Reviewing, what we have discussed so far:

Generalized coordinates

- **Any number of variables (say, n) sufficient to specify the configuration of the system at each instant to time (need not be the minimum number).**
- **In general, let** $q_1, q_2, \cdots, q_n -$ **be generalized coordinates. Then, the position vectors are described by**

$$\underline{r}_i = \underline{r}_i(q_1, q_2, ..., q_n),\ \ i = 1, 2, ..., N$$

In general, we also have some

- **geometric constraints**

$$\phi_i(\underline{r}_i, \cdots, \underline{r}_N, t) = 0, \;\; i = 1, 2, \cdots, d.$$

These are *d* equations in 3N (scalar) variables.

Let $q_1, q_2, \cdots, q_n$ **be the generalized coordinates or variables**

i.e., $\underline{r}_i = \underline{r}_i(q_1, q_2, ..., q_n, t), \;\; i = 1, 2, ..., N$

so that the geometric constraints are satisfied.

- **If we eliminate all geometric constraints**

$$n = n_o = 3N - d.$$

where *n* – number of generalized coordinates $\Rightarrow \ n > n_o.$

- **Sometimes, one may not want to solve for all the geometric constraints**

Then $q_1, \ldots, q_n$ **are more than the minimum needed and not all are independent.**

- **Now consider the work done by effective forces in any virtual displacement**

$$\delta W = \sum_{i=1}^{N} \underline{F}_i \bullet \delta \underline{r}_i$$

Now, consider the position vector, and its virtual displacement:

$$\underline{r}_i = \underline{r}_i(q_1, q_2, \ldots, q_n, t)$$

$$\Rightarrow \ \delta \underline{r}_i = \sum_{j=1}^{n} \frac{\partial \underline{r}_i}{\partial q_j} \delta q_j, \ \ i = 1, \ldots., N$$

where δq_j – **virtual displacement in** q_j.

Then, the virtual work is

$$\delta W = \sum_{i=1}^{N} \underline{F}_i \bullet \sum_{j=1}^{n} \frac{\partial \underline{r}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{N} \underline{F}_i \bullet \frac{\partial \underline{r}_i}{\partial q_j} \right) \delta q_j$$

$$\Rightarrow \ \delta W \equiv \sum_{j=1}^{n} Q_j \delta q_j \ \Rightarrow \ \boxed{Q_j = \sum_{i=1}^{N} \underline{F}_i \bullet \frac{\partial \underline{r}_i}{\partial q_j}}$$

Aside: consider the position vector:

$$\underline{r} = \underline{r}(q_1, \ldots., q_n, t)$$

Differentiating, we get

$$\dot{\underline{r}} = \frac{\partial \underline{r}}{\partial q_1}\dot{q}_1 + \frac{\partial \underline{r}}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \underline{r}}{\partial t}$$

$$\dot{\underline{r}}dt = \sum_{i=1}^{n} \frac{\partial \underline{r}}{\partial q_i}\dot{q}_2 dt + \frac{\partial \underline{r}}{\partial t}dt$$

$$d\,\underline{r} = \sum_{i=1}^{n} \frac{\partial \underline{r}}{\partial q_i} dq_i + \frac{\partial \underline{r}}{\partial t}dt$$

$$\delta\,\underline{r} = \sum_{i=1}^{n} \frac{\partial \underline{r}}{\partial q_i}\delta q_i \quad \Leftarrow$$

Here Q_j – is generalized force corresponding to the jth generalized coordinate q_j.

