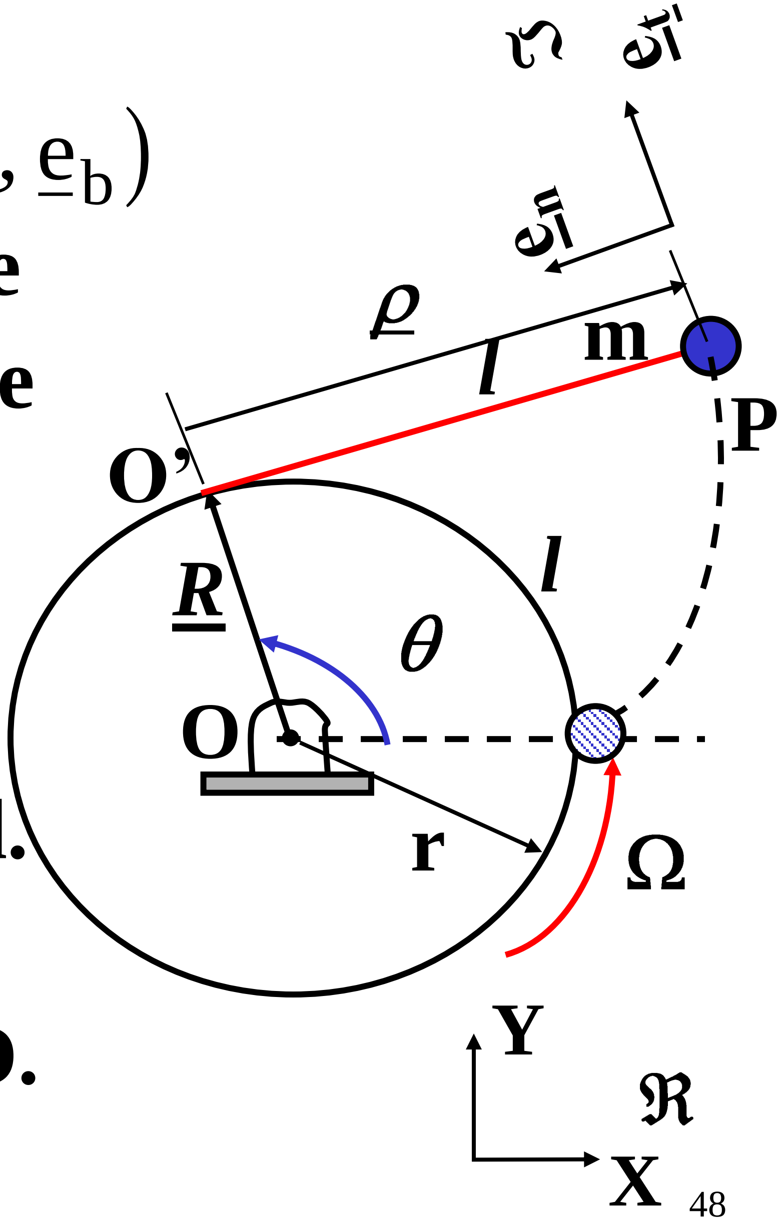
At $t = 0 \, , \, \ell(0) = 0 \, , \, \dot{\ell}(0) = r\Omega$

Find: Tension in the rope as a function of time.

