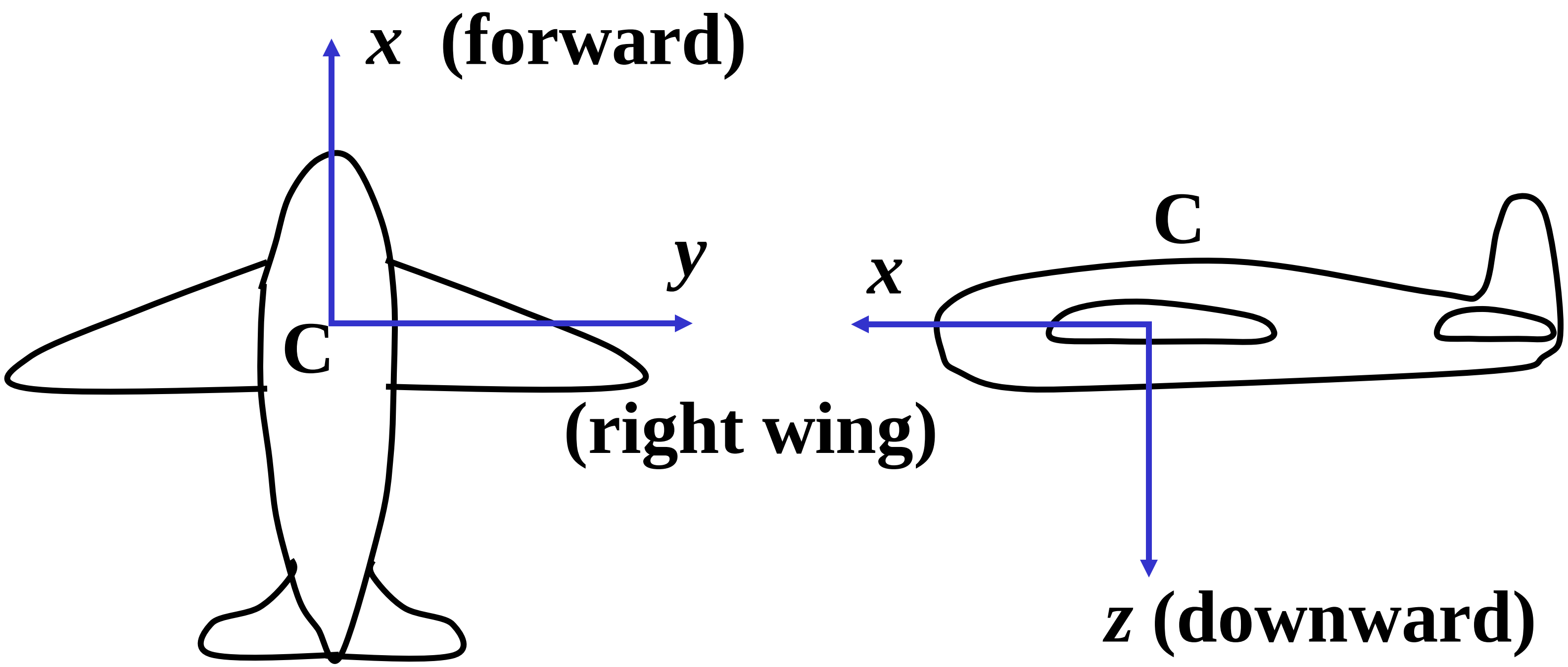
7.13 Eulerian Angles

- **Rotational degrees of freedom for a rigid body - <u>three rotations</u>.**
- **If one were to use Lagrange's equations to derive the equations for rotational motion - one needs three generalized coordinates.**
- **Nine direction cosines with six constraints given by $[l]^T[l]=[1]$ are a possible choice. That will need the six constraint relations to be carried along in the formulation. Not the most convenient.**

