CHAPTER 3
DYNAMICS OF A PARTICLE

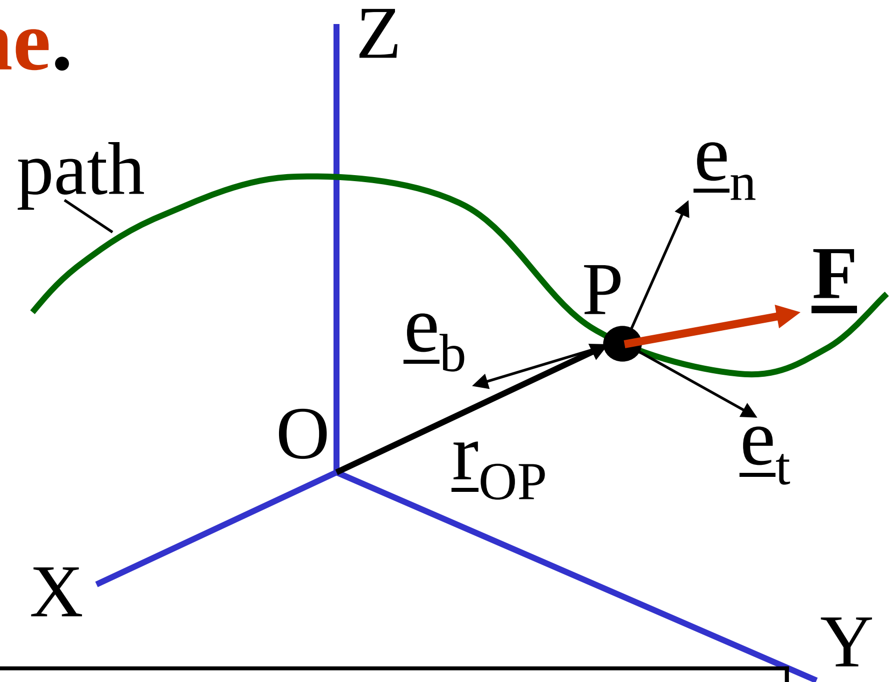
Newton's Second Law: It is an experimentally derived law, valid in a reference frame – Inertial reference frame.

XYZ - inertial reference frame

Let *m* be mass,

$\underline{r}_{OP}$- position vector.

Then



$$\underline{F} = \underline{F}(\underline{r}, \dot{\underline{r}}, t), \text{and} \quad \boxed{\underline{F}(\underline{r}, \dot{\underline{r}}, t) = d(m\dot{\underline{r}}_P)/dt = m\underline{a}_P}$$

3.1 Direct Integration of Equations of Motion

Newton's Law gives: $\underline{F}(\underline{r},\dot{\underline{r}},t)=m\underline{a}_P$

This needs to be solved, subject to

initial conditions: $\underline{r}(t=t_0)=\underline{r}_0;\ \dot{\underline{r}}(t=t_0)=\dot{\underline{r}}_0$

Case 1: The external force is a constant.

In Cartesian coordinate system

$$m\ddot{x}=F_x,\quad m\ddot{y}=F_y,\quad m\ddot{z}=F_z$$

Consider the system in x-direction :

$$x=F_x/m-\text{constant}\,(\text{say }'a')\rightarrow d(\dot{x})/dt=a$$

Integrating first by "separation of variables":

$$\int_{v_0}^{v} d(\dot{x}) = \int_{t=0}^{t} a\, d\tau \rightarrow v(t) - v_0 = at$$

or, $$\boxed{v(t) = v_0 + at}$$ **(speed vs. time)**

Integrating again: $$\int_{x_0}^{x} d(u) = \int_{t=0}^{t} (v_0 + a\tau)\, d\tau$$

or $$\boxed{x(t) = x_0 + v_0 t + at^2/2}$$ **(position vs. time)**

One can also approach the integration with position as the independent variable:

Let

$$\frac{d}{dt}(\dot{x}) = \frac{d(\dot{x})}{dx}\frac{d(x)}{dt} \quad (\text{chain rule}) = \dot{x}\frac{d(\dot{x})}{dx} = \frac{d(\dot{x}^2/2)}{dx}$$

$$\text{Then, Newton's 2nd Law} \rightarrow \quad \frac{d(\dot{x}^2/2)}{dx} = a$$

$$\text{Separation of variables} \rightarrow \int_{v_0}^{v} d(\dot{x}^2/2) = \int_{x_0}^{x} a\,du$$

$$or, \quad \boxed{v^2 = v_0^2 + 2a(x - x_0)}$$ **(position vs. speed)**

Reading Assignment: Motion of a particle in a uniform gravitational field.

Case 2: The external force is a function of time.

Then, in Cartesian coordinates,

$$m\ddot{x} = F_x(t),\ \ m\ddot{y} = F_y(t),\ m\ddot{z} = F_z(t).$$

$$\int_{v_0}^{v} d(\dot{x}) = \int_{t_0=0}^{t} (F_x(\tau)/m)\,d\tau \rightarrow \boxed{v(t) = v_0 + \int_{t_0=0}^{t} (F_x(\tau)/m)\,d\tau}$$

Similarly, $d(x) = \{v_0 + \int_{t_0=0}^{t} (F_x(\tau)/m)\,d\tau\}\,d\tau$

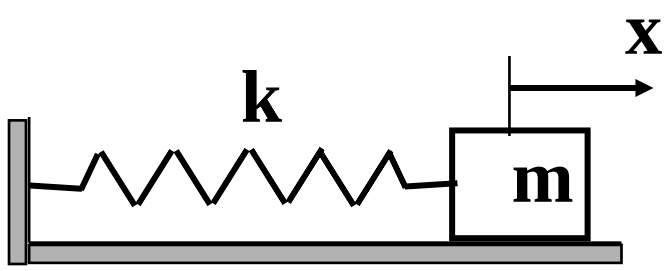
$$\rightarrow \quad \boxed{x(t) = x_0 + v_0 t + \int_0^t \Big(\int_{t_0=0}^{\tau} (F_x(s)/m)\,ds \Big) d\tau}$$

Case 3: The force is a function of position.