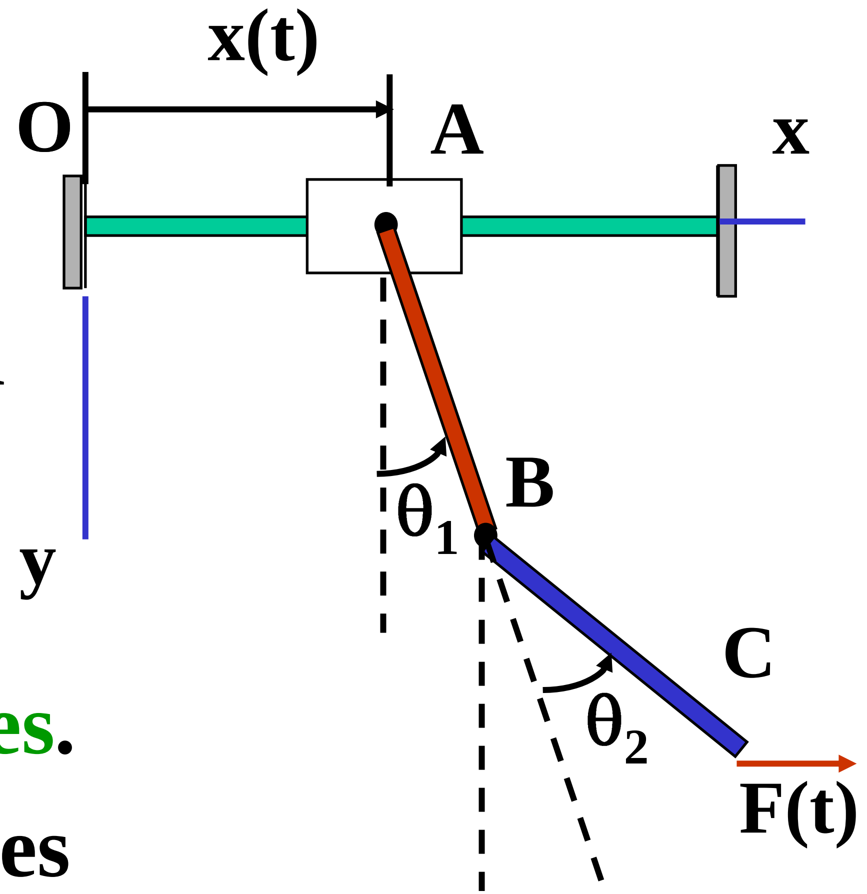
- **Thus,**

$$\delta W = \sum_{i=1}^{N} \underline{F}_i \bullet \delta \underline{r}_i = \sum_{j=1}^{n} Q_j \delta q_j$$

Ex 18: **Consider the compound pendulum.**

Let, x(t) – motion of slider. It is a specified function of time

Let θ_1, θ_2 – **be generalized coordinates.**



Find: generalized forces Q_1, Q_2 **(corresponding to the generalized coordinates** θ_1, θ_2).

Now, to find the generalized forces, we need to first define the position of C in terms of generalized coordinates:

$$\underline{r}_C = [x + L\sin\theta_1 + L\sin(\theta_1 + \theta_2)]\,\underline{i}$$
$$+L[\cos\theta_1 + \cos(\theta_1 + \theta_2)]\,\underline{j}$$

$$\delta\underline{r}_C = L[\cos\theta_1\delta\theta_1 + \cos(\theta_1 + \theta_2)(\delta\theta_1 + \delta\theta_2)]\,\underline{i}$$
$$+L[-\sin\theta_1\delta\theta_1 - \sin(\theta_1 + \theta_2)(\delta\theta_1 + \delta\theta_2)]\,\underline{j}$$

or

$$\delta\underline{r}_C = L[\{\cos\theta_1 + \cos(\theta_1 + \theta_2)\}\,\underline{i} - \{\sin\theta_1 +$$
$$\sin(\theta_1 + \theta_2)\}\,\underline{j}]\delta\theta_1 + L[\cos(\theta_1 + \theta_2)\,\underline{i}$$
$$-\sin(\theta_1 + \theta_2)\,\underline{j}]\delta\theta_2$$

Aside: **first find the velocity to find virtual dispalcement**:

$$\dot{\underline{r}}_c = [\dot{x} + L\dot{\theta}_1 \cos\theta_1 + L(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2)]\,\underline{i}$$
$$+L[-\dot{\theta}_1 \sin\theta_1 - (\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2)]\,\underline{j}$$

$$\delta\,\underline{r}_c = [\dot{x} + L\,\delta\theta_1 \cos\theta_1$$
$$+L(\delta\theta_1 + \delta\theta_2)\cos(\theta_1 + \theta_2)]\,\underline{i}$$
$$+L[-\delta\theta_1 \sin\theta - (\delta\theta_1 + \delta\theta_2)\sin(\theta_1 + \theta_2)]\,\underline{j}$$