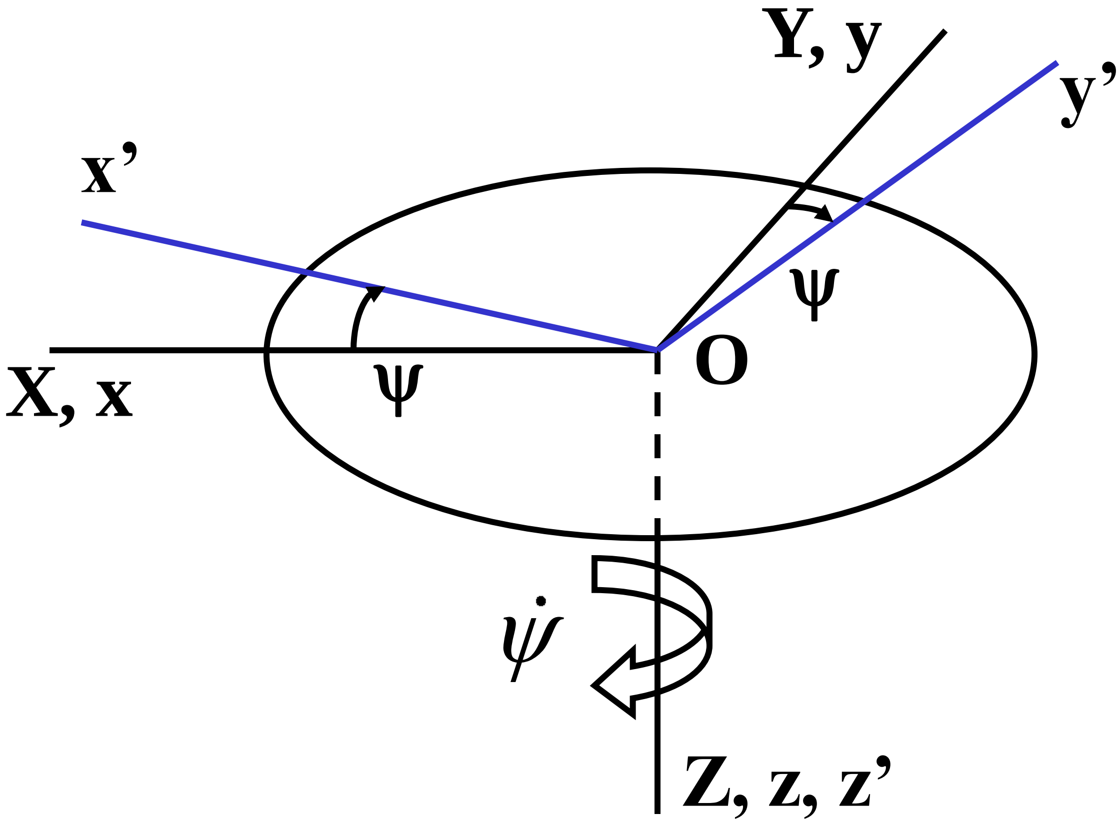
- **One could use the angular velocity components $\omega_x, \omega_y, \omega_z$ to define the rotational kinetic energy. <u>There do not exist</u> three variables which specify the orientation of the body and whose time derivatives are the angular velocity components $\omega_x, \omega_y, \omega_z$**

- **Need to search for a set of three coordinates which can define the orientation of a rigid body at every time instant;**

- **A set of three coordinates - Euler angles.**
- many choices exist in the definition of Euler angles - Aeronautical Engineering (Greenwood's)

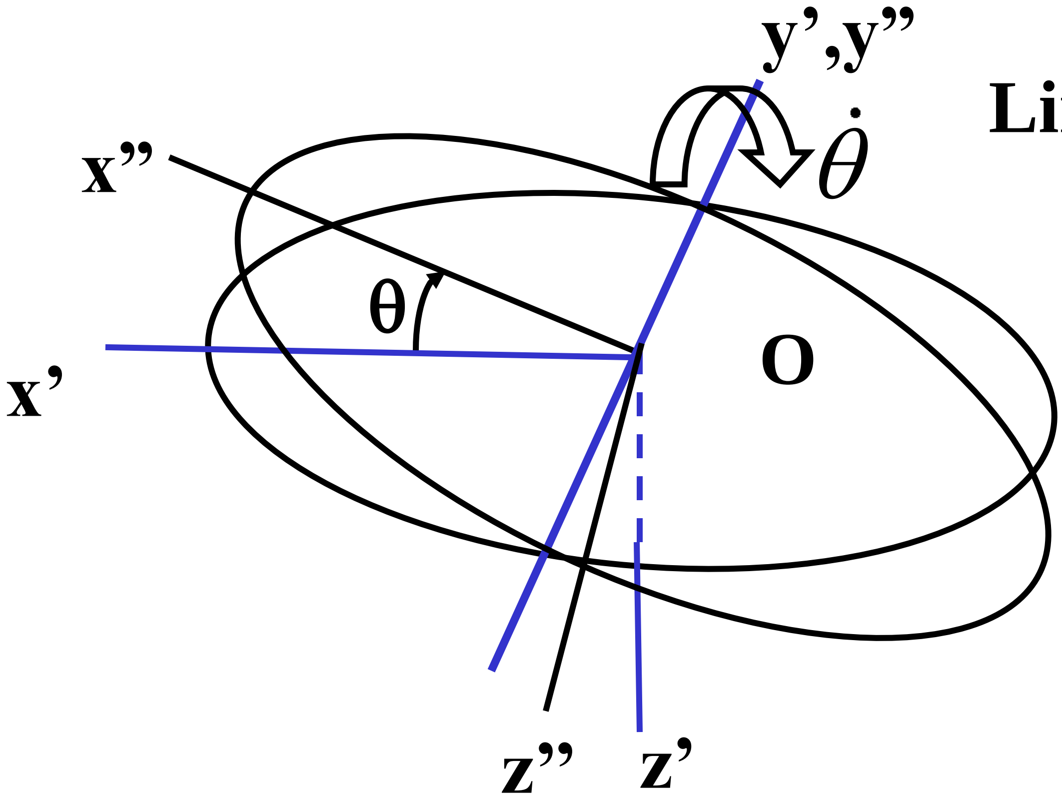


- **xyz system attached to the rigid body; XYZ system is the fixed system attached to ground**
- **Two systems are initially coincident; a series of three rotations about the body axes, performed in a proper sequence, allows one to reach any desired orientation of the body (or xyz) w.r.t. XYZ.**
- **First rotation: a positive rotation ψ about Z or z axis** $\rightarrow x'y'z'$ **system**
- **ψ - <u>heading angle</u>**



Clearly, the rotation is defined by:

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$



Line of nodes

2nd rotation:a positive rotation θ about y' axis → x''y''z''system

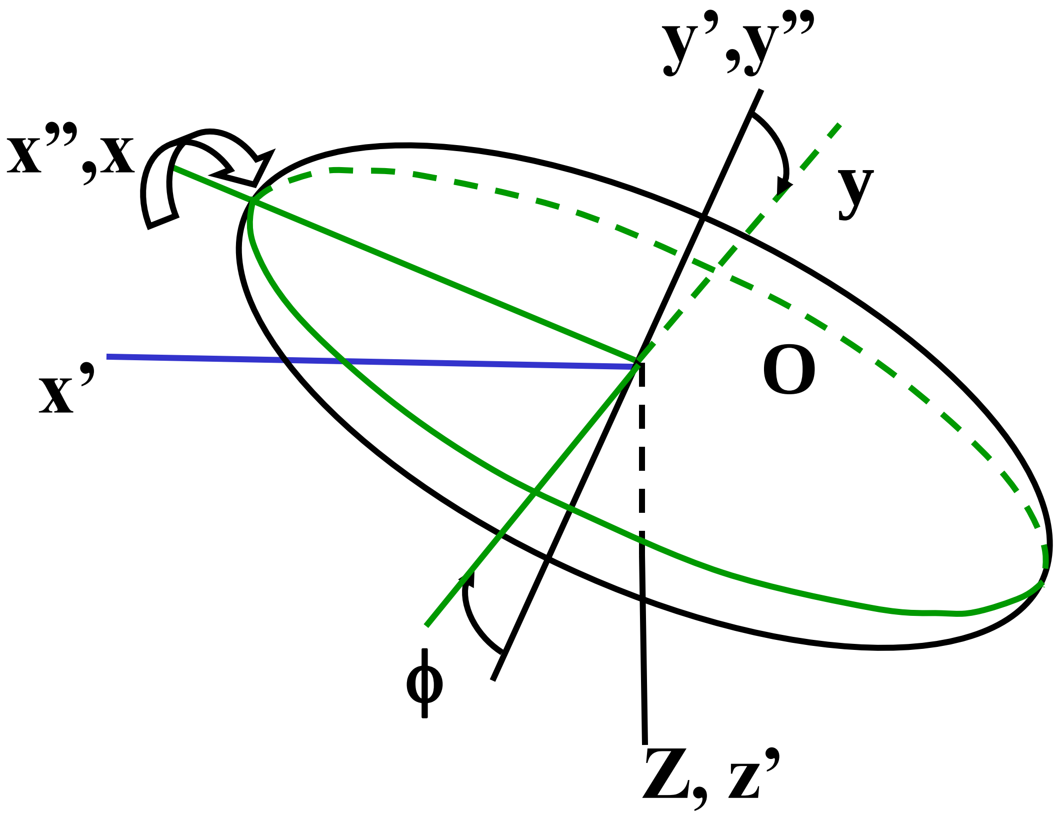
The resulting rotation is:

$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}$$

θ -<u>attitude angle</u> $-\pi/2 \le \theta \le \pi/2$

- **3rd rotation**: **a positive rotation ϕ about the x'' axis $\rightarrow$ xyz system**

ϕ - bank angle $0 \leq \phi \leq 2\pi$



Note that Oy” is in XY plane.
When $\phi=\pi/2 \Rightarrow$ xy plane vertical

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix}$$

The transformation equations in more compact form can be written as:

$$\{r'\} = [\psi]\{R\}$$

$$\{r''\} = [\theta]\{r'\}$$

$$\{r\} = [\phi]\{r''\}$$

$$or \quad \{r\} = [\phi][\theta][\psi]\{R\}$$

- **Since [ψ], [θ], [ϕ] are orthogonal, so is the final matrix [ϕ][θ][ψ].**

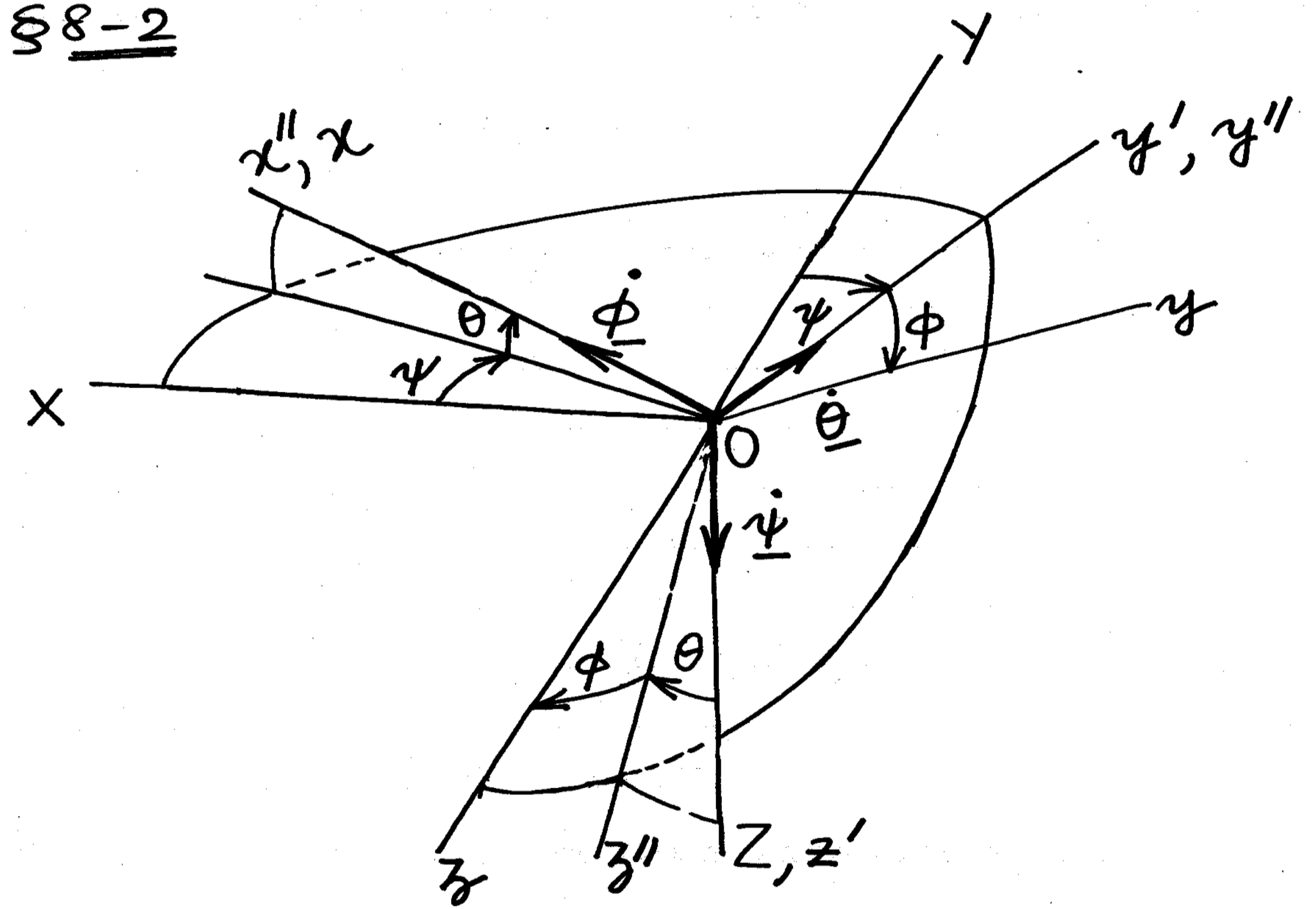
$$\{R\} = [x]^T [\theta]^T [\phi]^T \{r\}$$ **gives the inverse transformation.**

- **Any possible orientation of the body can be attained by performing the proper rotations in the given order.**
- **Important: if the orientation is such that x-axis is vertical → no unique set of values for ψ and ϕ can be found.**

i.e., if $\theta = \pm \pi/2$, the angles ψ and ϕ are undefined. However, $(\psi - \phi)$ is well defined when $\theta = \pi/2$ - it is the angle between the x and z axes. Similarly, $(\psi + \phi)$ well defined when $\theta = -\pi/2$.

- **In such situations, called <u>singular</u>, only two rotational degrees are represented.**

8.2 Angular Velocities in terms of Eulerian Angles



- $\underline{\dot{\psi}}$ along z'
- $\underline{\dot{\phi}}$ along x
- $\underline{\dot{\theta}}$ along y', y''

<u>Now</u>: $$\underline{\omega} = \underline{\dot{\psi}} + \underline{\dot{\theta}} + \underline{\dot{\phi}}$$

In component form

$$\{\omega\} = [\phi][\theta]\{\dot{\psi}\} + [\phi]\{\dot{\theta}\} + \{\dot{\phi}\}$$

where $\{\dot{\phi}\} = \{\dot{\phi}, 0, 0\}^T$

$$\{\dot{\theta}\} = \{0,\ \dot{\theta},\ 0\}^T$$

and

$$\{\dot{\psi}\} = \{0,\ 0,\ \dot{\psi}\}^T.$$

combining $\Rightarrow$

$$\omega_x = \dot{\phi} - \dot{\psi}\sin\theta$$

$$\omega_y = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi$$

$$\omega_z = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi$$

These are the relations in angular velocity and rates of change of Euler angles.

7.13 Eulerian Angles

Review:

To define the orientation of a body with respect to a fixed frame,

- let XYZ be the fixed frame
- let xyz be the frame attached to the body
- let xyz and XYZ coincide initially

- Perform a sequence of three rotations in a specified order about axes fixed to the body and arrive at the desired (final) position - angles called **Euler angles**.
- One possible sequence:

  1st rotation: a positive rotation x about z (Z) axis $\rightarrow$ body axes now $x'y'z'$

  ψ - heading angle.

2nd rotation: a positive rotation θ about the y' axis → body axes now $x''y''z''$.