Special case: $\underline{F}(\underline{r}) = F_x(x)\underline{i} + F_y(y)\underline{j} + F_z(z)\underline{k}$

e.g. (example 3.3) $F_x = -kx$

(linear, separable function)



Equation of motion :

$$m\ddot{x} = -kx$$

Integration : $m\int_{v_0}^{v} d(\dot{x}^2/2) = -\int_{x_0}^{x} kudu$

$$\rightarrow \boxed{m(v^2 - v_0^2)/2 = -k(x^2 - x_0^2)/2}$$

or $v^2 = v_0^2 - k(x^2 - x_0^2)/m$ **(position vs. speed)**

Then, to integrate again, we write as

$$dx/dt = [v_0^2 - k(x^2 - x_0^2)/m]^{1/2}$$

or $$\int_0^t d\tau = \int_{x_0}^{x} [v_0^2 - k(u^2 - x_0^2)/m]^{1/2} du$$

$$\rightarrow t = \sqrt{k/m}[\sin^{-1}\frac{x}{\sqrt{(m/k)v_0^2 + x_0^2}} - \sin^{-1}\frac{x_0}{\sqrt{(m/k)v_0^2 + x_0^2}}]$$

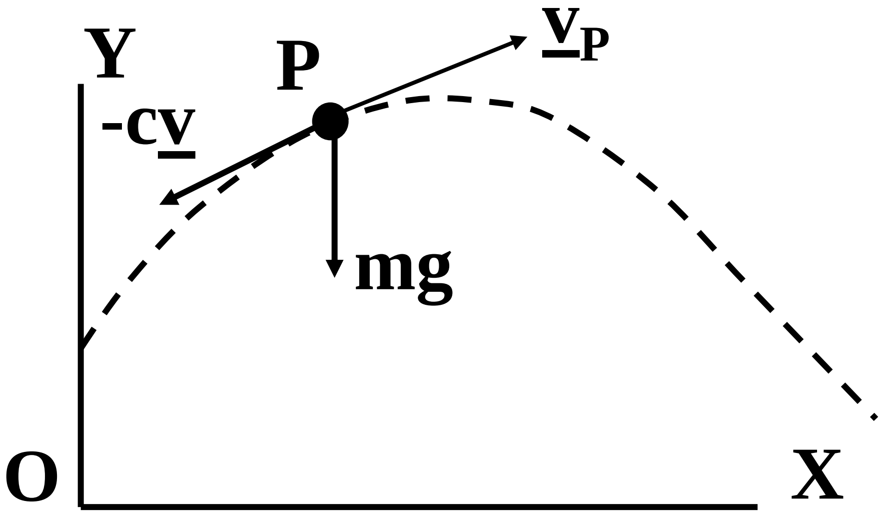
or $$\boxed{x(t) = \sqrt{(m/k)v_0^2 + x_0^2}\,\sin(\sqrt{k/m}\,t + \alpha)}$$

where $\alpha = \sin^{-1}(x_0/\sqrt{(m/k)v_0^2 + x_0^2})$ (position vs. time)

Case 4: The force is a function of velocity.

Special case: $\underline{F}(\dot{\underline{r}}) = F_x(\dot{x})\underline{i} + F_y(\dot{y})\underline{j} + F_z(\dot{z})\underline{k}$

Ex. 3.4: Projectile with air drag



drag force ~ velocity

$$\underline{F} = -mg\,\underline{j} - c\underline{v}$$

where $\underline{v} = \dot{x}\underline{i} + \dot{y}\underline{j}$

$\underline{\underline{i}}: \quad -c\dot{x} = m\ddot{x} \quad (1) \qquad \underline{\underline{j}}: \quad -c\dot{y} - mg = m\ddot{y} \quad (2)$

x-motion: $-c\dot{x} = m\ddot{x}$

initial conditions: $x(t=0) = x_0, \;\; \dot{x}(t=0) = \dot{x}_0$

Integrating:

$$d(\dot{x})/dt = -(c/m)\dot{x}$$

$$\int_{\dot{x}_0}^{\dot{x}} d(\dot{u})/\dot{u} = -\int_0^t (c/m)d\tau = -(c/m)t$$

$$\rightarrow \quad \ln(\dot{x}/\dot{x}_0) = -(c/m)t \rightarrow \boxed{\dot{x}(t) = \dot{x}_0 e^{-(c/m)t}}$$

Integrating again : $\displaystyle\int_{x_0}^{x} du = \int_0^t \dot{x}_0 e^{-(c/m)\tau} d\tau$

or $x(t) - x_0 = -(m/c)\dot{x}_0 e^{-(c/m)t}\Big|_0^t = (m/c)\dot{x}_0[1 - e^{-(c/m)t}]$

$$\rightarrow \quad \boxed{x(t) = x_0 + (m/c)\dot{x}_0[1 - e^{-(c/m)t}]}$$

y-motion: $m\ddot{y} + c\dot{y} = -mg$

initial conditions: $y(t=0) = y_0, \quad \dot{y}(t=0) = \dot{y}_0$

Integrating: $\int_{\dot{y}_0}^{\dot{y}} \frac{md(\dot{u})}{c\dot{u} + mg} = -\int_0^t d\tau = -t$

$$or \quad t = -(m/c)\int_{\dot{y}_0}^{\dot{y}} \frac{d(\dot{u})}{\dot{u} + (mg/c)} = -(m/c)\ln\left[\frac{\dot{y} + (mg/c)}{\dot{y}_0 + (mg/c)}\right]$$

or $\boxed{\dot{y}(t) = -(mg/c) + [\dot{y}_0 + (mg/c)]e^{-(c/m)t}}$

Integrating again →

$$\boxed{\begin{aligned} y(t) &= y_0 - (mg/c)t \\ &\quad + (m/c)\{\dot{y}_0 + (mg/c)\}[1 - e^{-(c/m)t}] \end{aligned}}$$

Summarizing: $x(t) = x_0 + (m/c)\dot{x}_0[1 - e^{-(c/m)t}]$

$$\dot{x}(t) = \dot{x}_0 e^{-(c/m)t}$$

$$as \;\; t \to \infty, \; \dot{x}(t) \to 0, \; x(t) \to x_0 + m\dot{x}_0 / c$$

(limiting x-displacement)

$$of \;\; c = 0, \quad \dot{x}(t) = \dot{x}_0 \;\; (remains\, constant)$$

***x(t)* $\rightarrow \infty$ *as t* $\rightarrow \infty$ (unbounded x-displacement)**

$$\dot{y}(t) = -(mg/c) + [\dot{y}_0 + (mg/c)]e^{-(c/m)t}$$

$$y(t) = y_0 - (mg/c)t$$

$$+ (m/c)\{\dot{y}_0 + (mg/c)\}[1 - e^{-(c/m)t}]$$

as *t* $\rightarrow \infty$, $\dot{y}(t) \rightarrow -(mg/c)$ **(terminal speed)**