Now, the force acting at C is: $\underline{F} = F\,\underline{i}$

Thus, the virtual work done is: $\delta W = \underline{F} \bullet \delta \underline{r}_C$

or $\delta W = FL[\{\cos\theta_1 + \cos(\theta_1+\theta_2)\}\delta\theta_1$

$$+\cos(\theta_1+\theta_2)\}\delta\theta_2]$$

$$= Q_1\delta\theta_1 + Q_2\delta\theta_2$$

$$\Rightarrow \left.\begin{aligned} Q_1 &= FL\{\cos\theta_1 + \cos(\theta_1+\theta_2)\} \\ Q_2 &= FL\cos(\theta_1+\theta_2) \end{aligned}\right\}$$

Note: if the forces are conservative, the generalized forces are: $Q_j^{con} = -\partial V / \partial q_j$

where the potential function is $V(q_1, \ldots, q_n)$

- **The potential function can be found as below:**

$$\frac{\partial V}{\partial \theta_1} = -FL\{\cos\theta_1 + \cos(\theta_1 + \theta_2)\} = -Q_1$$

$$\Rightarrow\ V = -FL\{\sin(\theta_1) + \sin(\theta_1 + \theta_2)\} + h_1(\theta_2)$$

- $$\frac{\partial V}{\partial \theta_2} = -FL\cos(\theta_1 + \theta_2) = -Q_2$$

$$\Rightarrow\ V = -FL\sin(\theta_1 + \theta_2) + h_2(\theta_1)$$

or

$$V(\theta_1, \theta_2) = -FL\{\sin\theta_1 + \sin(\theta_1 + \theta_2)\} + h_1(\theta_2) + h_2(\theta_1)$$

6.6 Lagrange's Equations

(Important: this derivation is different from the one in the text). The starting point is the

- **D'Alembert's principle:**

$$\sum_{i=1}^{N} (\underline{F}_i - m_i \ddot{\underline{r}}_i) \bullet \delta \underline{r}_i = 0$$

- **Recall that there are also *d* finite and *g* kinematic constraints to be satisfied by any virtual displacement of the system:**

$$\phi_i(\underline{r}_1, \underline{r}_2, ..., \underline{r}_N, t) = 0, \;\; i = 1, 2, ..., d$$

(**finite constraints**)

$$\sum_{j=1}^{N} \underline{l}_{ij}(\underline{r}_1, ..., \underline{r}_N, t) \bullet \dot{\underline{r}}_j + D_i = 0, \;\; i = 1, 2, ..., g$$

(**kinematic constraints**)

Let $q_1, ..., q_n -$ **generalized coordinates. (need not be all independent; i.e., need not satisfy finite constraints identically) or** $n \geq n_0 = 3N - d$.

Then $\underline{r}_i = \underline{r}_i(q_1, \ldots, q_n, t), \; i = 1, 2, \ldots, N$

Now, we calculate the different terms in

D'Alembert's principle:

- $\sum_{i=1}^{N} \underline{F}_i \bullet \delta \underline{r}_i = \sum_{i=1}^{N} \underline{F}_i \bullet \left(\sum_{j=1}^{n} \frac{\partial \underline{r}_i}{\partial q_j} \delta q_j\right)$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{N} \underline{F}_i \bullet \frac{\partial \underline{r}_i}{\partial q_j}\right) \delta q_j = \sum_{j=1}^{n} Q_j \delta q_j$$

effective forces

- $\sum_{i=1}^{N} m_i \bullet \delta \underline{r}_i = \sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \bullet \sum_{j=1}^{n} \frac{\partial \underline{r}_i}{\partial q_j} \delta q_j$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \bullet \frac{\partial \underline{r}_i}{\partial q_j}\right) \delta q_j$$

accleration

We now define the kinetic energy of the system of particles to be $T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\underline{r}}_i \bullet \dot{\underline{r}}_i$

but $\dot{\underline{r}}_i = \sum_{j=1}^{n} \frac{\partial \underline{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \underline{r}_i}{\partial t} \Rightarrow$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{j=1}^{n} \frac{\partial \underline{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \underline{r}_i}{\partial t}\right) \bullet \left(\sum_{k=1}^{n} \frac{\partial \underline{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \underline{r}_i}{\partial t}\right)$$

or

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \{\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial \underline{r}_i}{\partial q_j} \bullet \frac{\partial \underline{r}_i}{\partial q_k}\right)\dot{q}_j \dot{q}_k +$$

