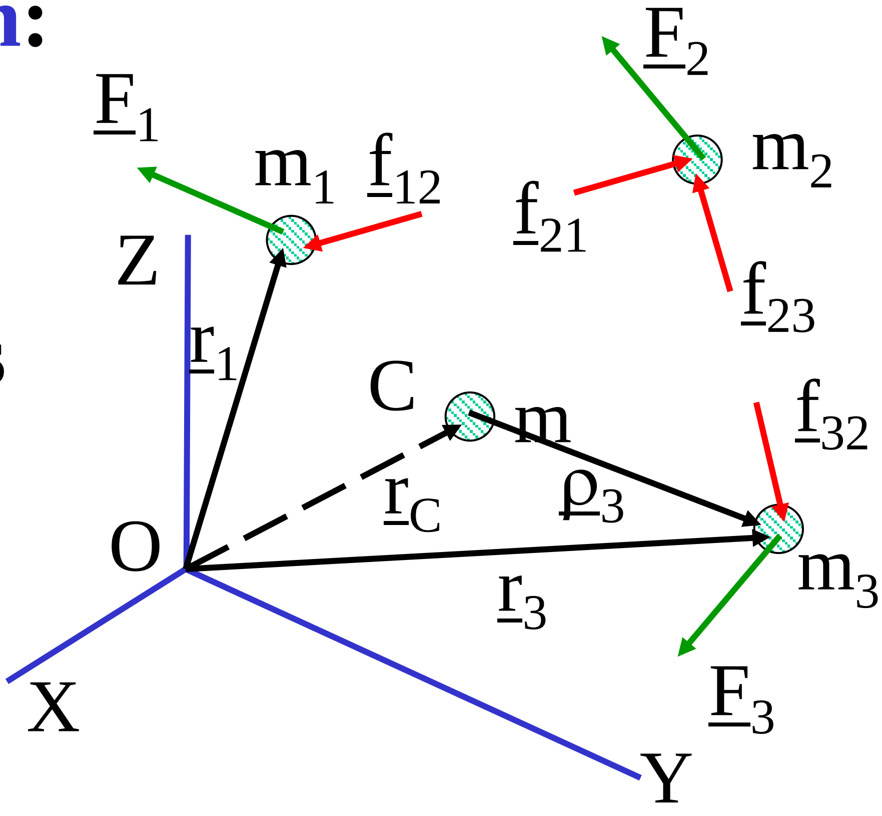
CHAPTER 7

Basic Concepts and Kinematics of Rigid Body Motion

7.1 Degrees-of-freedom:

(of a rigid body)

Consider three unconstrained particles



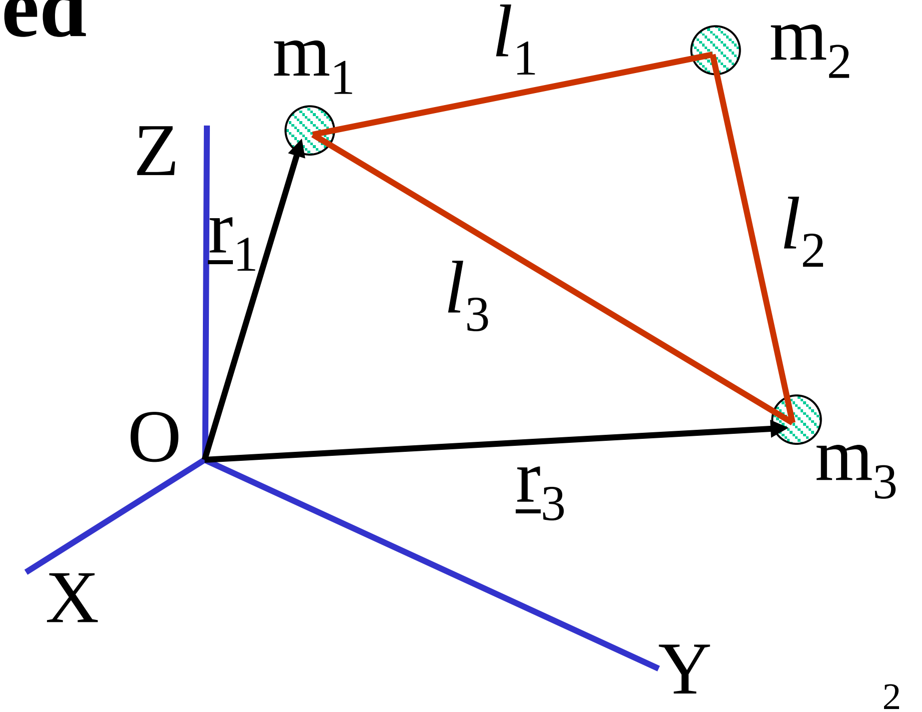
The positions are defined by

$$\underline{r}_i = x_i \underline{i} + y_i \underline{j} + z_i \underline{k} \qquad \textbf{i = 1, 2, 3.}$$

$\Rightarrow$ **degrees of freedom = n = 9 = 3N**

Now, constrain the particles $\Rightarrow$

(three particles placed at the corners of a triangle whose sides are formed by rigid massless rods)



Now: there are three constraints

$$\left|\underline{r}_1 - \underline{r}_2\right| = l_1; \quad \left|\underline{r}_3 - \underline{r}_2\right| = l_2; \quad \left|\underline{r}_1 - \underline{r}_3\right| = l_3$$

or $(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - l_1^2 = 0, \quad etc.$

$\Rightarrow$ n = 3(N) - 3 = 6 $\Leftarrow$ degrees of freedom.

Now: as another particle is introduced, its position is specified by 3 additional coordinates, but also have 3 additional constraints.

$\Rightarrow$ four rigidly connected particles also have only 6 degrees of freedom.

In general: a rigid body (any collection of particles whose relative positions are fixed) has 6 degrees of freedom.

- **translational motion of a point on the body - specified by <u>3 translational degrees of freedom</u>.**
- **rotational motion about the specified point - <u>3 rotational degrees of freedom</u>.**

Laws of motion for a system of particles (extended to a rigid body)

a) $\sum \underline{F} = m\ddot{\underline{r}}_c$

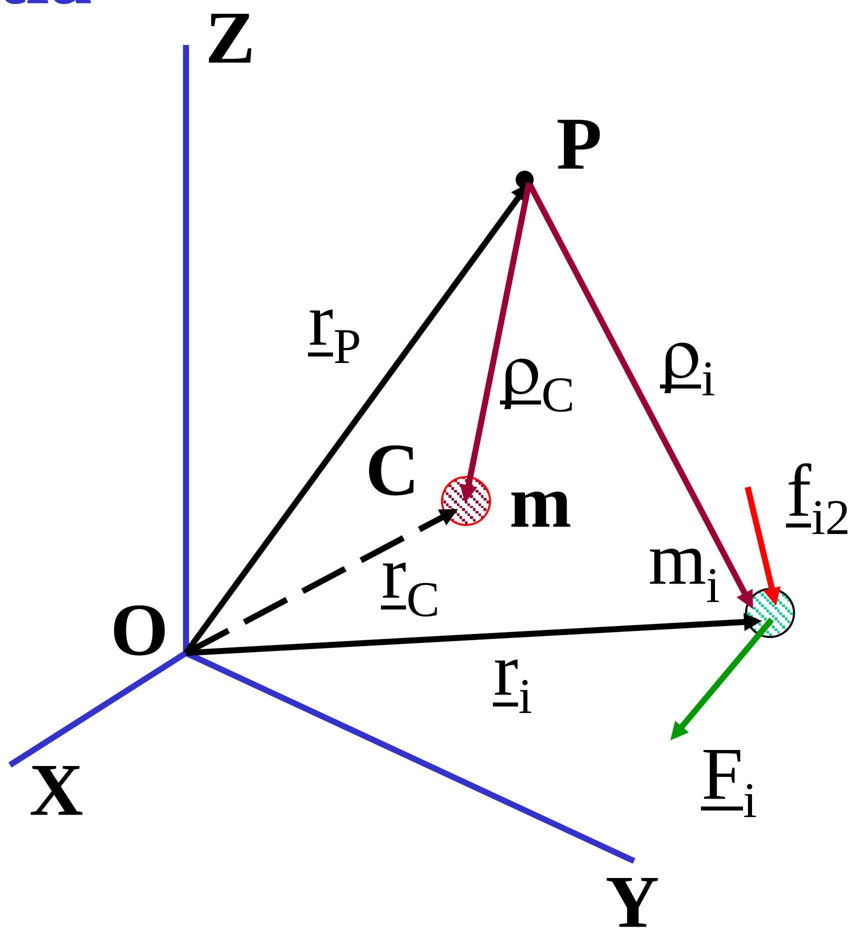
$\Rightarrow$ **translational motion of the C.M. (3 degrees of freedom)**

b) $\sum \underline{M} = \frac{d}{dt}\underline{H}$ **about C.M. or an inertially fixed point**

$\Rightarrow$ **rotational motion about the C.M. (3 rotational degrees of freedom)**

7.2 Moments of Inertia

Recall: the notation and definitions



The equation for moment about an arbitrary point P is:

$$\underline{M}_p = \frac{d}{dt}\underline{H}_p + \underline{\rho}_c \times m\ddot{\underline{r}}_p$$

Now, the angular momentum about the point P is

$$\underline{H}_p = \sum_{i=1}^{N} \underline{\rho}_i \times m_i \dot{\underline{\rho}}_i$$

where $\dot{\underline{\rho}}_i$ **- velocity of** m_i **as viewed by a non-rotating observer translating with P. (relative velocity in an inertial frame)**