$y(t) \rightarrow -\infty$ **(unbounded)**

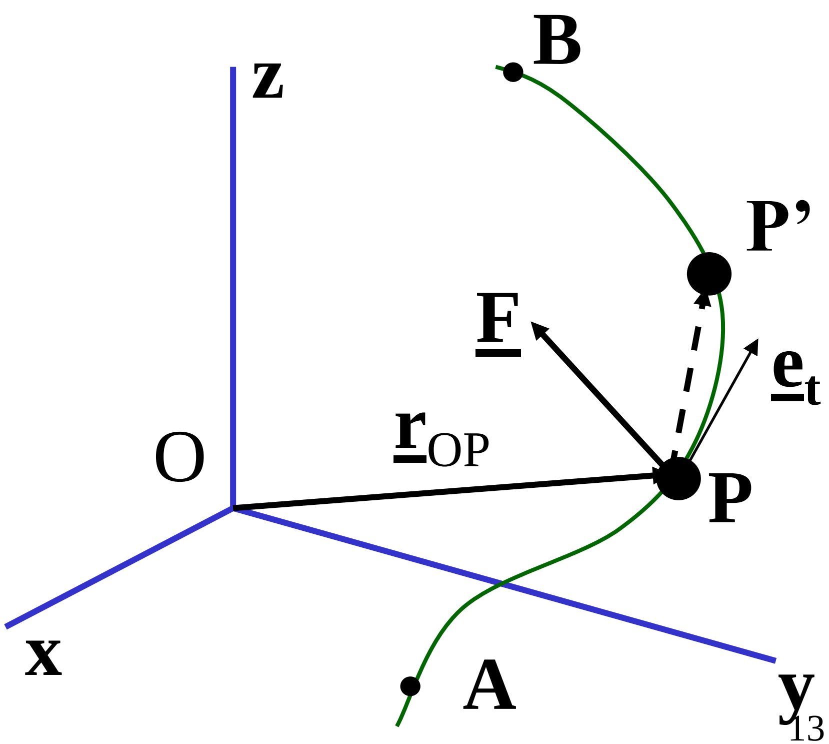
if c=0, $\dot{y}(t) \rightarrow -\infty,\ y(t) \rightarrow -\infty$. **(both unbounded)**

3.2 Work and Kinetic Energy

Imp: **for a particle, the work-energy approach is derivable from Newton's law for the particle, and gives no new information; it provides further insight.**

Consider a particle moving along a path, starts at A, goes to B.

Let, when at position P, a force $\underline{F}$ act on the particle.



***d*r̲** - small change in position;

$\underline{F} = m\ddot{\underline{r}}$ **Newton's 2nd Law**

Defn: Work done by the force acting on the particle in a small (infinitesimal) displacement is: ***dW*=*F̲*•*d*r̲*.**

Dot product with Newton's law $\rightarrow \int_{\underline{r}_A}^{\underline{r}_B} \underline{F} \bullet d\underline{r}$

$$= \int_A^B m\ddot{\underline{r}} \bullet d\underline{r} = m\int_A^B [d(\dot{\underline{r}} \bullet \dot{\underline{r}})/2dt]dt = m\int_A^B d(v^2)/2$$

$$\rightarrow \int_{\underline{r}_A}^{\underline{r}_B} \underline{F} \bullet d\underline{r} = W_{A\rightarrow B} = m\int_A^B d(v^2)/2 = m(v_B^2 - v_A^2)/2$$

Let $T \equiv mv^2/2 = m\underline{v} \bullet \underline{v}/2$ - **kinetic energy of the particle**

$\rightarrow W_{A\rightarrow B} = T_B - T_A \equiv m(v_B^2 - v_A^2)/2$ **principle of work and kinetic energy**

3.3 Conservative Forces:

Suppose that the force $\underline{F}$ acting is such that

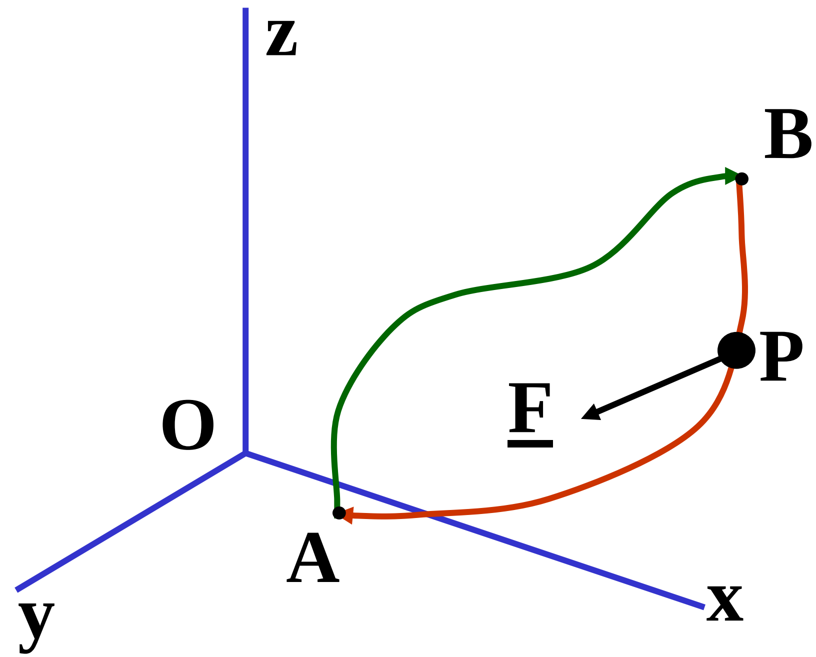
1) it is a single-valued function only of position, that is, $\underline{F}$ does not explicitly depend on t;

2) the line integral $\int_A^B \underline{F} \cdot d\underline{r}$ only depends on end points

$$\rightarrow \oint \underline{F} \bullet d\underline{r} = 0$$

One then says that:

The force is conservative or the mechanical process is reversible



- $dW = \underline{F} \bullet d\underline{r}$ **must be an exact differential**

$= -d(V)$ **(- ve sign is for convenience)**

V **- potential energy associated with the force.**

Then, $$W_{A\rightarrow B} = \int_A^B dW = \int_A^B \underline{F} \bullet d\underline{r} = -\int_A^B dV = V_A - V_B$$

- **Decrease in potential energy in moving the particle from A to B equals the work done on the particle.**

Let E = T + V; it is called the total energy.