θ - attitude angle

3rd rotation: a positive rotation ϕ about the x'' axis body axes now xyz (final position of the body). ϕ - bank angle

- cumulatively $\{r\} = [\phi][\theta][\psi]\{R\}$.

cumulatively

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ \begin{array}{l}(-\sin\psi\cos\phi \\ +\cos\psi\sin\theta\sin\phi)\end{array} & \begin{array}{l}(\cos\psi\cos\phi \\ +\sin\psi\sin\theta\sin\phi)\end{array} & \cos\theta\sin\phi \\ \begin{array}{l}(\sin\psi\sin\phi \\ +\cos\psi\sin\theta\cos\phi)\end{array} & \begin{array}{l}(-\cos\psi\sin\phi \\ +\sin\psi\sin\theta\cos\phi)\end{array} & \cos\theta\cos\phi \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$

7.14 Rigid Body Motion in a Plane

The equations of motion for a rigid body are:

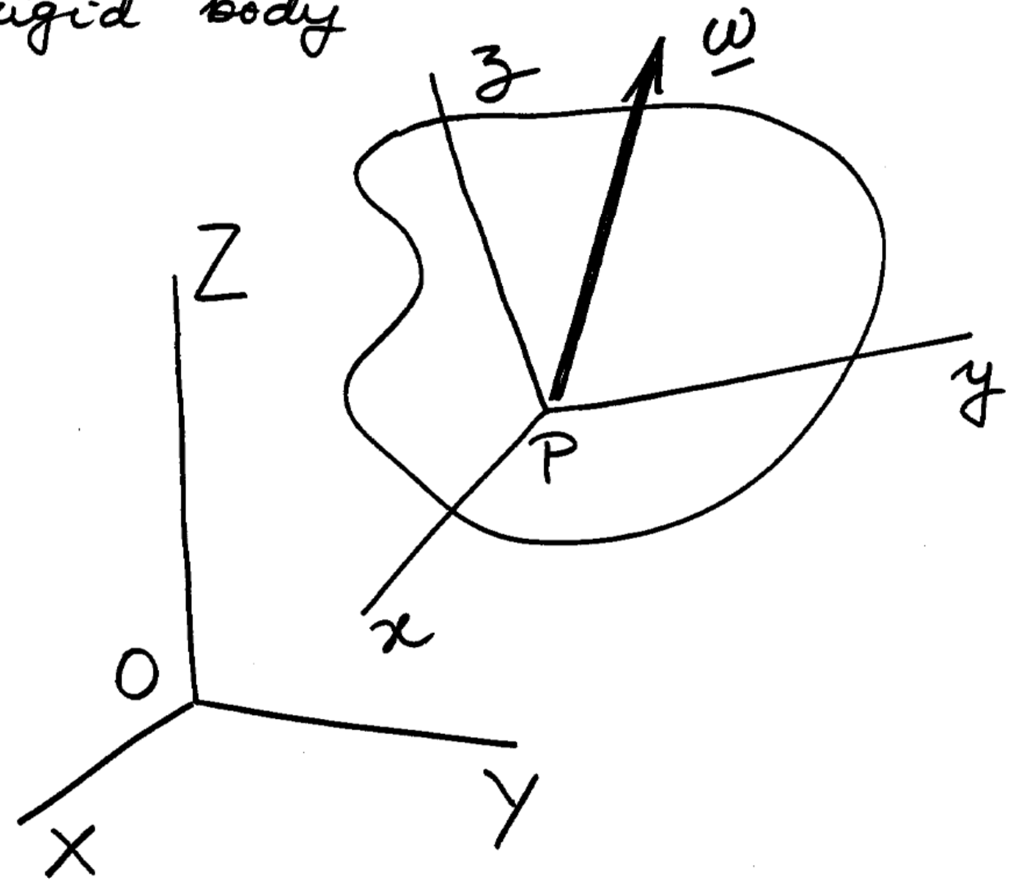
1. $\underline{F} = m\ddot{\underline{r}}_c$ Translational motion of the center of mass

2. $\underline{M}_p = \frac{d}{dt}\underline{H}_p + \underline{\rho}_c \times m\ddot{\underline{r}}_p$

P = an arbitrary point,

$\underline{\rho}_c$ - position of center of mass relative to P.

For a rigid body



- $\underline{\omega} = \omega_x \underline{i} + \omega_y \underline{j} + \omega_z \underline{k}$ – angular vel. in <u>body fixed axes</u>

- $\underline{H} = H_x \underline{i} + H_y \underline{j} + H_z \underline{k}$

$$H_x = I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z$$

$$H_y = I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z$$

$$H_z = I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z.$$

- Inertia properties relative to xyz at P.

Planar motion: $\omega_y = \omega_x = 0$

i.e., all rotations about z axis, which remains parallel to Z axis.

$$\Rightarrow \quad H_x = I_{xz}\,\omega_z \quad , \quad H_y = I_{yz}\,\omega_z \quad , \quad H_z = I_{zz}\,\omega_z$$

- Suppose that the products of inertia $I_{xz} = I_{yz} = 0$, i.e, <u>xy plane</u> is a <u>plane of symmetry</u> of the body

$$\Rightarrow \qquad \underline{H} = I_{zz}\,\omega_z\,\underline{k}$$

$$\frac{d}{dt}\underline{H} = I_{zz}\,\alpha_z\,\underline{k}$$ for <u>body fixed axes</u>

⇒ Equations for planar motion:

$$\underline{F} = m\,\ddot{\underline{r}}_c$$

$$\underline{M}_p = (I_{zz})_p\,\alpha_z\,\underline{k} + \underline{\rho}_c \times m\,\ddot{\underline{r}}_p \Leftarrow$$