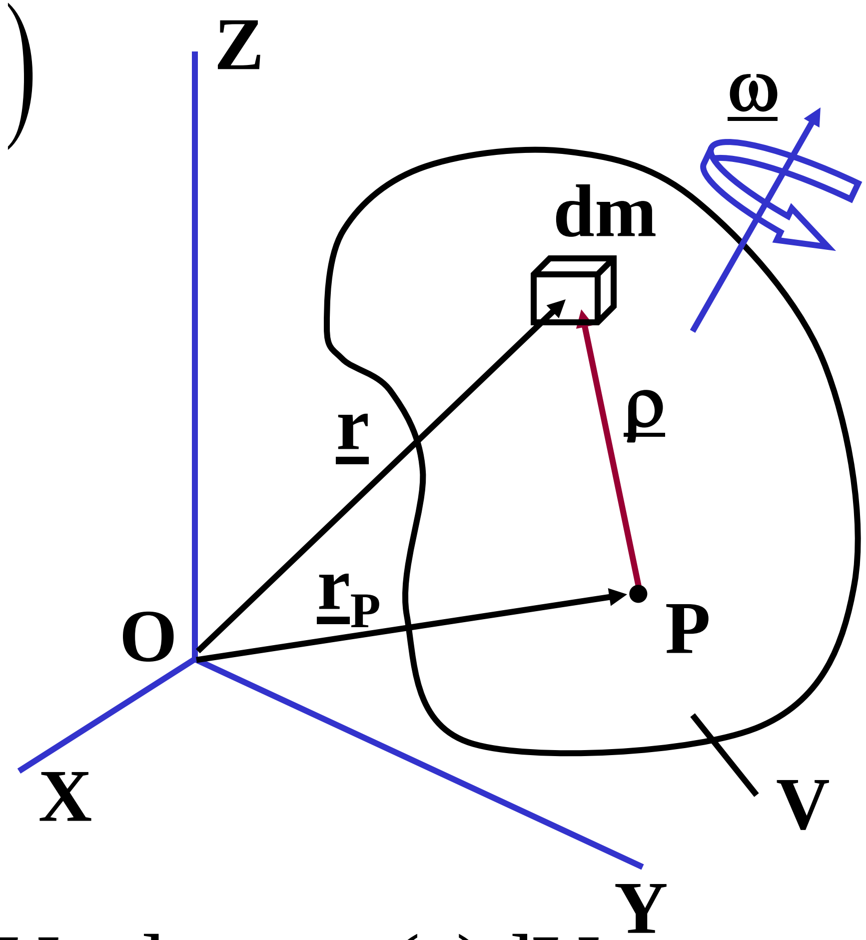
For a rigid body - suppose that P is fixed in the body $\Rightarrow$ $\left|\underline{\rho}_i\right|$ **= constant, and** $\dot{\underline{\rho}}_i = \underline{\omega} \times \underline{\rho}_i$
where $\underline{\omega}$ **- angular velocity of the body.**

Thus, the angular momentum about P is ⇒

$$\underline{\mathrm{H}}_{\mathrm{p}} = \sum_{t=1}^{N} \mathrm{m}_{\mathrm{i}}\, \underline{\rho}_{\mathrm{i}} \times \left(\underline{\omega} \times \underline{\rho}_{\mathrm{i}}\right)$$

By analogy:

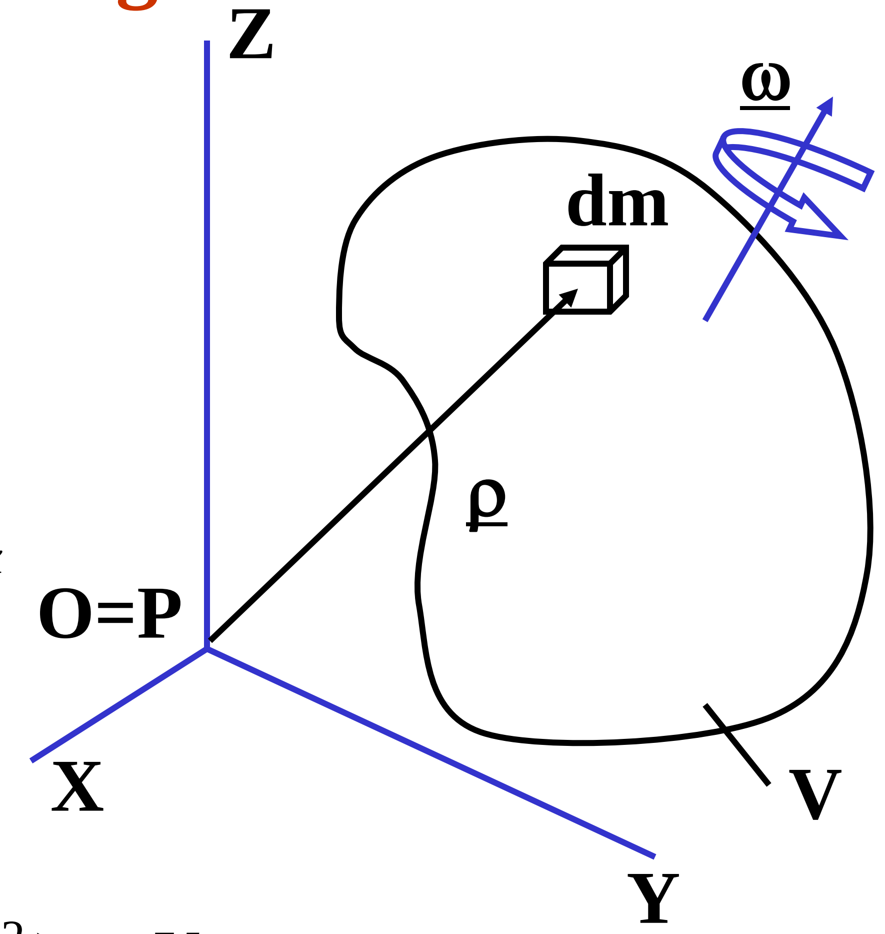
for a rigid body rotating with angular velocity $\left(\underline{\omega}\right)$



$$\underline{\mathrm{H}}_{\rho} = \int_{\mathrm{V}} \rho(\underline{r})\, \underline{\rho} \times (\underline{\omega} \times \underline{\rho})\, \mathrm{dV}\ ; \ \mathrm{dm} = \rho(\underline{\mathrm{r}})\, \mathrm{dV}$$

We now consider the various cases:

Reference point P is at origin:



$$\underline{\rho} = x\underline{i} + y\underline{j} + z\underline{k}$$

$$\underline{\omega} = \omega_x \underline{i} + \omega_y \underline{j} + \omega_z \underline{k}$$

Then

$$\begin{aligned}\underline{\rho} \times (\underline{\omega} \times \underline{\rho}) &= [(y^2 + z^2)\omega_x \\ &- xy\omega_y - xz\omega_z]\underline{i} + [-xy\omega_x \\ &+ (x^2 + z^2)\omega_y - yz\omega_z]\underline{j} \\ &+ [-xz\omega_x - yz\omega_y + (x^2 + y^2)\omega_z]\underline{k}\end{aligned}$$

Let us define: $$I_{xx} = \int_V \rho(y^2 + z^2)dV$$

$$I_{yy} = \int_V \rho(x^2 + z^2)dV \quad ; \quad I_{zz} = \int_V \rho(x^2 + z^2)dV$$

These are the moments of inertia

$$\left.\begin{array}{l} I_{xx} - \text{about x axis} \\ I_{yy} - \text{about y axis} \\ I_{zz} - \text{about z axis} \end{array}\right\} \text{through the reference point } O \equiv P$$

Similarly, we define products of Inertia:

$$I_{xy} = I_{yx} = -\int_V \rho\, xy\, dV$$

$$I_{xz} = I_{zx} = -\int_V \rho\, xz\, dV$$

$$I_{yz} = I_{zy} = -\int_V \rho\, yz\, dV$$

$$\Rightarrow \underline{H}_p = H_x \underline{i} + H_y \underline{j} + H_z \underline{k}$$

(angular momentum vector for the body, or angular momentum about P)

Here, $$H_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$H_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$H_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

In compact notation:

$$\underline{\mathrm{H}}_{\mathrm{P}} = \sum_{\mathrm{i}=1}^{3} \sum_{\mathrm{j}=1}^{3} \mathrm{I}_{\mathrm{ij}}\ \omega_{\mathrm{j}}\ \ \underline{\mathrm{e}}_{\mathrm{i}}$$

where $\underline{\mathrm{e}}_1 = \underline{\mathrm{i}}$, $\underline{\mathrm{e}}_2 = \underline{\mathrm{j}}$, $\underline{\mathrm{e}}_3 = \underline{\mathrm{k}}$

and symbols 1, 2, 3 ≡ x, y, z.

The components of the angular momentum are

$$H_i = \sum_{j=1}^{3} I_{ij}\omega_j, \qquad i = 1, 2, 3$$

Note that I_{ii} **is the second moment of the mass distribution with respect to a Cartesian axis.**

Radius of gyration

(effective location of a mass point)

$$k_i \equiv \sqrt{\frac{I_{ii}}{m}}, \quad i = 1, 2, 3$$

or

$$I_{ii} = mk_i^2$$

Three moments of inertia and three products of inertia specify the inertia properties of a rigid body with regards to rotational motion.

- **Note that P is fixed in the rigid body and $\underline{\omega}$ is the angular velocity of the rigid body.**
- **No assumption is made concerning the rotational motion of the xyz system. The angular velocity in terms of the xyz system,**

$$\underline{\omega} = \omega_x \underline{i} + \omega_y \underline{j} + \omega_z \underline{k}$$

is valid at the instant considered.

- **$\omega_x, \omega_y, \omega_z$ <u>as well as</u> I_{ij} are, in general, functions of time depending on the orientation of xyz relative to the rigid body.**

- **To avoid the difficulties associated with treating I_{ij} as functions of time, <u>often</u> one chooses xyz system that is fixed to the rigid body and rotates with it.**

- **Called a body-fixed coordinate system**

7.3 MATRIX NOTATION:

Consider $\underline{H} = H_x \underline{i} + H_y \underline{j} + H_z \underline{k}$

If $\underline{i}, \underline{j}, \underline{k}$ **are known, the three scalar components** H_x, H_y, H_z **can be used to represent** $\underline{H}$

- **We write** $\{H\} = \begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix}$ **as a column vector.**
- **A force** $\underline{F}$ **can be represented as**

$\lfloor F \rfloor = \lfloor F_x, F_y, F_z \rfloor$, **a row vector.**

or $\{F\}=\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}$ **as a column vector.**

A square matrix is a n × n array of elements:
e.g. the elements of inertia I_{ij} can be written as a square matrix

$$[I]=\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

This is called an inertia matrix for the body.