If the only force acting on the particle is a conservative force: $W_{A\to B} = V_A - V_B = T_B - T_A$

$$\rightarrow \quad \boxed{T_A + V_A = T_B + V_B}$$

principle of conservation of mechanical energy

3.4 Potential Energy

Recall: for a force dependent only on position, if a potential function exists: V depends only on position - V = V(x, y, z)   (in Cartesian system)

$$dV = (\partial V/\partial x)dx + (\partial V/\partial y)dy + (\partial V/\partial z)dz$$

For the work done by the force:

$$\underline{F} = F_x \underline{i} + F_y \underline{j} + F_z \underline{k},\ d\underline{r} = dx\,\underline{i} + dy\underline{j} + dz\underline{k}$$

$$\rightarrow dW = \underline{F} \bullet d\underline{r} = -dV \rightarrow F_x = -\frac{\partial V}{\partial x}, F_y = -\frac{\partial V}{\partial y}, F_z = -\frac{\partial V}{\partial z}$$

$$\rightarrow \boxed{\underline{F} = -\frac{\partial V}{\partial x}\underline{i} - \frac{\partial V}{\partial y}\underline{j} - \frac{\partial V}{\partial z}\underline{k} \equiv -\nabla V} \ (gradient\ of\ V)$$

Ex: Inverse-square law of attraction:

The force exerted by the attracting field is radial:

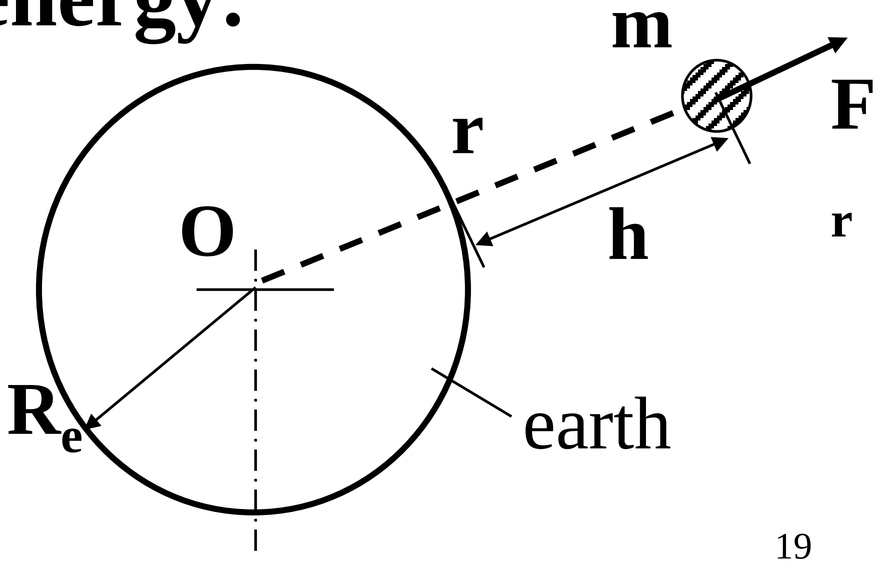
$$F_r = -\frac{\partial V}{\partial r} = -\frac{K}{r^2}$$

$$\rightarrow V = -\frac{K}{r} + C \quad (may\ choose\ C = 0)$$

gravitational potential energy:

$$F_r = -K/r^2;\ r \geq R_e$$

$$weight = w = mg_0 = K/r^2$$



$$K = mg_o R^2 \quad \rightarrow \quad F_r = -\frac{mg_o R^2}{r^2} \quad (r \geq R)$$

$$V = -mg_o R^2 / r \quad (r \geq R)$$

(potential energy due to the gravitational field)

Now, r = R+h → $V = -mg_o R / \{1 + (h/R)\}$

- **Near earth's surface h < < R → h/R < < 1.**

$$V \simeq -\, mg_o R\{1 - (h/R)\}$$

- **If we choose constant C (reference) so that the potential energy at the surface is zero →** $V \simeq mg_o h.$

Ex: Potential energy of a linear spring:

The force in the spring is

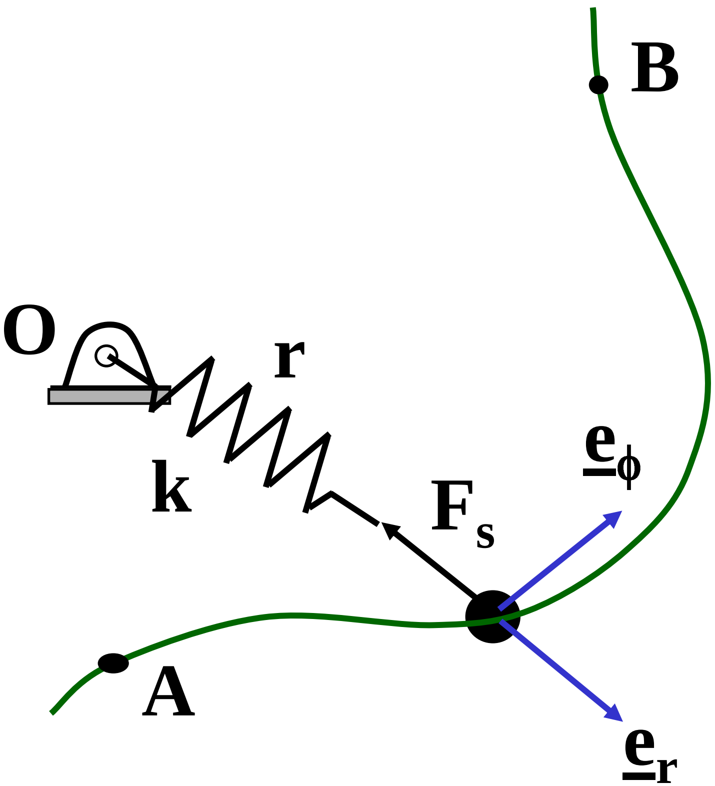
$$\underline{F}_s = -F_s \underline{e}_r = -k(r - l_0)\underline{e}_r$$