$$\frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{j=1}^{n} \frac{\partial \underline{r}_i}{\partial q_j} \bullet \frac{\partial \underline{r}_i}{\partial t}\dot{q}_j + \sum_{k=1}^{n} \frac{\partial \underline{r}_i}{\partial q_k} \bullet \frac{\partial \underline{r}_i}{\partial t}\dot{q}_k\right)$$

$$+\frac{1}{2}\sum_{i=1}^{N} m_i \frac{\partial \underline{r}_i}{\partial t} \bullet \frac{\partial \underline{r}_i}{\partial t}$$

or

$$T = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{N} m_i \frac{\partial \underline{r}_i}{\partial q_j} \bullet \frac{\partial \underline{r}_i}{\partial q_k}\right)\dot{q}_j \dot{q}_k$$

$$+\sum_{j=1}^{n}\left(\sum_{i=1}^{N} m_i \frac{\partial \underline{r}_i}{\partial q_j} \bullet \frac{\partial \underline{r}_i}{\partial t}\right)\dot{q}_j + \frac{1}{2}\sum_{i=1}^{N} m_i \frac{\partial \underline{r}_i}{\partial t} \bullet \frac{\partial \underline{r}_i}{\partial t}$$

$$T \equiv T_2(\dot{q}_j, \dot{q}_k) + T_1(\dot{q}_j) + T_0$$

Review: Derivation of Lagrange's Equations

$$\sum_{i=1}^{N} (\underline{F}_i - m_i \ddot{\underline{r}}_i) \bullet \delta \underline{r}_i = 0$$

$$\phi_i(\underline{r}_1, \ldots, \underline{r}_N, t) = 0, \;\; i = 1, 2, \ldots d,$$

$$\sum_{j=1}^{N} \underline{l}_{ij} \bullet \dot{\underline{r}}_j + D_i = 0, \;\; i = 1, 2, \ldots, g$$

- **Now express** $\underline{r}_i(q_1, q_2, \ldots, q_n, t)$

- $$\sum_{i=1}^{N} \underline{F}_i \bullet \delta \underline{r}_i = \sum_{i=1}^{n} \underbrace{\left(\sum_{j=1}^{N} \underline{F}_j \bullet \frac{\partial \underline{r}_j}{\partial q_i} \right)}_{Q_i} \delta q_i$$

- $$\sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \bullet \delta \underline{r}_i = \sum_{i=1}^{n} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} \right\} \delta q_i$$

where $$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\underline{r}}_i \bullet \dot{\underline{r}}_i$$

Algebraic manipulations

$$\delta \underline{r}_j = \sum_{k=1}^{n} \frac{\partial \underline{r}_j}{\partial q_k} \delta q_k , \quad j = 1, 2, \ldots, N$$

$$\sum_{j=1}^{N} \frac{\partial \phi_i}{\partial \underline{r}_j} \bullet \sum_{k=1}^{n} \frac{\partial \underline{r}_j}{\partial q_k} \delta q_k = 0, \; i = 1, 2, \ldots, d$$

$$\sum_{k=1}^{n} \left(\sum_{j=1}^{N} \frac{\partial \phi_i}{\partial \underline{r}_j} \bullet \frac{\partial \underline{r}_j}{\partial q_k} \right) \delta q_k = 0$$

$$\sum_{k=1}^{n} a_{ik}(q,t) \delta q_k = 0, \; i = 1, 2, \ldots, d$$

constraints in differential form

$$\sum_{j=1}^{N} \frac{\partial \phi_i}{\partial \underline{r}_j} \bullet \delta \underline{r}_j = 0, \;\; i = 1, 2, \ldots, d$$

$$\sum_{j=1}^{N} \underline{l}_{ij} \bullet \delta \underline{r}_j = 0, \;\; i = 1, 2, \ldots, g$$

These are *d* + *g* relations in 3N virtual displ.

$\Rightarrow$ **D. O. F. = 3N – (*d* + *g*) = (**$\mathrm{n}_0 - \mathrm{g}$**)**

n = number of generalized coordinates

$\geq$ D. O. F.