The angular velocity vector can be represented as

$$\{\omega\} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix},$$ **a column vector.**

- $\{\omega\}^T \equiv \lfloor \omega_x \quad \omega_y \quad \omega_z \rfloor$ **,the transpose of a column vector gives a row vector, etc.**
- **consider** $\lceil \omega \rceil \ \{H\}$ **, the product of a row vector with a column vector**

$$\lfloor \omega \rfloor \ \{H\} = \omega_x H_x + \omega_y H_y + \omega_z H_z \equiv \underline{\omega} \cdot \underline{H}$$

In scalar components, it is easy to see that $\underline{H} = \Sigma\Sigma I_{ij}\omega_j \underline{e}_i$ **can be expressed as**

$$\{H\} = [I]\{\omega\}$$

Clearly, multiplication of $\{\omega\}$ with $[I]$ **transforms** $\{\omega\}$ **into the vector** $\{H\}$, **usually with a different magnitude as well as direction.**

⇒In general, The angular velocity and the angular momentum vectors for a rigid body are in different directions

Assignment: **Complete the review of matrix operations in 7.3**.

7.4 Kinetic Energy

For a system of n particles, the kinetic energy is

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2} \sum_{i=1}^{n} m_i \underline{\dot{\rho}}_i \cdot \underline{\dot{\rho}}_i$$

where

- v_C **- speed of the center of mass**
- $\underline{\dot{\rho}}_i$ **- velocity of the** i^{th} **particle as viewed from the C.M.**

For a set of particles rigidly connected and the assemblage rotating with angular velocity $\underline{\omega}$,

$$\dot{\underline{\rho}}_i = \underline{\omega} \times \underline{\rho}_i$$

$$\dot{\underline{\rho}}_i \cdot \dot{\underline{\rho}}_i = (\underline{\omega} \times \underline{\rho}_i) \cdot (\underline{\omega} \times \underline{\rho}_i) = \dot{\underline{\rho}}_i \cdot (\underline{\omega} \times \underline{\rho}_i)$$

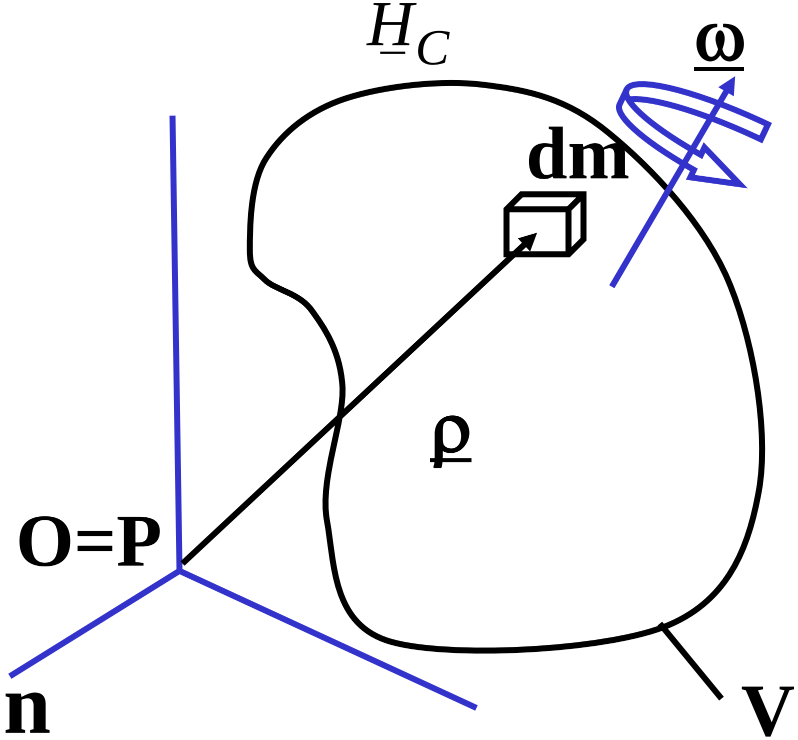
$$\Rightarrow \quad T_{rot} = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\underline{\rho}}_i \cdot \dot{\underline{\rho}}_i = \frac{1}{2}\sum_{i=1}^{n} \underline{\omega} \cdot \underline{\rho}_i \times m_i \dot{\underline{\rho}}_i$$

(using permutation in scalar triple product)

Now, for a continuous mass distribution

$$T_{rot} = \frac{1}{2}\int_V \rho\, \underline{\omega} \cdot \underline{\rho} \times \dot{\underline{\rho}}\, dV = \frac{1}{2}\underline{\omega} \cdot \underbrace{\int_V \rho\, \underline{\rho} \times \dot{\underline{\rho}}\, dV}_{\underline{H}_C}$$

$$or \quad T_{rot} = \frac{1}{2}\underline{\omega} \cdot \underline{H}_C$$



If P is a fixed point:

$$T_{rot} = \frac{1}{2}\underline{\omega} \cdot \underline{H}_P$$

In vector matrix notation

$$T_{rot} = \frac{1}{2}\lfloor \omega \rfloor \{H_P\} = \frac{1}{2}\{\omega\}^T \{H_P\}$$

Since $\{H_P\}=[I]\{\omega\}, \quad T_{rot}=\frac{1}{2}\{\omega\}^T[I]\{\omega\}$

If one uses xyz coordinate system located at the center of mass of the rigid body,

$$T_{rot}=\frac{1}{2}\{\omega\}^T[I]\{\omega\}=[I_{xx}\omega_x^2+I_{yy}\omega_y^2+I_{zz}\omega_z^2$$

$$+2I_{xy}\omega_x\omega_y+2I_{xz}\omega_x\omega_z+2I_{yz}\omega_y\omega_z]/2$$

In summation, notation

$$T_{rot}=\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}I_{ij}\omega_i\omega_j$$

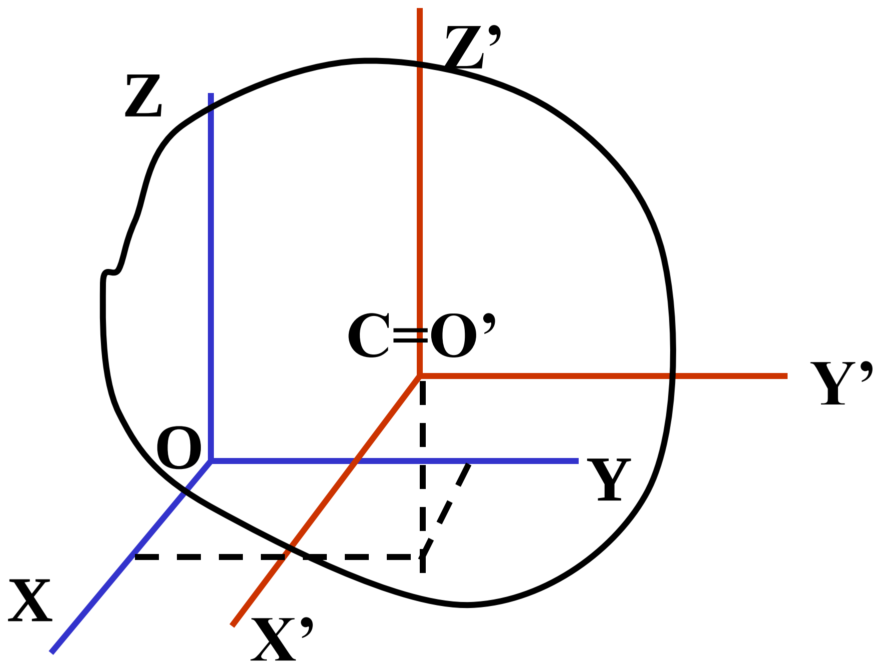
Ex: If $\underline{\omega}$ has the direction of one of the coordinate axes at an instant,

$$T_{rot} = \frac{1}{2} I \omega^2$$

Here, I - moment of inertia about the axis of rotation,

$\underline{\omega}$ - instantaneous angular velocity of the rigid body.

7-6 Translation of Coordinate Axes



O'x'y'z'- coordinate system located at the centroid of the body; Oxyz-any other coord system; O'=C-the centroid

O'**: center of mass: the coordinates of the center of mass in Oxyz system are** $\left(x_c, y_c, z_c\right)$

Let I_{xx}, I_{yy}, I_{zz} - moments of inertia about xyz axes system

$I_{x'x'}, I_{y'y'}, I_{z'z'}$ = moments of inertia about centroidal axes

m - mass of the body

Easy to show (read 7.6) that

for moments of inertia

$$\left.\begin{aligned} I_{xx} &= I_{x'x'} + m(y_c^2 + z_c^2) \\ I_{yy} &= I_{y'y'} + m(x_c^2 + z_c^2) \\ I_{zz} &= I_{z'z'} + m(x_c^2 + y_c^2) \end{aligned}\right\} \text{parallel axis theorem}$$

For products of inertia

$$I_{xy} = I_{x'y'} - m x_c y_c$$

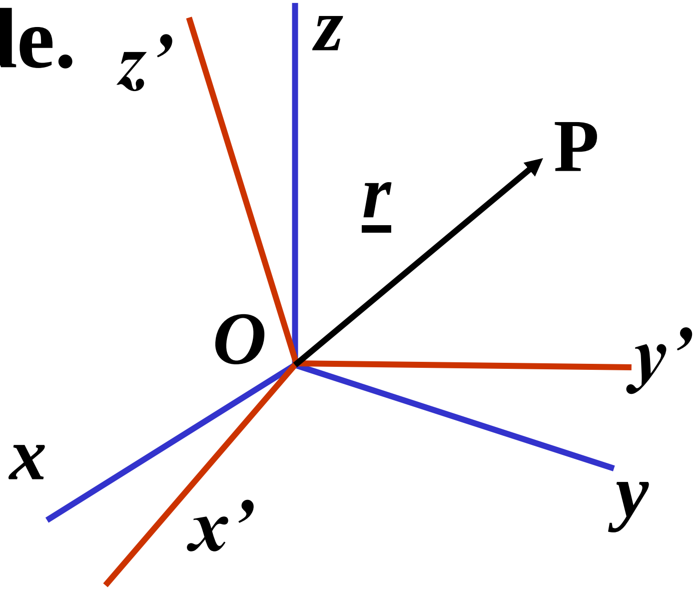
$$I_{xz} = I_{x'z'} - m x_c z_c$$

$$I_{yz} = I_{y'z'} - m y_c z_c$$

- **Note that the moments of inertia about the centroidal axes are the smallest.**
- **The products of inertia may increase or decrease compared to those about centroidal axes depending on the particular case.**

7.7 Rotation of Coordinate Axes

Consider two different coordinate systems. We assume that the origins for the two systems coincide.



consider a vector:

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$
$$= x'\underline{i}' + y'\underline{j}' + z'\underline{k}'$$

These are two ways of expressing the same vector $\underline{r}$.