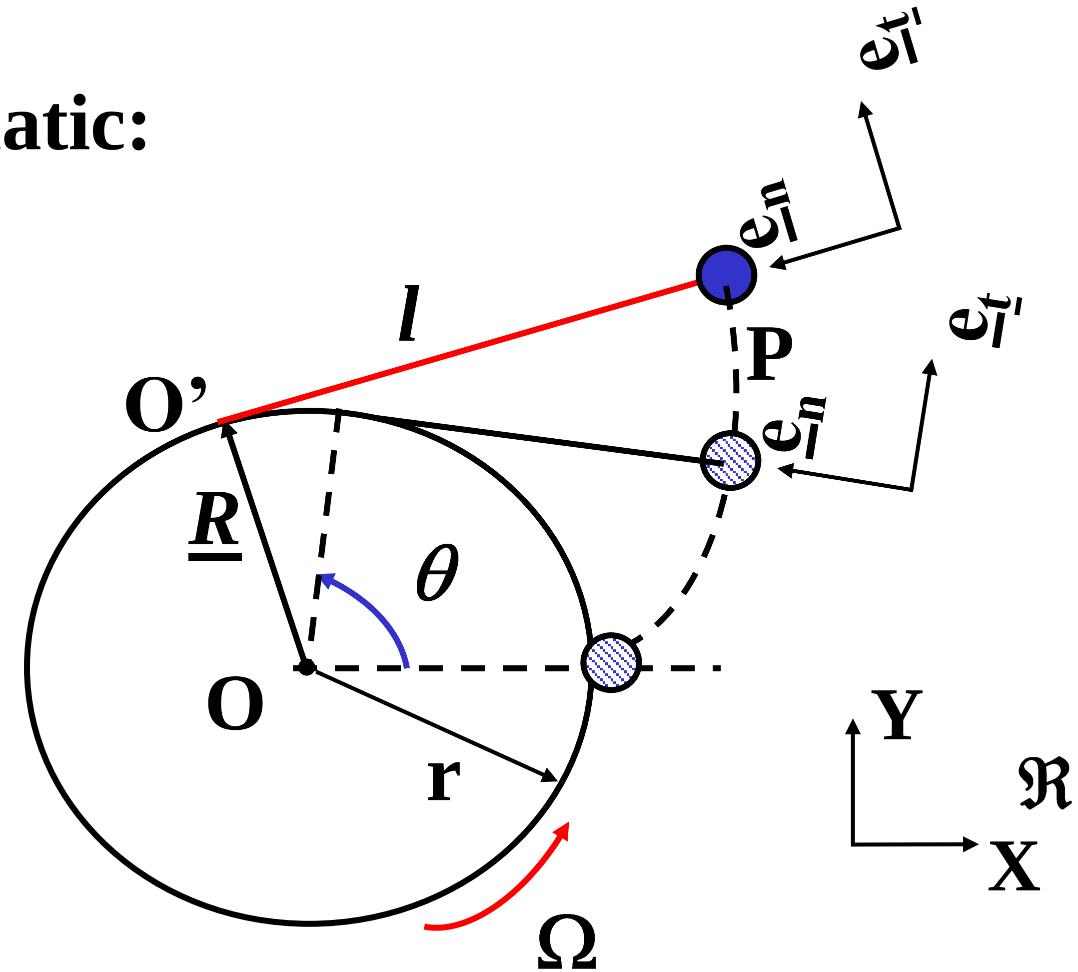
Setup:

Consider a triad $(\underline{e}_t , \underline{e}_n , \underline{e}_b)$ **for the moving reference frame** $\mathfrak{I}$**, with coordinate system located at** O'.

- **Let** $\underline{\omega}$ **be angular velocity of the moving reference frame or triad.**
- **The fixed reference frame is** $\mathfrak{R}$ **with origin O.**



Schematic:



$(\underline{e}_t, \underline{e}_n, \underline{e}_b)$**- triad for moving coordinate system**

Let $\underline{\omega}$ be angular velocity of the moving frame.