The distance moved by the particle is :

$$d\underline{r} = dr\underline{e}_r + rd\phi\underline{e}_\phi$$

$\rightarrow$ The work done is :

$$dW = \underline{F}_s \bullet d\underline{r} = -k(r - l_0)dr$$



$$W_{A\to B} = -\int_A^B k(r - l_0)dr = [k(r_A - l_0)^2 - k(r_B - l_0)^2]/2$$

- **V = elastic energy or spring potential energy** $\equiv \frac{1}{2}k\Delta^2$, $\Delta = (r - l_o)$ **- stretch in the spring**

Reading Assignment Ex: 3.6

General form of work-energy principle:

$$W_{A\to B} = V_A - V_B + W^{nc}_{A\to B} = T_B - T_A$$

or

$$\boxed{T_A + V_A + W^{nc}_{A\to B} = T_B + V_B}$$

3.5 Linear Impulse and Momentum:

Newton's second law $\rightarrow$ $$\underline{F} = \frac{d}{dt}(m\underline{v}_P) = \frac{d}{dt}(\underline{p})$$

where $\underline{p} \equiv m\underline{v}_P$ – linear momentum of the particle

Newton's law used → the velocity is measured relative to an inertial frame.

Suppose that $\underline{\mathrm{F}}$ is given as a function of time.

Integrating: $$\int_{t_1}^{t_2} \underline{F}(\tau)d\tau = \int_{t_1}^{t_2} d(\underline{p}(\tau)) = \underline{p}(t_2) - \underline{p}(t_1)$$

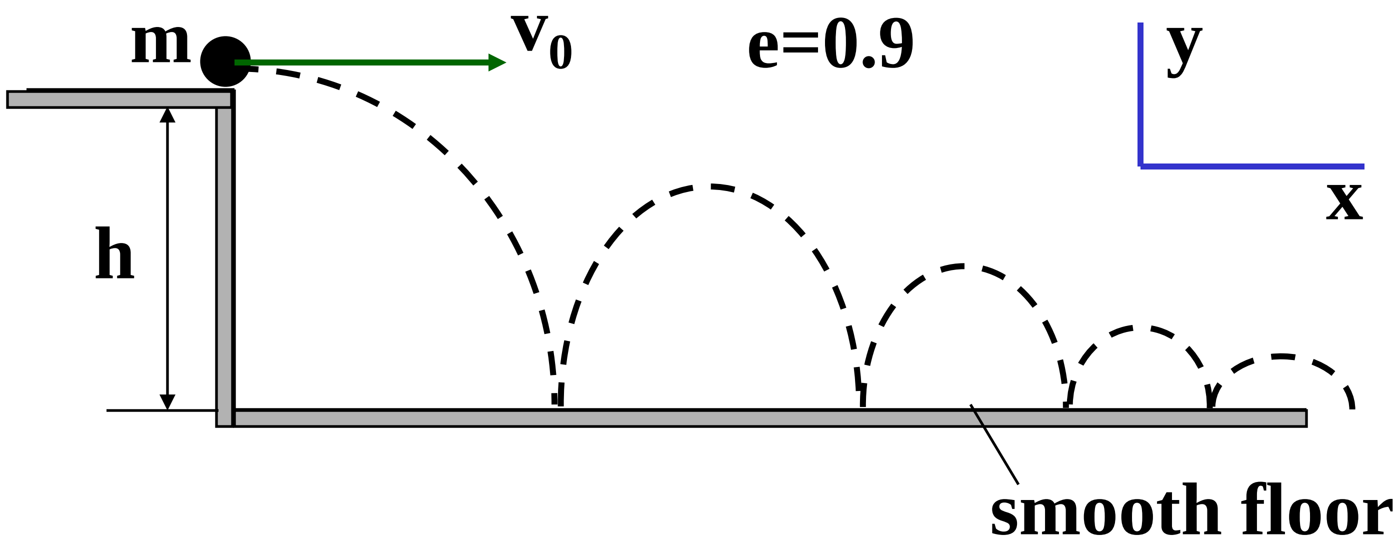
$$\rightarrow \boxed{\hat{\underline{F}} \equiv \int_{t_1}^{t_2} \underline{F}(\tau)d\tau = \underline{p}(t_2) - \underline{p}(t_1)}$$

$\hat{\underline{F}}$ - **impulse of the force** $\underline{F}$ **over the time interval** $(t_1 - t_2)$.

- **The change in linear momentum of a particle during a given time interval equals the total impulse (linear) of the forces acting on the particle.**

- **When the time interval of action is very small, the force is called an impulsive force. Then** $\hat{\underline{F}} = \int_{t_1}^{t_2} \underline{F}(t)dt$ **is finite even though** $(t_2 - t_1) \rightarrow 0$. **Ex. Forces during impact.**

Ex: (Example 4.5)



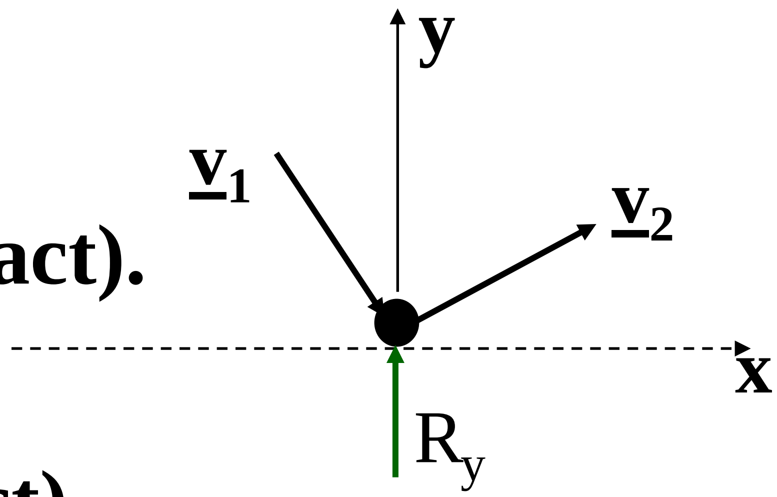
Find: Total horizontal distance x_{tot} till the ball continues to bounce; Also, the total time taken.

- **Key - consideration of the impact with the floor and the impulsive action. It needs to be used repeatedly.**

FBD during an impact:

$$\underline{v}_1 = v_{1x}\underline{i} + v_{1y}\underline{j}$$

(velocity just before impact).



$$\underline{v}_2 = v_{2x}\underline{i} + v_{2y}\underline{j}$$

(velocity just after impact).