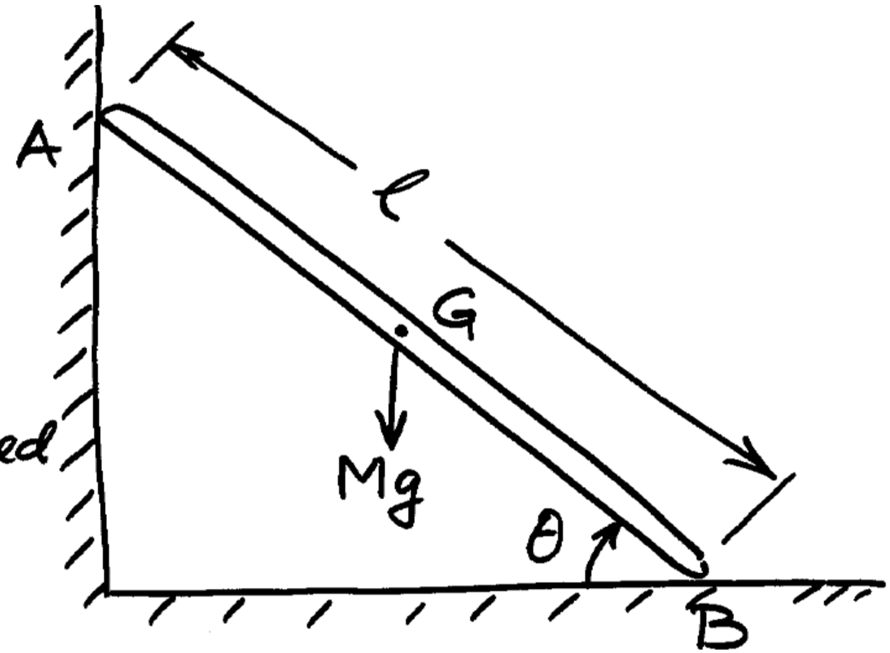
<u>special case</u>:

$P = C$ or $P = O$ (a fixed point)

$$\Rightarrow \quad \underline{M}_p = (I_{zz})_p\,\alpha_z\,\underline{k}$$

Reading assignment: Examples <u>7.5 to 7.10</u>.

Ex (Ex 7-8 in text)

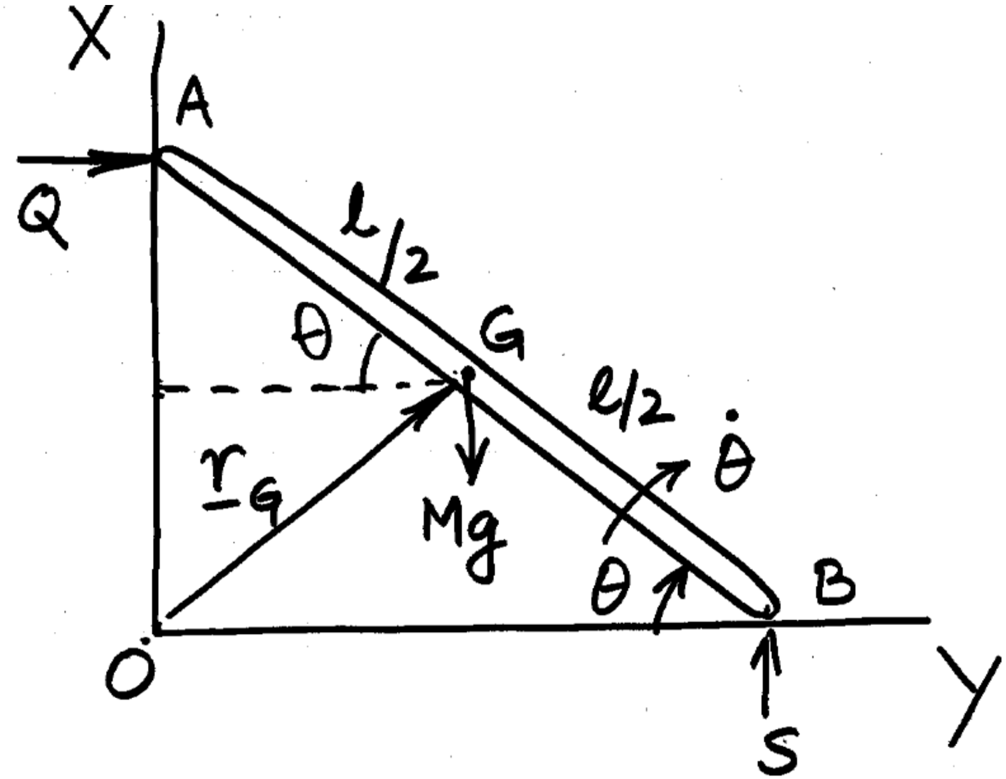


A thin homogeneous rod of length ℓ and mass M displaced slightly from its initial position $\theta = 90°$ (vertical position).