•**Use the general formulation to express $\underline{\mathbf{a}}_{\mathbf{P}}$:**

$$\underline{a}_P = \ddot{\underline{R}} + \dot{\underline{\omega}} \times \underline{\rho} + \underline{\omega} \times (\underline{\omega} \times \underline{\rho}) + (\ddot{\underline{\rho}})_r + 2\underline{\omega} \times (\dot{\underline{\rho}})_r$$

Let us now consider the various terms:

$$\underline{\omega} = (\Omega + \dot{\theta})\underline{e}_b \quad \text{and} \quad l = r\theta$$

$$\rightarrow \quad \dot{l} = r\dot{\theta} \quad \rightarrow \quad \dot{\theta} = \dot{l} / r$$

$$\rightarrow \quad \boxed{\underline{\omega} = (\Omega + \dot{l} / r)\underline{e}_b}, \quad \rightarrow \quad \boxed{\dot{\underline{\omega}} = (\ddot{l} / r)\underline{e}_b}$$

$$\text{Position}: \ \underline{R} = r\underline{e}_t, \quad \dot{\underline{R}} = r\frac{d\underline{e}_t}{dt} = r\underline{\omega} \times \underline{e}_t$$

$$\rightarrow \quad \dot{\underline{R}} = r(\Omega + \dot{l} / r)\underline{e}_b \times \underline{e}_t = r(\Omega + \dot{l} / r)\underline{e}_n$$

$$\text{Also} \ \ \ddot{\underline{R}} = r(\ddot{l} / r)\underline{e}_n + r(\Omega + \dot{l} / r)\underline{\omega} \times \underline{e}_n$$

Thus $\boxed{\underline{\ddot{R}} = (\ddot{l})\underline{e}_n - r(\Omega + \dot{l}/r)^2 \underline{e}_t}$

Now, $\underline{\rho} = -l\underline{e}_n \rightarrow \boxed{(\dot{\underline{\rho}})_r = -\dot{l}\underline{e}_n} \rightarrow \boxed{(\ddot{\underline{\rho}})_r = -\ddot{l}\underline{e}_n}$

Also $\dot{\underline{\omega}} \times \underline{\rho} = (\ddot{l}/r)\underline{e}_b \times (-l\underline{e}_n) = (l\ddot{l}/r)\underline{e}_t$

$$\underline{\omega} \times (\underline{\omega} \times \underline{\rho}) = l(\Omega + \dot{l}/r)^2 \underline{e}_n$$

$$2\underline{\omega} \times (\dot{\underline{\rho}})_r = 2\dot{l}(\Omega + \dot{l}/r)\underline{e}_t$$

$$\rightarrow \boxed{\begin{aligned} \underline{a}_P = {} & [-r(\Omega + \dot{l}/r)^2 + l\ddot{l}/r + 2\dot{l}(\Omega + \dot{l}/r)]\underline{e}_t \\ & + l(\Omega + \dot{l}/r)^2 \underline{e}_n \end{aligned}}$$

Imp: The is acceleration relative to $\Re$

Now, applying Newton's Second Law:

$$\underline{F} = m\underline{a} \rightarrow T\underline{e}_n = m\underline{a}$$

$$\underline{e}_t : -r(\Omega + \dot{l}/r)^2 + l\ddot{l}/r + 2\dot{l}(\Omega + \dot{l}/r) = 0$$

or $$\boxed{l\ddot{l} + \dot{l}^2 - r^2\Omega^2 = 0}$$

$$\rightarrow \quad d(l\dot{l})\big/dt = r^2\Omega^2 \;\rightarrow\; d(l\dot{l}) = r^2\Omega^2 dt$$

Initial conditions: $\ell(0) = 0\,,\ \dot{\ell}(0) = \mathrm{r}\Omega$

Integration $\rightarrow$ $\boxed{\ell\dot{\ell} = r^2\Omega^2 t}$

Integrating once again,

$$\int_0^l \ell \, d\ell = r^2\Omega^2 \int_0^t \tau \, d\tau \rightarrow l^2 = r^2\Omega^2 t^2 \rightarrow \boxed{l = r\Omega t}$$

$$\underline{e}_n: \quad T = ml(\Omega + \dot{l}/r)^2 = 4ml\Omega^2$$

$$\text{or} \quad \boxed{T = 4mr\Omega^3 t}$$