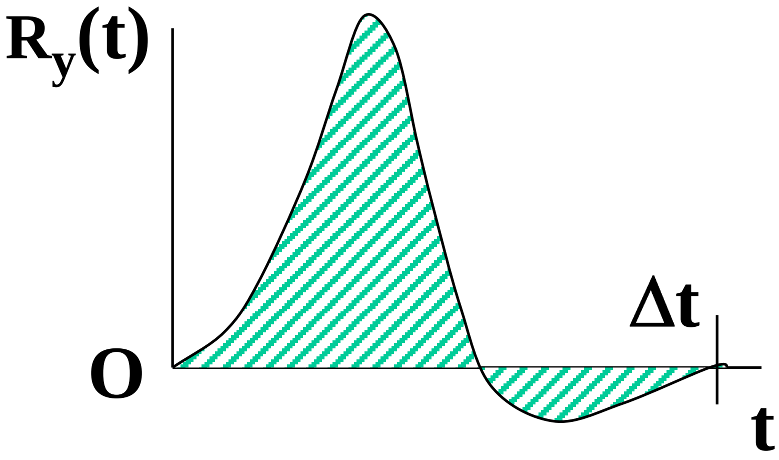
Impact → Δt → 0.

Applying Impulse-momentum principle:

$$\sum F_x = 0 \rightarrow mv_{2x} = mv_{1x} \rightarrow v_{2x} = v_{1x} = v_0$$

$$\sum F_y = R_y \neq 0 \rightarrow -mv_{1x} + \int_{\Delta t} R_y d\tau = mv_{2x}$$

- **Nature of the impulsive force R_y not known**



$$\int_{\Delta t} R_y(\tau)d\tau =$$ area under the curve

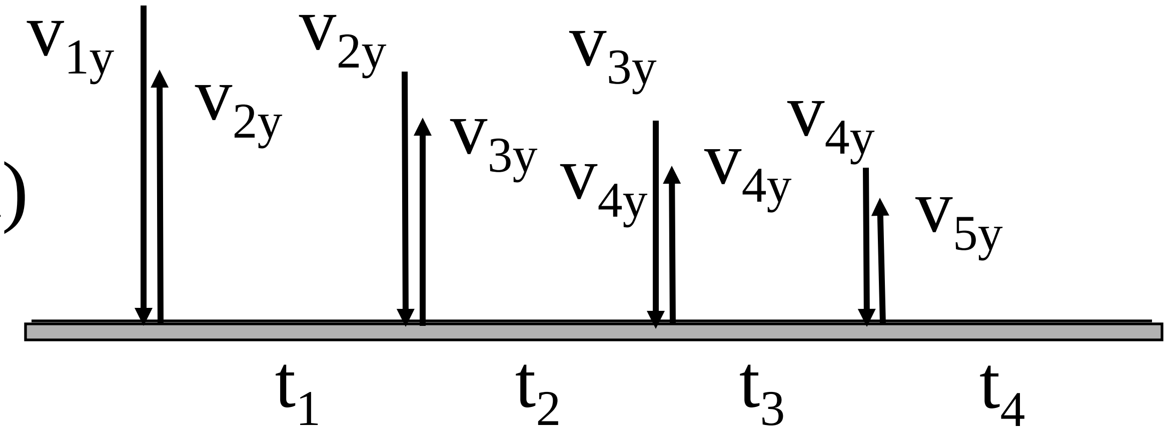
- **coefficient of restitution**

$$e = \frac{|v_{2y}|}{|v_{1y}|} \rightarrow |v_{2y}| = e|v_{1y}|$$ relates velocities in direction of impact

$$\text{In general: } |v_{(n+1)y}| = e|v_{ny}|$$ during nth impact

Schematic representation of motion:

$V_{1y} = \sqrt{(2gh)}$



Now,

$$v_{2y} = e v_{1y},\ v_{3y} = e v_{2y} = e^2 v_{1y} = e^2 \sqrt{2gh}$$

$$v_{4y} = e v_{3y} = e^3 \sqrt{2gh}, ...$$

Time of flight: t_n **= time between** n^{th} **and** $(n+1)^{th}$ **bounce**

$$t_n = 2 v_{(n+1)y} / g = 2 e v_{ny} / g$$

$$t_0 = \text{time to hit the floor 1st time} = \sqrt{2gh}$$

Total time it takes before ball stop bouncing:

$$T = t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + \ldots\ldots\ldots\ldots$$

$$= t_0 + 2ev_{1y}/g + 2ev_{2y}/g + 2ev_{3y}/g + 2ev_{4y}/g + \ldots$$

$$= \sqrt{2h/g} + 2e\sqrt{2h/g} + 2e^2\sqrt{2h/g} + 2e^3\sqrt{2h/g} + \ldots$$

$$= \sqrt{2h/g}\{1 + 2e + 2e^2 + 2e^3 + \ldots\ldots\ldots\}$$

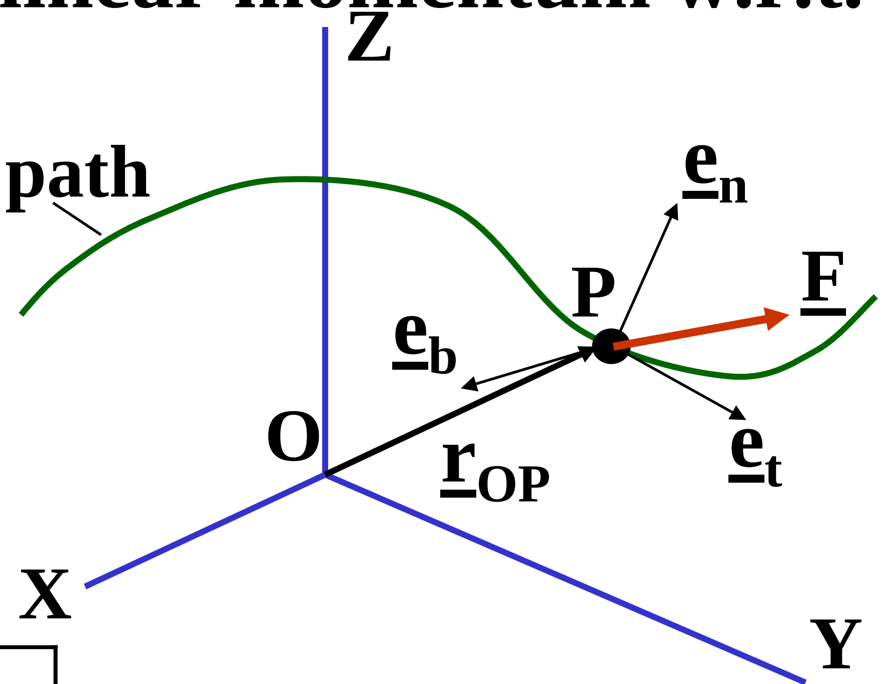
$$\rightarrow \boxed{T = \sqrt{2h/g}\{\frac{1+e}{1-e}\}} \quad - \text{ finite time (for } e < 1)$$

$$\boxed{x_{tot} = v_0 T}$$

3.6 Angular Momentum and Angular Impulse

Recall the definition of linear momentum w.r.t. an inertial frame:

$$\underline{p} = m\underline{v}_P$$



Moment of linear momentum about a point O:

$$\underline{H}_O = \underline{r}_{OP} \times \underline{p} = \underline{r}_{OP} \times m\underline{v}_P$$