The two systems are characterized by unit vectors $\underline{i}, \underline{j}, \underline{k}$ **and** $\underline{i}', \underline{j}', \underline{k}'$

Let $\underline{i} = l_{x'x}\underline{i}' + l_{y'x}\underline{j}' + l_{z'x}\underline{k}'$

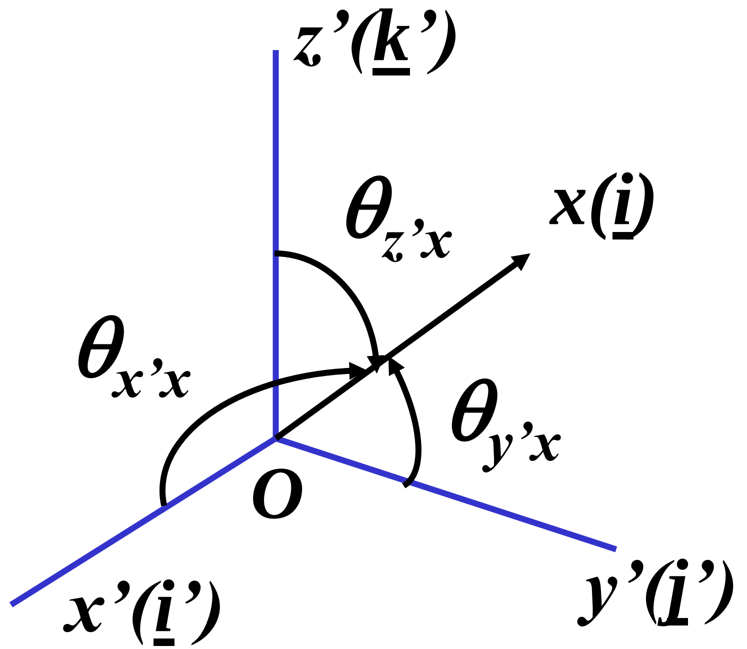
- $l_{x'x}, l_{y'x}, l_{z'x}$ **are the cosines of the angles made by the x axis with the** x', y' and z' **directions, respectively.**

Note that,

$$\left|\underline{i}\right| = \left|l_{x'x}\underline{i}' + l_{y'x}\underline{j}' + l_{z'x}\underline{k}'\right| = 1$$

$$\Rightarrow \quad l_{x'x}^2 + l_{y'y}^2 + l_{z'z}^2 = 1$$

Consider the example:



$$\left.\begin{array}{l}\theta_{x'x}\\ \theta_{y'x}\\ \theta_{z'x}\end{array}\right\}$$ **direction angles of x-axis**

$$Also, \quad l_{x'x} = \cos\theta_{x'x}, \quad l_{y'x} = \cos\theta_{y'x},$$

$$l_{z'x} = \cos\theta_{z'x}$$

Similarily, we can write

$$\underline{j} = l_{x'y}\,\underline{i'} + l_{y'y}\,\underline{j'} + l_{z'y}\,\underline{k'}$$

$$\underline{k} = l_{x'z}\,\underline{i'} + l_{y'z}\,\underline{j'} + l_{z'z}\,\underline{k'}$$

Thus, the vector $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ **can be written as**

$$\begin{aligned}\underline{r} &= x(l_{x'x}\,\underline{i'} + l_{y'x}\,\underline{j'} + l_{z'x}\,\underline{k'}) \\ &\quad + y(l_{x'y}\,\underline{i'} + l_{y'y}\,\underline{j'} + l_{z'y}\,\underline{k'}) \\ &\quad + z(l_{x'z}\,\underline{i'} + l_{y'z}\,\underline{j'} + l_{z'z}\,\underline{k'}) \\ &\equiv x'\underline{i'} + y'\underline{j'} + z'\underline{k'}\end{aligned}$$

In vector-matrix notation

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} l_{x'x} & l_{x'y} & l_{x'z} \\ l_{y'x} & l_{y'y} & l_{y'z} \\ l_{z'x} & l_{z'y} & l_{z'z} \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

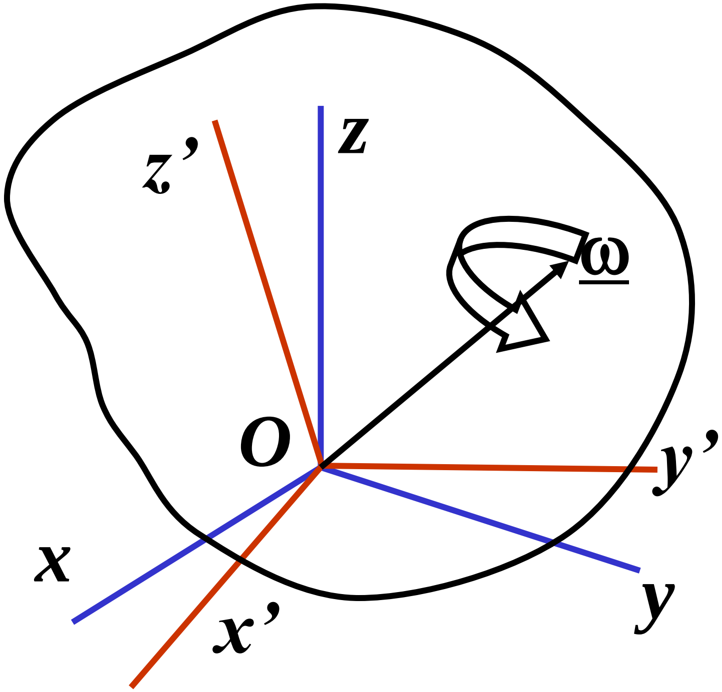
In more compact notation

$$\{r'\} = [l]\{r\}$$

where $\{r\}$ and $\{r'\}$ **are representations of** $\underline{r}$ and $\underline{r'}$ **in the two coordinate systems.**

It provides relation between components of the same vector in two coordinate systems.

Ex: As another example, consider the Kinetic energy of a rigid body



O =C, the center of mass of the body. There are two coordinate axes, xyz, x'y'z'

$$\underline{\omega} = \omega_x \underline{i} + \omega_y \underline{j} + \omega_z \underline{k}$$
$$= \omega_{x'} \underline{i}' + \omega_{y'} \underline{j}' + \omega_{z'} \underline{k}'$$

Here $\{\omega\}$ and $\{\omega'\}$ **are two representations of the same angular velocity vector.**

Then, the kinetic energy expressions in the two coordinate systems are

$$T_{rot} = \frac{1}{2}\{\omega\}^T [I]\{\omega\} = \frac{1}{2}\{\omega'\}^T [I']\{\omega'\}$$

$$Also, \; \{\omega'\} = [l]\{\omega\} \quad or \quad \{\omega'\}^T = \{\omega\}^T [l]^T$$

$$\Rightarrow \; T_{rot} = \frac{1}{2}\{\omega\}^T [I]\{\omega\} = \frac{1}{2}\{\omega\}^T [l]^T [I'][l]\{\omega\}$$

$$and \quad [I] = [l]^T [I'][l]$$

In component form, this transformation of inertia matrices is $I_{ij} = \frac{1}{2}\sum_{m'=1}^{3}\sum_{n'=1}^{3} l_{m'i} l_{n'j} I'_{m'n'}$

Some properties of the matrix [*l*].

It is an orthogonal matrix, i.e.,

$$[1] = [l]^T [l] \qquad \textbf{(the identity matrix)}$$

$$or \quad [l]^T = [l]^{-1}$$

Since $\det[l] = \det([l]^T)$, **(true for any matrix)**

$(\det[l])^2 = 1 \;\Rightarrow$ **the matrix operation with [*l*] only rotates a given vector.**

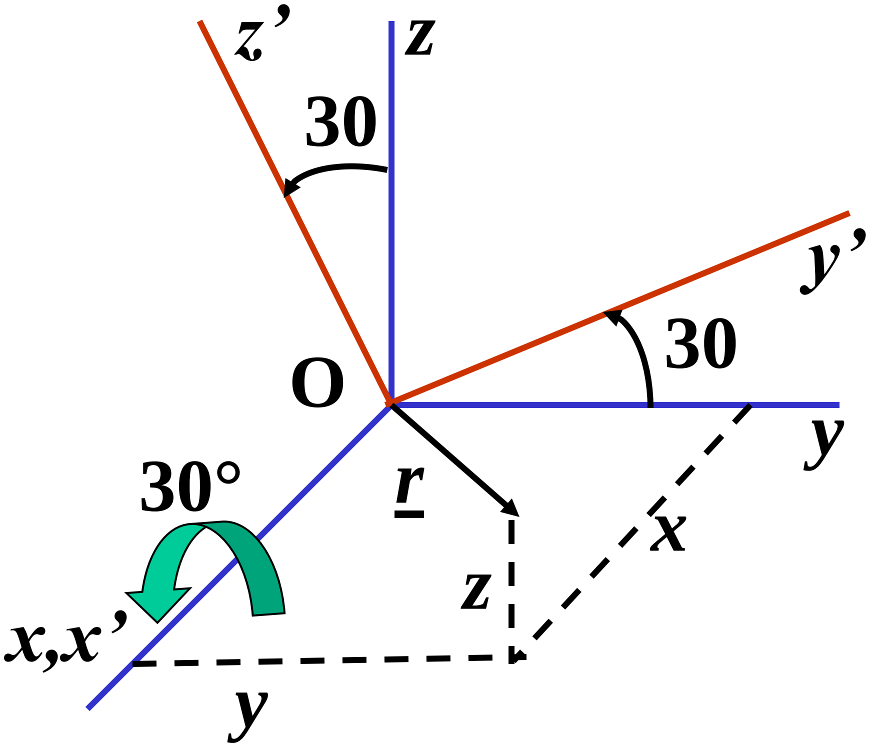
The relation $[1]=[l]^T[l]$ can be explicitly written as

$$\sum_{m'=1}^{3} l_{m'i} l_{m'j} = \begin{cases} 1 & if \;\; i=j \\ 0 & if \;\; i \neq j \end{cases}$$

These are <u>nine</u> relations in the <u>nine</u> components of the matrix [*l*]. Six of the these equations are linearly independent.