Neglect friction

a) Find angle θ at which end A leaves the wall.

b) Find expressions for $\underline{\alpha}$ and $\underline{\omega}$ after end A has left the wall (subsequent motion)



<u>Case a</u>: ends A and B remain in contact.

$$\underline{r}_G = \frac{l}{2}\sin\theta\,\underline{i} + \frac{l}{2}\cos\theta\,\underline{j}$$

$$\dot{\underline{r}}_G = \dot{\theta}\frac{l}{2}(\cos\theta\,\underline{i} - \sin\theta\,\underline{j})$$

$$\ddot{\underline{r}}_G = \left(\ddot{\theta}\frac{l}{2}\cos\theta - \dot{\theta}^2\frac{l}{2}\sin\theta\right)\underline{i}$$
$$-\left(\ddot{\theta}\frac{l}{2}\sin\theta + \dot{\theta}^2\frac{l}{2}\cos\theta\right)\underline{j}$$

<u>Translation</u>:

$$\left[S\underline{i} + Q\underline{j} - Mg\underline{i} = \cancel{m}M\ddot{\underline{r}}_G \right] \quad (1)$$

Taking dot product with $\underline{j}$ $\Rightarrow$

$$\underline{Q = -M\left(\frac{l}{2}\sin\theta\,\ddot{\theta} + \frac{l}{2}\cos\theta\,\dot{\theta}^2\right)} \quad (2)$$

- Rod leaves wall $\Rightarrow$ <u>when $Q = 0$</u> or

$$\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2 = 0 \quad (3)$$

Taking dot product with $\underline{i}$ $\Rightarrow$

$$\underline{S - Mg = M\left(\frac{l}{2}\cos\theta\,\ddot{\theta} - \frac{l}{2}\sin\theta\,\dot{\theta}^2\right)} \qquad (4)$$

<u>Rotational Motion</u> :

$\ddot{\theta}$

$$\sum \underline{M}_G = I_{zz}\,\alpha\,\underline{k}$$

$$\underline{Q\frac{l}{2}\sin\theta - S\frac{l}{2}\cos\theta = \frac{1}{12}Ml^2\ddot{\theta}} \qquad (5)$$

Eliminate Q and S from equations (2), (4) and (5) $\Rightarrow$

(5) $\Rightarrow$

$$-\cancel{M}\left(\frac{l}{2}\sin\theta\,\ddot{\theta}+\frac{l}{2}\cos\theta\,\dot{\theta}^2\right)\frac{l}{2}\sin\theta$$

$$-\frac{l}{2}\cos\theta\left[\cancel{M}g+\cancel{M}\left(\frac{l}{2}\cos\theta\,\ddot{\theta}-\frac{l}{2}\sin\theta\,\dot{\theta}^2\right)\right]$$

$$=\frac{1}{12}\cancel{M}l^2\ddot{\theta}$$

or

$$-\left(\frac{l}{2}\sin\theta\right)^2\ddot{\theta}-\left(\frac{l}{2}\right)^2\cancel{\sin\theta\cos\theta\,\dot{\theta}^2}$$

$$-g\frac{l}{2}\cos\theta-\left(\frac{l}{2}\right)^2\cos^2\theta\;\ddot{\theta}+\left(\frac{l}{2}\right)^2\cancel{\sin\theta\cos\theta\,\dot{\theta}^2}$$

$$=\frac{1}{12}l^2\ddot{\theta}$$

or

$$-\frac{l^2}{4}\ddot{\theta} - \frac{1}{12}l^2\ddot{\theta} = g\frac{l}{2}\cos\theta$$

$$\Rightarrow \qquad \boxed{\ddot{\theta} = -\frac{3g}{2l}\cos\theta} \qquad (6)$$

we want to integrate this equation to find how $\dot{\theta}^2$ depends on θ, (why?) Since we want to find angle θ at which the bar leaves the wall at A, given by (3).

- $\ddot{\theta} = \dot{\theta}\dfrac{d(\dot{\theta})}{d\theta}$ (change of independent variable)

$$(6) \Rightarrow \frac{1}{2}\frac{d}{d\theta}\dot{\theta}^2 = -\frac{3g}{2l}\cos\theta$$

Integrating $\Rightarrow$ $\dot{\theta}^2 = -\dfrac{3g}{l}\sin\theta + \underline{\underline{C}}$

at $\theta = \pi/2$, $\dot{\theta} = 0$ $\Rightarrow$ $C = 3g/l$

or $\Rightarrow$ $\boxed{\dot{\theta}^2 = \dfrac{3g}{l}(1-\sin\theta)}$ (7)

- condition for bar leaving the wall

$(Q = 0)$:

$$\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2 = 0 \qquad (3)$$

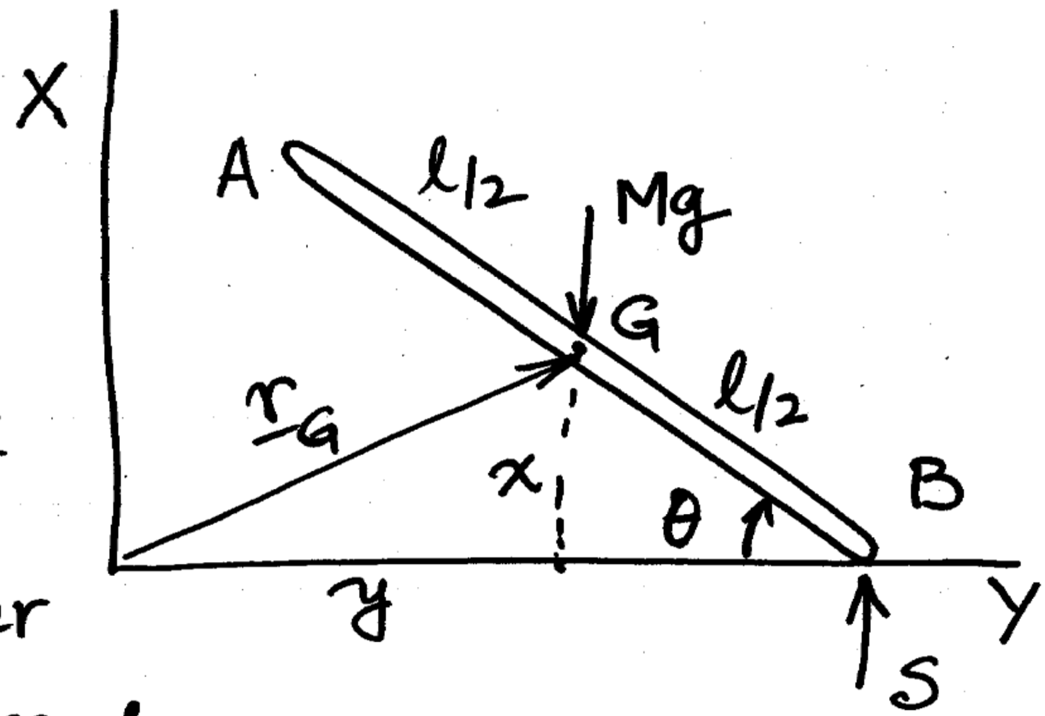
(6), (7) in (3) $\Rightarrow$

$$-\frac{3g}{2\cancel{l}}\sin\theta\,\cancel{\cos\theta} + \frac{\cancel{3g}}{\cancel{l}}\cancel{\cos\theta} - \frac{\cancel{3g}}{\cancel{l}}\sin\theta\,\cancel{\cos\theta} = 0$$

or $\quad -\frac{3}{2}\sin\theta = \overset{-1}{\cancel{-1}} \quad \Rightarrow \quad \boxed{\theta_0 = \sin^{-1}(2/3)}$

Then (7) $\Rightarrow$ $\boxed{\dot{\theta}_0 = -\sqrt{g/l}}$ (8)

B After leaving the wall



$$\underline{r}_G = \frac{l}{2}\sin\theta\,\underline{i} + y\,\underline{j}$$

(Note: the number of degrees of freedom has increased by one)

⇒ we use θ and y as the independ. coordinates.

$$\dot{\underline{r}}_G = \dot{y}\,\underline{j} + \frac{l}{2}\cos\theta\,\dot{\theta}\,\underline{i}$$

$$\ddot{\underline{r}}_G = \ddot{y}\,\underline{j} + \frac{l}{2}\left\{\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right\}\underline{i} \quad (9)$$

Translation :

$$\sum \underline{F} = S\,\underline{i} - Mg\,\underline{i} = M\ddot{\underline{r}}_G$$

$$= \frac{Ml}{2}\left(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right)\underline{i} + M\ddot{y}\,\underline{j} \quad (10)$$

$\underline{j}$: $\quad M\ddot{y} = 0 \Rightarrow \boxed{\ddot{y} = 0} \quad (11)$

$$\underline{\underline{j}} \qquad S - Mg = \frac{Ml}{2}\left\{\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right\} \qquad (12)$$

<u>Rotational</u>:

$$\sum \underline{M}_G = -\frac{l}{2}\cos\theta\, S\ \underline{k} = \frac{1}{12} Ml^2\ddot{\theta}\ \underline{k}$$

$$\Rightarrow \quad S = -Ml\ddot{\theta}/6\cos\theta \qquad (13).$$

(12), (13) $\Rightarrow$

$$-\frac{1}{6}Ml\ddot{\theta} - Mg\cos\theta = \frac{Ml}{2}\left\{\cos^2\theta\,\ddot{\theta} - \sin\theta\cos\theta \times \dot{\theta}^2\right\}$$

or

$$\Rightarrow \boxed{-\frac{6g}{l}\cos\theta = (1+3\cos^2\theta)\ddot{\theta} - 3\sin\theta\cos\theta\,\dot{\theta}^2} \quad (14)$$

Again: $\ddot{\theta} = \frac{1}{2}\frac{d}{d\theta}\dot{\theta}^2$

$$(14) \Rightarrow \quad -\frac{6g}{l}\cos\theta = \frac{(1+3\cos^2\theta)}{2}\frac{d}{d\theta}(\dot{\theta}^2)$$

$$- 3\sin\theta\cos\theta\,\dot{\theta}^2$$

$$= \frac{d}{d\theta}\left(\left(\frac{1+3\cos^2\theta}{2}\right)\dot{\theta}^2\right)$$

Integrating $\Rightarrow$ $(1+3\cos^2\theta)\dot{\theta}^2 = -\frac{12g}{l}\sin\theta + C$ (15)

Initial conditions :

$$\theta = \theta_0 = \sin^{-1}(2/3) \quad , \quad \dot{\theta}_0 = -\sqrt{g/l}$$

$$\Rightarrow \quad C = \frac{32}{3}\frac{g}{l}$$

or

$$\boxed{\dot{\theta}^2(1+3\cos^2\theta) + \frac{12g}{l}\sin\theta = \frac{32}{3}\frac{g}{l}} \quad (16)$$

This relates $\dot{\theta}$ to θ at any position.

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