(angular momentum about O). Its rate of change is $\dot{\underline{H}}_O = d(\underline{H}_O)/dt = \dot{\underline{r}}_{OP} \times m\underline{v}_P + \underline{r}_{OP} \times m\dot{\underline{v}}_P$

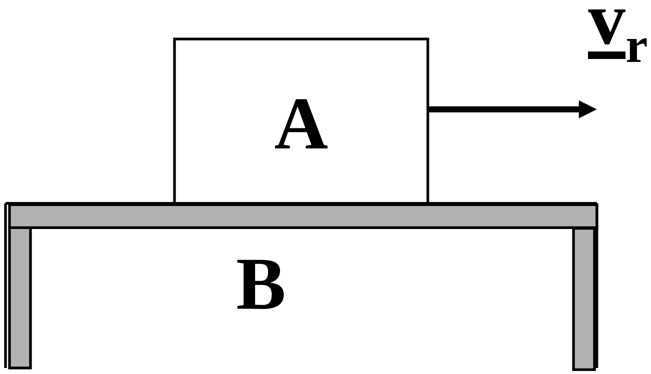
- Now, $\dot{\underline{H}}_O = d(\underline{H}_O)/dt = \underline{r}_{OP} \times m\dot{\underline{v}}_P$
- **Newton's Second law:** $\underline{F} = m\dot{\underline{v}}_P$

$$\rightarrow \dot{\underline{H}}_O = \underline{r}_{OP} \times \underline{F} = \underline{M}_O$$

- $\boxed{\underline{M}_O = \dot{\underline{H}}_O}$ **moment of the net force about a fixed point equals the time rate of change of angular momentum about the same fixed point.**
- $\int_{t_1}^{t_2} \underline{M}_O(\tau)d\tau = \hat{\underline{M}}$ **- angular impulse**

Principle of angular impulse and momentum: $\boxed{\hat{\underline{M}} = \underline{H}(t_2) - \underline{H}(t_1)}$

3.8 Coulomb Friction:

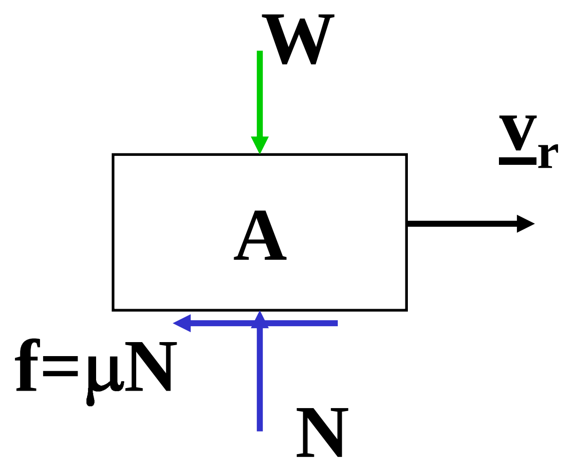


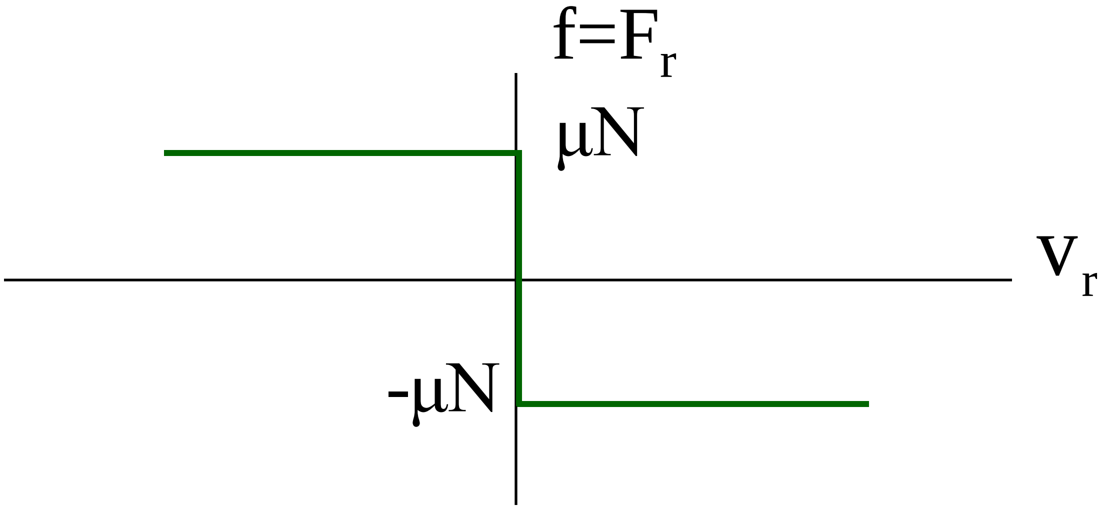
Consider a block A sliding on block B with a velocity $\underline{v}_r$ relative to B.

(i.e. $\underline{v}_r = \underline{v}_A - \underline{v}_B$)

- **Classical Coulomb law:**

when sliding, the friction force = μ (normal force)



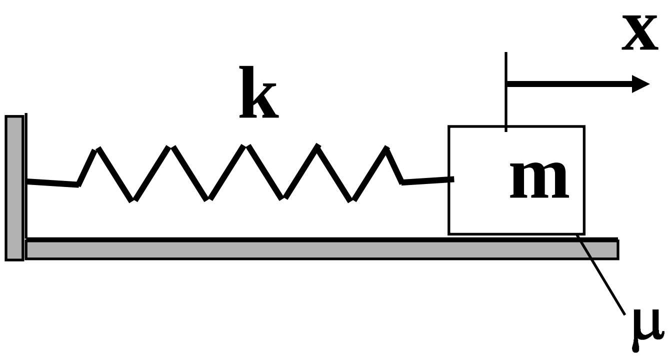


μ - coefficient of sliding friction (depends on the materials and roughness of the sliding surfaces).