$\rightarrow$ There are 9 - 6 = 3 rotational degrees of freedom for a set of orthogonal coordinate axes (or for a rigid body).

Ex: Rotation of axes



Consider a vector $\underline{r}$ that is represented in xyz and x'y'z' axes. The two systems are rotated by 30° with respect to each other.

Then,

$$\underline{r} = x'\underline{i}' + y'\underline{j}' + z'\underline{k}'$$

$$\{r'\} = [l]\{r\}$$

$$\{r'\} = [l]\{r\}$$

Now

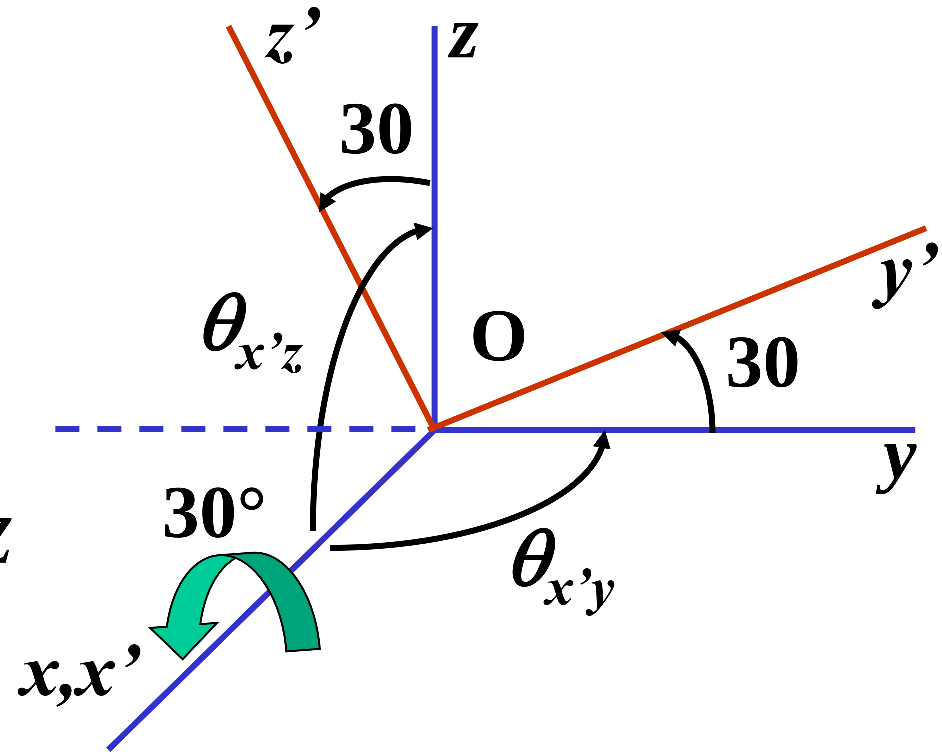
$$x' = l_{x'x}x + l_{x'y}y + l_{x'z}z$$

$$l_{x'x} = \cos 0 = 1,$$

$$l_{x'y} = \cos\theta_{x'y} = \cos 90 = 0$$

$$l_{x'z} = \cos\theta_{x'z} = \cos 90 = 0$$

$$y' = l_{y'x}x + l_{y'y}y + l_{y'z}z$$



$$l_{y'x} = \cos\theta_{y'x} = \cos 90 = 0,$$

$$l_{y'y} = \cos\theta_{y'y} = \cos 30 = \sqrt{3}/2$$

$$l_{y'z} = \cos\theta_{y'z} = \cos 60 = 1/2$$

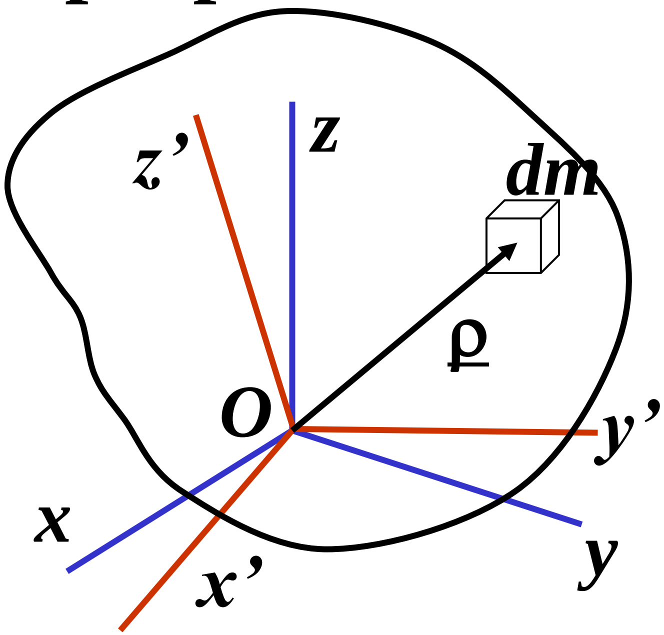
$$z' = l_{z'x}x + l_{z'y}y + l_{z'z}z$$

$$l_{z'x} = 0, \quad l_{z'y} = \cos 120 = -1/2$$

$$l_{z'z} = \cos 30 = \sqrt{3}/2$$

$$\Rightarrow \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

7.8 Principal Axes consider the inertia properties of a rigid body



$$I_{xx} = \int_V \rho(y^2 + z^2)dV$$

$$I_{yy} = \int_V \rho(x^2 + z^2)dV$$

$$I_{zz} = \int_V \rho(x^2 + y^2)dV$$

Then, it is easy to see that

$$I_{xx} + I_{yy} + I_{zz} = \int_V 2\rho(r^2)dV$$

where $r^2 = x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 = |\underline{\rho}|^2$

$$\Rightarrow \mathrm{I}_{xx} + \mathrm{I}_{yy} + \mathrm{I}_{zz} = \mathrm{I}_{x'x'} + \mathrm{I}_{y'y'} + \mathrm{I}_{z'z'}$$

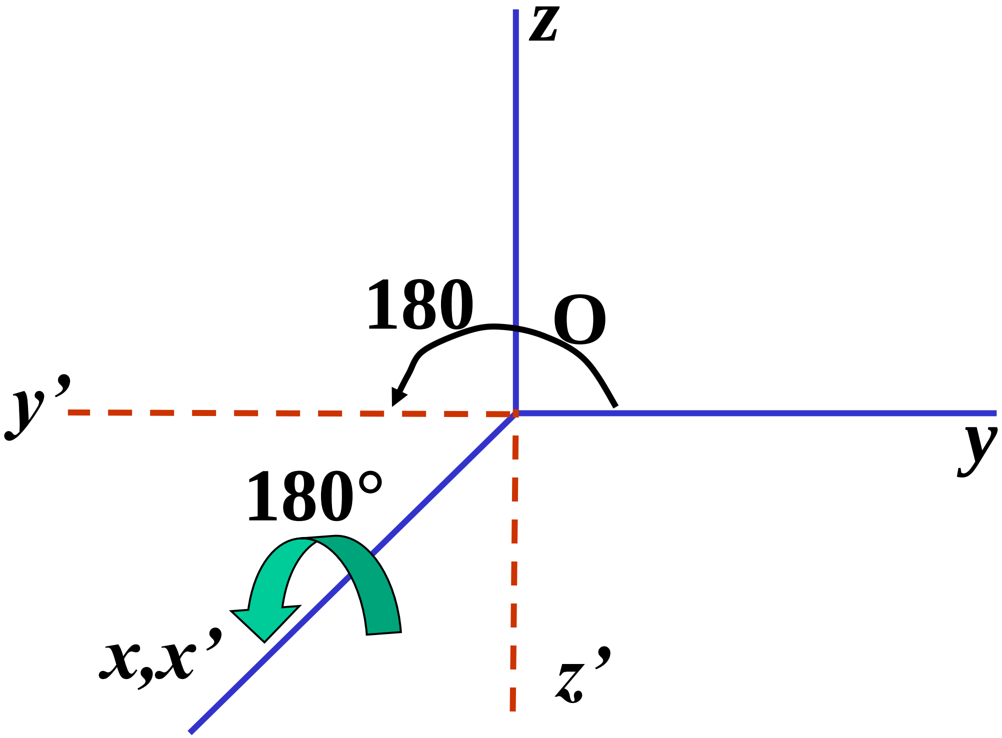
i.e., the sum of moments of inertia is invariant to coordinate system rotation.

More generally, *tr*[I]≡*sum of diagonal terms* is unchanged due to coordinate rotation (an orthogonal transform).

Consider products of inertia:

As an example: $I_{xy} = -\int_V \rho \, \mathrm{xy} \, \mathrm{dV}$

A 180 ° rotation about the x-axis →



$$\Rightarrow I_{x'y'} = -I_{xy}$$

$$I_{x'z'} = -I_{xz}$$

$$I_{y'z'} = I_{yz}$$

- **In general, the products of inertia have no preferred sign; the sign depends on the orientation of the body with respect to the coordinate system.**

For a body with random orientation, positive and negative values of products of inertia are equally likely to occur.

- **The moments and products of inertia are a smooth function of the orientation of the coordinate system orientation since**

$$[I'] = [l][I][l]^T$$

relates the inertia properties in two systems

$\rightarrow$ **It is possible to find a coordinate system in which the products of inertia vanish simultaneously**

$\rightarrow$ **Such a coordinate system is called the** **principal axes of the rigid body.**

- **Consider the relation (alternate way to think)**

$$\{H\} = I\{\omega\}$$

Q: Is it possible to find a coordinate system in which the angular momentum vector is instantaneously parallel to the angular velocity vector?

i.e., can we write $\{H\} = I\{\omega\}$ **for some system?**

If one can do that, then

$$[I]\{\omega\} = I\{\omega\} = I[1]\{\omega\}$$

or
$$\begin{bmatrix} (I_{xx} - I) & I_{xy} & I_{xz} \\ I_{xy} & (I_{yy} - I) & I_{yz} \\ I_{zx} & I_{zy} & (I_{zz} - I) \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = 0$$

(This is really the eigenvalue problem for the inertia matrix [I]).