One can then show (with some manipulation) that

$$\sum_{i=1}^{N} m_i \underline{\ddot{r}}_i \bullet \frac{\partial \underline{r}_i}{\partial q_j} \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}, \quad j = 1, 2, \ldots, n$$

$$\Rightarrow \sum_{i=1}^{N} (\underline{F}_i - m_i \underline{\ddot{r}}_i) \bullet \delta \underline{r}_i \equiv \sum_{j=1}^{n} [Q_j - \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right\}] \delta q_j = 0$$

Case 1: Number of generalized coordinates – $\mathrm{n = n_0 = 3N - d =}$ **degrees of freedom of the system, i.e., all holonomic (geometric) constraints are automatically satisfied by the choice of generalized coordinates; <u>and</u> there are no kinematic or velocity constraints.**

$\Rightarrow$**The generalized coordinates (hence δq_j) are independent. Then**

$$\sum_{j=1}^{n}\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}-Q_j\right\}\delta q_j=0 \quad \Rightarrow$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j}, \quad j=1,2,\ldots,n$$

This is one form of Lagrange's equations for a holonomic <u>n</u> degrees of freedom mechanical system. These are a system of n equations. (2nd order differential equations)

- **Lagrange's equations are 2nd-order nonlinear ordinary differential equations for n generalized coordinates** q_i
- **We need to specify** $q_i(0), \dot{q}_i(0), \ i = 1, 2, ..., n$

  (2n initial conditions)
- **Their solution gives** $q_i(t), \ i = 1, 2, ..., n.$
- **One can then find the positions**

  $$\underline{r}_i(t), \ i = 1, 2, ..., N$$

  and the constraint forces

  $$\underline{R}_i = -\underline{F}_i + m_i \ddot{\underline{r}}_i, \ i = 1, 2, ..., N$$

Ex. 19: Consider a plane pendulum with oscillating support:

With the coordinate system shown, the position:

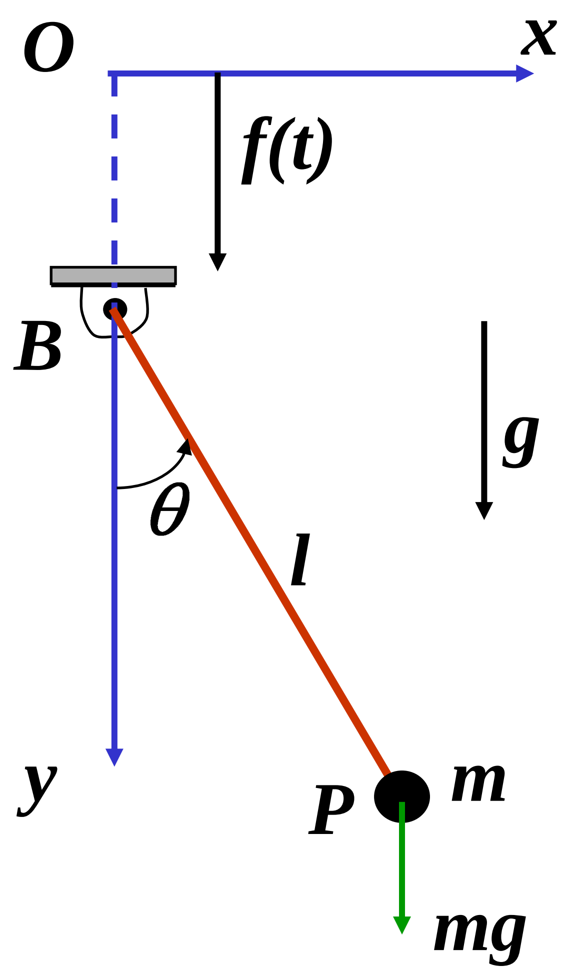
$$\underline{r}_P = f(t)\,\underline{j} + l(\cos\theta\,\underline{j} + \sin\theta\,\underline{i})$$

$$= l\sin\theta\,\underline{i} + (l\cos\theta + f(t))\,\underline{j}$$

The velocity is:

$$\underline{\dot{r}}_P = l\dot{\theta}\cos\theta\,\underline{i} + (-l\dot{\theta}\sin\theta + \dot{f}(t))\,\underline{j}$$

θ **- generalized coordinate – no constraint.**



The kinetic energy is: $T = \frac{1}{2} m \dot{\underline{r}}_P \bullet \dot{\underline{r}}_P$

$$= \frac{1}{2} m [l^2 \dot{\theta}^2 + \dot{f}^2(t) - 2l\dot{\theta}\ \dot{f}(t) \sin\theta]$$

- **Need to find generalized force** Q_θ.