- **In reality, f also depends on the slip velocity v_r.**
- **When $v_r = 0$, the force $|f| \le \mu|N|$, is determined by Static Equilibrium.**

Ex. (Ex. 3.11)

Consider a spring - mass system. There is coulomb friction between the block and the horizontal surface.



Let, initial conditions: $x\ (t = 0) = x_o,\ \dot{x}(t = 0) = 0.$

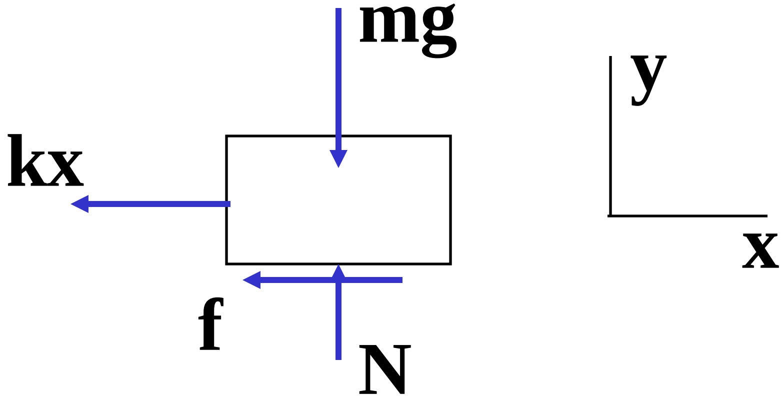
Find: $x(t)$.

Consider the FBD.

Assume initial x_o **such that**

$k\,x_o > f = \mu N = \mu mg$

$\rightarrow$ **block moves**



- When $\dot{x} < 0$, friction force is to the right;

  So, $f = \mu mg$ and $\rightarrow m\ddot{x} + kx = \mu mg$

- **When** $\dot{x} > 0$, friction force is to the left;

  So, $f = -\mu mg$ and $\rightarrow m\ddot{x} + kx = -\mu mg$

Note that both the equations are linear. The overall system is nonlinear as the equation used needs to be switched depending on the choice of the sign of $\dot{x}$**.**

Solution process: Consider $\dot{x} < 0$,

$\rightarrow m\ddot{x} + kx = \mu mg$ with $I.Cs: x(0) = x_0, \dot{x}(0) = 0.$

The solution is: $x(t) = x_h(t) + x_p(t)$

$$x_p(t) = \mu mg / k; \;\; x_h(t) = A\cos\omega_n t + B\sin\omega_n t$$

$$where\, \omega_n = \sqrt{k/m}$$

Thus, $x(t) = \mu mg / k + A\cos\omega_n t + B\sin\omega_n t$

$$\dot{x}(t) = -\omega_n A\sin\omega_n t + \omega_n B\cos\omega_n t$$

Using initial conditions : $\dot{x}(0) = 0 \rightarrow B = 0$

$$\rightarrow x(t) = \mu mg / k + A\cos\omega_n t$$

Now $x(0) = x_0 \rightarrow A = x_0 - \mu mg / k$

Thus, $\boxed{x(t) = \mu mg / k + (x_0 - \mu mg / k)\cos\omega_n t}$ $for\ \dot{x} < 0$

This solution is valid till velocity first becomes zero.

The velocity first becomes zero when:

$$\dot{x} = 0 = -\omega_n (x_0 - \mu mg / k) \sin \omega_n t \rightarrow t_1 = \pi / \omega_n$$

The time t_1 is period of 'half cycle'.

Then, $x(t_1) = -x_0 + 2\mu mg / k; \quad \dot{x}(t_1) = 0.$

Now consider $\underline{\dot{x} > 0}$: $m\ddot{x} + kx = -\mu mg \quad (t \geq t_1)$

The solution is : $x(t) = x_h(t) + x_p(t)$

or, $x(t) = -\mu mg / k + A \cos \omega_n t + B \sin \omega_n t$

$$\dot{x}(t) = \omega_n (-A \sin \omega_n t + B \cos \omega_n t)$$

Using initial conditions at $t = t_1$,

$$B = 0,\ A = x_0 - 3\mu mg / k$$

Thus, $x(t) = (x_0 - 3\mu mg/k)\cos\omega_n t - \mu mg/k$

$$\text{for}\,(\pi/\omega_n < t < 2\pi/\omega_n)$$

The velocity vanishes again when

$$\dot{x}(t) = -\omega_n (x_0 - 3\mu mg/k)\sin\omega_n t = 0$$

$$\rightarrow t_2 = 2\pi/\omega_n \rightarrow x(t_2) = x_0 - 4\mu mg/k$$

The period of 'one cycle' $T = t_1 + t_2 - t_1 = 2\pi/\omega_n.$

In this duration, the amplitude of oscillation decreases by $x_0 - (x_0 - 4\mu mg/k) = 4\mu mg/k.$

- **In same manner, one can find solutions for succeeding half-cycles.**

In the nth **half-cycle:**

$$x(t) = (-1)^{n-1}\mu mg/k + \{x_0 - (2n-1)\mu mg/k\}\cos\omega_n t$$

in the time interval $\{(n-1)\pi/\omega_n < t < n\pi/\omega_n\}$

- Clearly, this motion exists provided the spring has enough force to overcome friction.
- *F*or the first half-cycle: $\boxed{x_0 > \mu mg/k}$.
- For the second half-cycle: $x_0 - 2\mu mg/k > \mu mg/k$

or $\boxed{x_0 > 3\mu mg/k}$.

- For the third half-cycle: $x_0 - 4\mu mg/k > \mu mg/k$

$$\text{or} \quad \boxed{x_0 > 5\mu mg/k}.$$

- For the nth half-cycle: $x_0 - 2(n-1)\mu mg/k > \mu mg/k$

$$\text{or} \quad \boxed{x_0 > (2n-1)\mu mg/k}.$$

- **If the spring force is not sufficient, the motion stops permanently at**

$$x = (-1)^n \{x_0 - (2n)\mu mg / k\}$$

at the time $t > n\pi / \omega_n$

where n is the total number of half-cycles.