If $\{\omega\} \neq 0$, i.e., $\underline{\omega} \neq 0$ **, then det ([I] - I [1]) = 0**

Or

$$\begin{vmatrix} (I_{xx}-I) & I_{xy} & I_{xz} \\ I_{xy} & (I_{yy}-I) & I_{yz} \\ I_{zx} & I_{zy} & (I_{zz}-I) \end{vmatrix} = 0$$

(this is a characteristic equation, a cubic in I with coefficients I_{ij} :

$$a_o I^3 + a_1 I^2 + a_2 I + a_3 = 0.$$

Let I_1, I_2, I_3 be the roots of the cubic and $\{\omega\}^1,\ \{\omega\}^2,\ \{\omega\}^3$ be the eigenvectors associated with the eigenvalues:

They satisfy

$$[I]\ \{\omega\}^{i} = I_i\ \{\omega\}^{i} \qquad \textbf{i = 1, 2, 3.}$$

Clearly, the eigenvectors are known only up to an arbitrary constant, i.e., only the ratio of the components are fixed; in particular, if we assume ω_x to be arbitrary and non-zero, the ratios satisfy

$$\begin{bmatrix} (I_{yy} - I_i) & I_{yz} \\ I_{zy} & (I_{zz} - I_i) \end{bmatrix} \begin{Bmatrix} \omega_y / \omega_x \\ \omega_z / \omega_x \end{Bmatrix} = \begin{Bmatrix} -I_{xy} \\ -I_{zx} \end{Bmatrix}$$

If one chooses the constraint

$$\omega_x^2 + \omega_y^2 + \omega_z^2 = 1,$$

then the values of ω_x, ω_y and ω_z **are direction cosines of** $\underline{\omega} \Rightarrow$ **they determine the direction of the corresponding principal axis at the reference point P.**

It is easy to show that if two principal moments of inertia are distinct, the corresponding eigenvectors are orthogonal, i.e.,

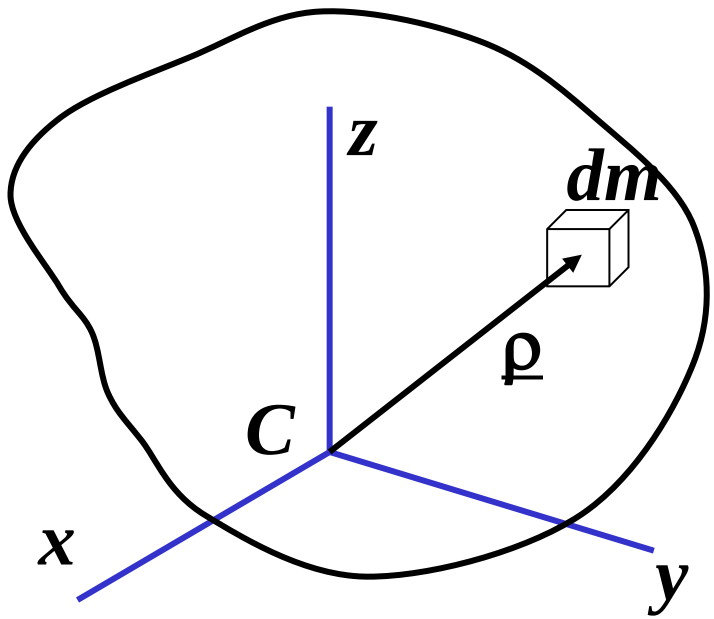
$$\{\omega\}^{1^T} \; \{\omega\}^2 = 0 \text{ if } \; I_1 \neq I_2.$$

THE ELLPSOID OF INERTIA

Read from the text.

Ex. 7.4 (text)

Consider a given body with a coordinate system located at its centroid.



In this coordinate system, the inertia properties of the body are given by a matrix [I].

$$[I]=\begin{bmatrix} 150 & 0 & -100 \\ 0 & 250 & 0 \\ -100 & 0 & 300 \end{bmatrix} kg-m^2$$

We want to find the principal moments of inertia and the associated directions; also, the coordinate transformation [*l*], specified by the rotation matrix, which diagonalizes the inertia matrix.

Essentially, we are considering the eigenvalue problem $[I]\{\omega\}=I\ \{\omega\}\ \text{for}\ [I]$.

The characteristic equation for the matrix [*I*] is

$$\left|\begin{bmatrix} 150-I & 0 & -100 \\ 0 & 250-I & 0 \\ -100 & 0 & 300-I \end{bmatrix}\right| = 0$$

$$\Rightarrow \quad (250-\mathrm{I})\,(\mathrm{I}^2 - 450\mathrm{I} + 3.5\times 10^4) = 0$$

The roots, the principal moments of inertia, are

$$\mathrm{I}_1 = 100\mathrm{kg}-\mathrm{m}^2, \mathrm{I}_2 = 250\mathrm{kg}-\mathrm{m}^2, \mathrm{I}_3 = 350\mathrm{kg}-\mathrm{m}^2$$

We now need to find the eigenvectors, which give directions of the principal axes.

The eigenvalue problem is

$$\left[\,[\,\mathrm{I}\,] - \mathrm{I}\,[1]\,\right]\{\omega\} = 0$$

or $(150 - I_i)\ \ \omega_x - 100\ \ \omega_z = 0$ **(1)**

$$(250 - \mathrm{I}_i)\ \ \omega_y = 0 \quad \textbf{(2)}$$

$$-100\omega_x\ \ + (300 - I_i)\ \ \omega_z = 0 \quad \textbf{(3)}$$

a) Consider the 1st eig.val.: $\mathrm{I}_1 = 100\ kg - m^2$:

(2) $\Rightarrow$ $\omega_y = 0$

(1) $\Rightarrow$ $50\,\omega_x = 100\,\omega_z$ or $\omega_z / \omega_x = 1/2.$

Thus $\{\omega\}^1 = \omega_x\ \ \{1\,,\ 0\ ,\ 1/2\}^T$.

a) Consider the 2nd eig.val.: $I_2 = 250 \;\; kg - m^2$:

(2) $\Rightarrow \;\; \omega_y$ is arbitrary

(1) $\Rightarrow \; -100\,\omega_x - 100\,\omega_z = 0$

(3) $\Rightarrow \; -100\,\omega_x + 50\,\omega_z = 0$

The only possible solution to these equations is

$$\omega_x = \omega_z = 0 \Rightarrow \{\omega\}^2 = \omega_y \;\; \{0 \; , \; 1 \; , \; 0\}^T \; .$$

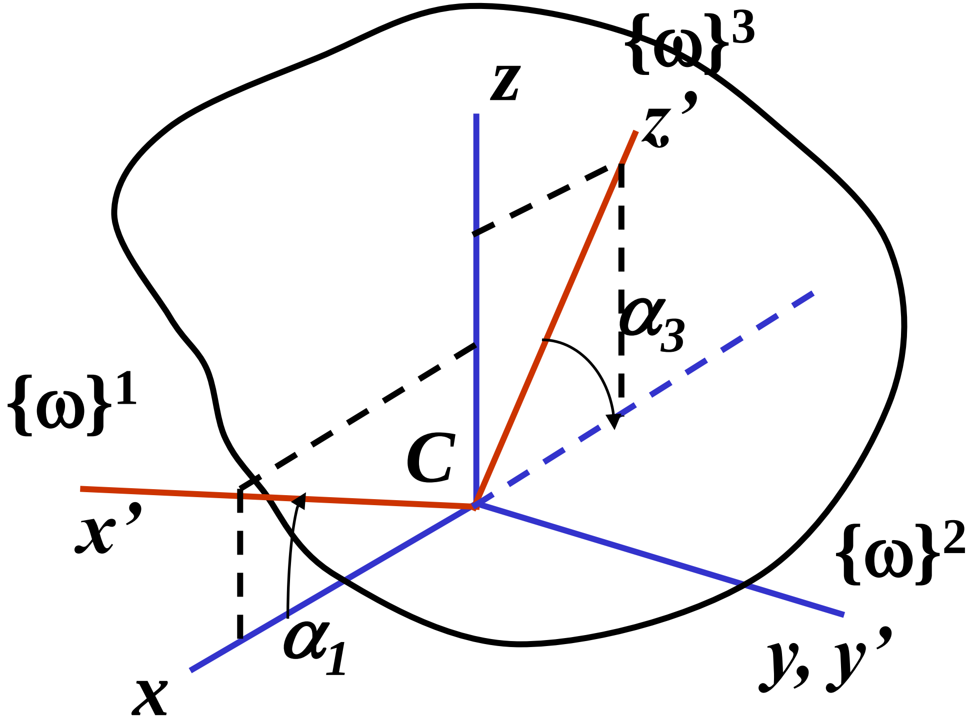
c) Consider the 3rd eig.val.: $I_3 = 350\,kg - m^2$:

(2) $\Rightarrow \; \omega_y = 0$ **Then (1), (3)** $\Rightarrow \omega_z / \omega_x = -2.$

Thus, $\{\omega\}^3 = \omega_x \;\; \{1, \; 0 \; , \; -2\}^T \; .$

Note: The three eigenvectors are, as expected, orthogonal to each other.

$$\{\omega\}^{iT}\{\omega\}^{j}=0,\ i\neq j.$$



Note that

$$\{\omega\}^{1},\ \{\omega\}^{3}$$

are in xz plane

Clearly, x'y'z' is the principal coordinate system

The direction cosines for the $x'\ y'\ z'$ system are:

$$l_{x'x} = \cos\theta_{x'x} = \cos\alpha_1 = 2/\sqrt{5}$$

$$l_{x'y} = \cos\theta_{x'y} = \cos 90 = 0$$

$$l_{x'z} = \cos\theta_{x'z} = \sin\alpha_1 = 1/\sqrt{5}$$

$$Similarly, \quad l_{y'x} = \cos\theta_{y'x} = \cos 90 = 0,$$

$$l_{y'y} = \cos\theta_{y'y} = \cos 0 = 1,\ l_{y'z} = \cos\theta_{y'z} = \cos 90 = 0$$

$$and \quad l_{z'x} = -1/\sqrt{5},\ l_{z'y} = 0,\ l_{z'z} = 2/\sqrt{5}$$

These allow us to construct the rotation matrix

Thus,

$$[l]=\begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \end{bmatrix}$$

- **It is easy to check that**

$$[I']=[l][I][l]^T=\begin{bmatrix} 100 & 0 & 0 \\ 0 & 250 & 0 \\ 0 & 0 & 350 \end{bmatrix}\ kg-m^2$$

7.9 Displacements of a Rigid Body

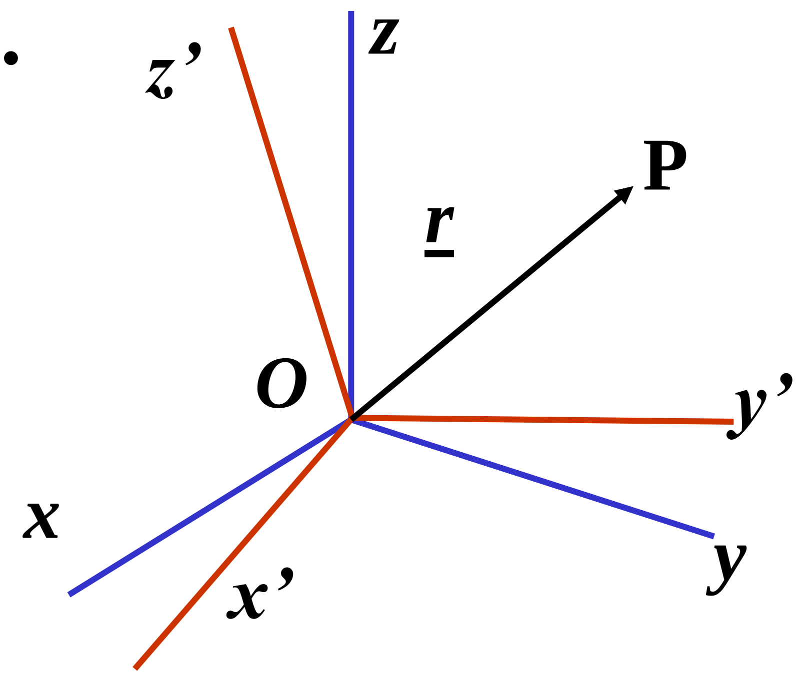
Euler's Theorem: The most general displacement of a rigid body with one point fixed is equivalent to a single rotation about some axis through that fixed point

Chasles' Theorem: The most general displacement of a rigid body is equivalent to a screw displacement, i.e., translational motion of a reference point followed by rotation about an axis through the ref. point

Assignment: Read the Section 7.9

7.10 Axis and Angle of Rotation

- **We know that the components of a vector in two different coordinate systems are obtained by the application of a rotation matrix [*l*] whose elements are the direction cosines.**



Then, the vector $\underline{r}$ can be written as

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = x'\underline{i}' + y'\underline{j}' + z'\underline{k}'$$

$$\Rightarrow \quad \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} l_{x'x} & l_{x'y} & l_{x'z} \\ l_{y'x} & l_{y'y} & l_{y'z} \\ l_{z'x} & l_{z'y} & l_{z'z} \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

A natural question: Is there a vector whose coordinates remain unchanged as the coordinate system xyz is rotated to the coordinate system $x'y'z'$?

If such a vector $\underline{r}$ exists,

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = x\underline{i'} + y\underline{j'} + z\underline{k'}$$

where $\underline{i},\ \underline{j},\ \underline{k}$ *and* $\underline{i'},\ \underline{j'},\ \underline{k'}$ **are the basis vectors that define the two coordinate systems, then, given the rotation matrix [*l*], the relation** $\{r'\} = [l]\{r\}$ **gives**

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} l_{x'x} & l_{x'y} & l_{x'z} \\ l_{y'x} & l_{y'y} & l_{y'z} \\ l_{z'x} & l_{z'y} & l_{z'z} \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

$$or \qquad 1\{r\}-[l]\{r\}=0 \quad \text{must be satisfied.}$$

This is an eigenvalue question for matrix [*l*].

- **Thus, the existence of such a vector $\{r\}$ (or $\underline{r}$) is associated with matrix [*l*] having an eigenvalue of 1. The corresponding eigenvector will then define the direction which remains unchanged due to rotation, and hence, represents the axis of rotation.**

7.10 Axis and Angle of Rotation

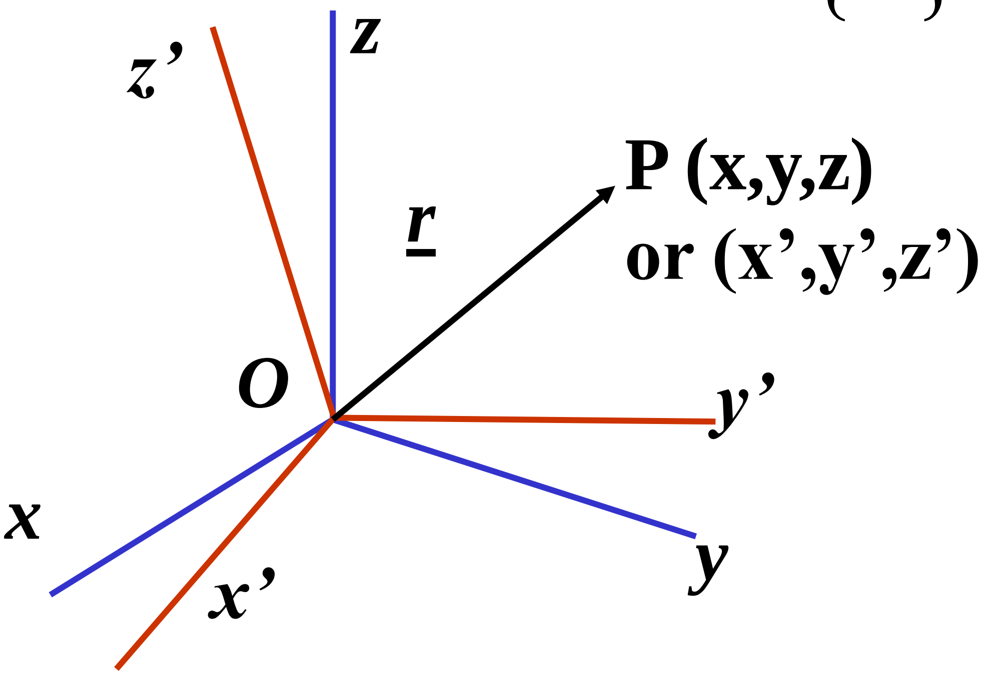
7.11 Reduction of Forces - equivalent forces and couples

Reading Assignments

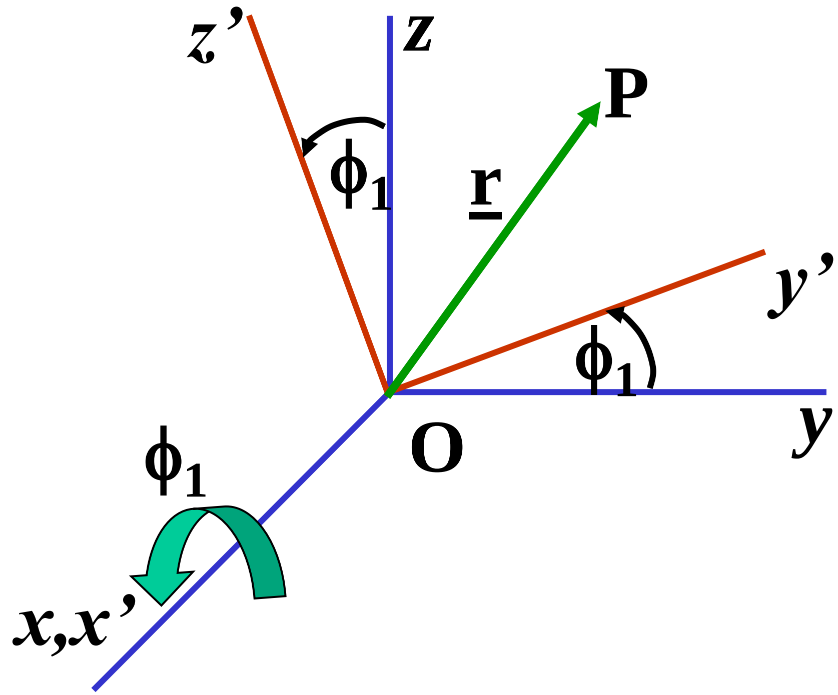
7.12 Infinitesimal Rotations

- **Consider a sequence of rotations: the new and old representations of the vector *r* are related by**

$$\{r'\} = [l]\{r\} \equiv [\Phi]\{r\}$$



Ex: Consider a counterclockwise rotation about the x axis.



Then,

$$\{r'\} = [\Phi_1]\{r\}$$

where,

the rotation matrix is

$$[\Phi_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & \sin\phi_1 \\ 0 & -\sin\phi_1 & \cos\phi_1 \end{bmatrix}$$

Suppose we perform two rotations in a sequence:

$$\{r\} \to \{r'\} = [\Phi_1]\{r\}$$

$$\{r'\} \to \{r''\} = [\Phi_2]\{r'\} = [\Phi_2][\Phi_1]\{r\}$$

combining $\Rightarrow \{r\} \to \{r''\} = [\Phi]\{r\}$

where $[\Phi] = [\Phi_2][\Phi_1]$ (combined rotation matrix)

Since matrix multiplication does commute,

$$\left[\Phi\right]=\left[\Phi_2\right]\left[\Phi_1\right]\neq\left[\Phi_1\right]\left[\Phi_2\right]$$

Thus, the order in which rotations are accomplished is crucial to know for finite rotations.

We now show that infinitesimal rotations commute!!

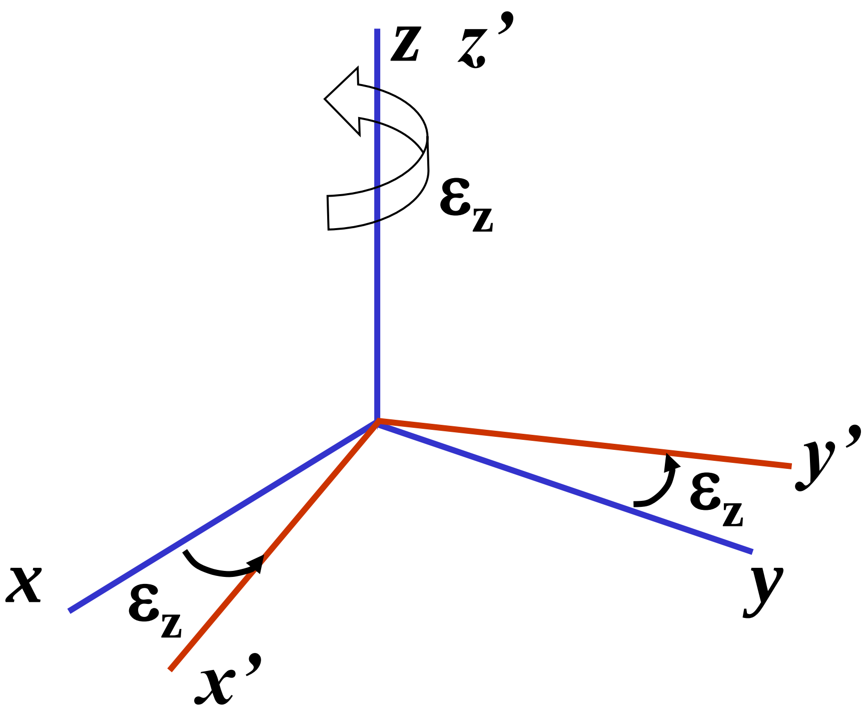
Ex: Consider a sequence of two rotations given below:

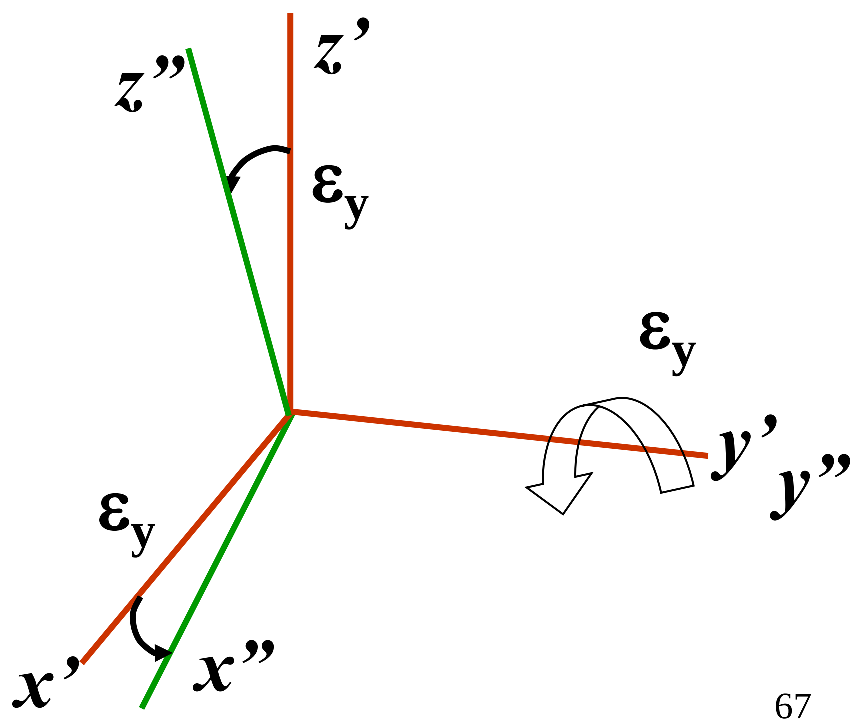
1. xyz→x'y'z'.

Let us call it $\left[\Phi_1\right]$

2. x'y'z' →x"y"z"

Let us call it $\left[\Phi_2\right]$





$$[\Phi_1] \equiv [\varepsilon_z] = \begin{bmatrix} \cos\varepsilon_z & \sin\varepsilon_z & 0 \\ -\sin\varepsilon_z & \cos\varepsilon_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & \varepsilon_z & 0 \\ -\varepsilon_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [1] + [\varepsilon_1]$$

$$[\Phi_2] \equiv [\varepsilon_y] = \begin{bmatrix} \cos\varepsilon_y & 0 & -\sin\varepsilon_z \\ 0 & 1 & 0 \\ \sin\varepsilon_z & 0 & \cos\varepsilon_z \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & -\varepsilon_y \\ 0 & 1 & 0 \\ \varepsilon_y & 0 & 1 \end{bmatrix}$$

$$= [1] + [\varepsilon_2]$$

Thus, the two sequences give:

$$
\begin{aligned}
[\Phi_2][\Phi_1] &= ([1]+[\varepsilon_2])([1]+[\varepsilon_1]) \\
&= [1]+[\varepsilon_1]+[\varepsilon_2]+[\varepsilon_2][\varepsilon_1] \\
&\cong [1]+[\varepsilon_1]+[\varepsilon_2]
\end{aligned}
$$

$$
\begin{aligned}
[\Phi_1][\Phi_2] &= ([1]+[\varepsilon_1])([1]+[\varepsilon_2]) \\
&= [1]+[\varepsilon_1]+[\varepsilon_2]+[\varepsilon_1][\varepsilon_2] \\
&\cong [1]+[\varepsilon_1]+[\varepsilon_2]
\end{aligned}
$$

Order not important and rotations can be added vectorially. Infinitesimal rotations → angular velocities add as vectors

Ex: Consider now a sequence of three infinitesimal rotations:

1. ε_x - about x axis

$$\{r\} \rightarrow \{r'\} = [\varepsilon_x]\{r\}$$

2. ε_y - about y' axis

$$\{r'\} \rightarrow \{r''\} = \left[\varepsilon_y\right]\{r'\} = \left[\varepsilon_y\right][\varepsilon_x]\{r\}$$

3. ε_z - about z" axis

$$\{r''\} \rightarrow \{r'''\} = [\varepsilon_z]\{r''\} = [\varepsilon_z]\left[\varepsilon_y\right][\varepsilon_x]\{r\}$$

$$\rightarrow \{r'''\} \cong \left[[1] + [\hat{\varepsilon}_z] + \left[\hat{\varepsilon}_y\right] + [\hat{\varepsilon}_x]\right]\{r\}$$

Here,

$$\left[\hat{\varepsilon}_x\right]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon_x \\ 0 & -\varepsilon_x & 0 \end{bmatrix}, \quad \left[\hat{\varepsilon}_y\right]=\begin{bmatrix} 0 & 0 & -\varepsilon_y \\ 0 & 0 & 0 \\ \varepsilon_y & 0 & 0 \end{bmatrix}$$

$$\left[\hat{\varepsilon}_z\right]=\begin{bmatrix} 0 & \varepsilon_z & 0 \\ -\varepsilon_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let

$$[\hat{\varepsilon}] = \left[\hat{\varepsilon}_x\right] + \left[\hat{\varepsilon}_y\right] + \left[\hat{\varepsilon}_z\right]$$

$$Then,\ [\hat{\varepsilon}] = \begin{bmatrix} 0 & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 0 & \varepsilon_x \\ \varepsilon_z & -\varepsilon_x & 0 \end{bmatrix}$$

and the complete relation is

$$\left\{r'''\right\} \cong \left[[1] + \left[\hat{\varepsilon}_z\right] + \left[\hat{\varepsilon}_y\right] + \left[\hat{\varepsilon}_x\right]\right]\left\{r\right\} = \left[[1] + \left[\hat{\varepsilon}\right]\right]\left\{r\right\}$$

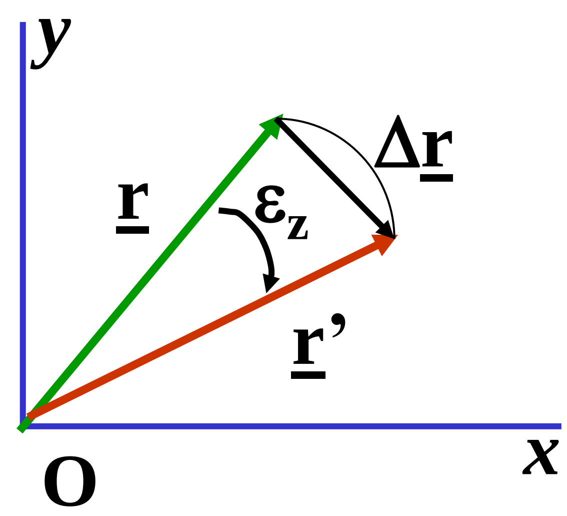
Now, apply an infinitesimal rotation to a vector $\underline{r}$ of constant length.

e.g.: consider a rotation about z axis.

Then,

$$\{r'\} = \left[[1] + [\varepsilon]\right]\{r\}$$

$$or \quad \{r'\} - \{r\} = [\varepsilon]\{r\}$$



Note: in the above, we considered rotation of coordinate axes with vector fixed in space

$\rightarrow$ any changes in components of $\underline{r}$ are entirely due to coordinate axes rotations.

Now: Consider coordinate system fixed and let the vector $\underline{r}$ rotate in opposite direction.

Suppose that this rotation takes place in time Δt. Then,

$$\{\dot{r}\} = \lim_{\Delta t \to 0} \frac{\{r'\} - \{r\}}{\Delta t}$$

or $\{\dot{r}\} = \lim_{\Delta t \to 0} \frac{[-\varepsilon]\{r\}}{\Delta t} = [\omega]\{r\}$ *or* $\{\dot{r}\} = [\omega]\{r\}$

Here $[\omega] = \lim_{\Delta t \to 0} \frac{[-\varepsilon]}{\Delta t}$ **the angular velocity matrix**

The negative sign is introduced so that [ω] refers to rotation of $\underline{r}$ and not to that of the coordinate system.

Recall: for a vector of constant length,

$$\underline{\dot{r}} = \underline{\omega} \times \underline{r}$$

Thus: $\dot{\underline{r}} = \underline{\omega} \times \underline{r}$

and $\{\dot{r}\} = [\omega]\{r\}$

are statements of the same fact in different forms.

Note: $[\omega]$ **is a skew symmetric matrix**

consider the matrix: $[\omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$

Then:

$$\{\dot{r}\} = [\omega]\{r\}\begin{Bmatrix} (-\omega_z y + \omega_y z) \\ (-\omega_z x + \omega_x z) \\ (-\omega_y x + \omega_x y) \end{Bmatrix} = \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix}$$

Also:

$$\underline{\dot{r}} = \underline{\omega} \times \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}$$

$$= (-\omega_z y + \omega_y z)\underline{i} + (\omega_z x - \omega_x z)\underline{j}$$

$$+ (-\omega_y x + \omega_x y)\underline{k}$$