$$\delta W = \underline{F} \bullet \delta \underline{r}_P = Q_\theta \delta\theta; \quad \underline{F} = mg\,\underline{j}$$

$\delta \underline{r}_P = d\underline{r}_P$ ***with time frozen*** $(set\ \delta t = 0)$

$$d\underline{r}_P = l\cos\theta\,\underline{i}\,d\theta + (-l\sin\theta d\theta + \frac{df}{dt} dt)\,\underline{j}$$

$$\Rightarrow \delta \underline{r}_P = (l\cos\theta \underline{i} - l\sin\theta\,\underline{j})\delta\theta$$

$$\delta W = -mgl\sin\theta\,\delta\theta \;\Rightarrow\; Q_\theta = -mgl\sin\theta$$

Thus, the Lagrange's equation is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

Computing the various terms:

$$\frac{\partial T}{\partial \dot{\theta}} = ml^2\dot{\theta} - ml\,\dot{f}(t)\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} - ml\,\ddot{f}\sin\theta - ml\,\dot{f}\,\dot{\theta}\cos\theta$$

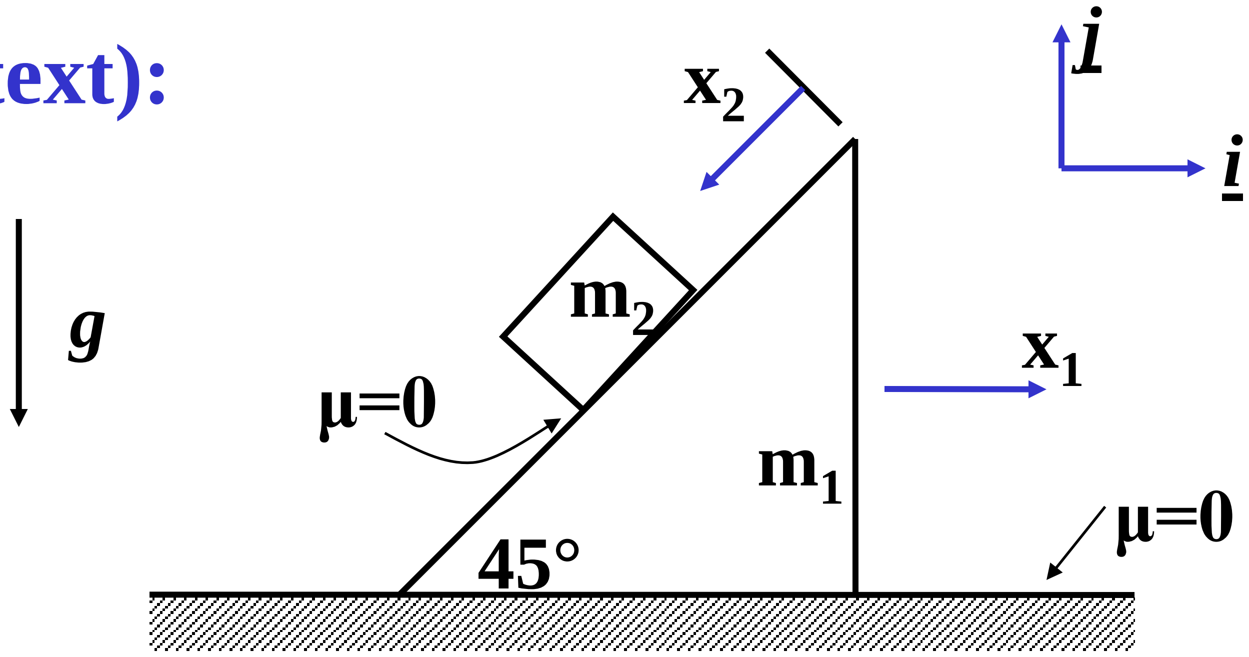
$$\frac{\partial T}{\partial \theta} = -ml\dot{\theta}\,\dot{f}\cos\theta$$

$$\Rightarrow\; ml^2\ddot{\theta} - ml\,\ddot{f}\sin\theta - ml\,\dot{f}\,\dot{\theta}\cos\theta$$

$$+ml\,\dot{f}\,\dot{\theta}\cos\theta = -mgl\sin\theta$$

or $$\ddot{\theta} + [\{g - \ddot{f}(t)\}\sin\theta]/l = 0$$

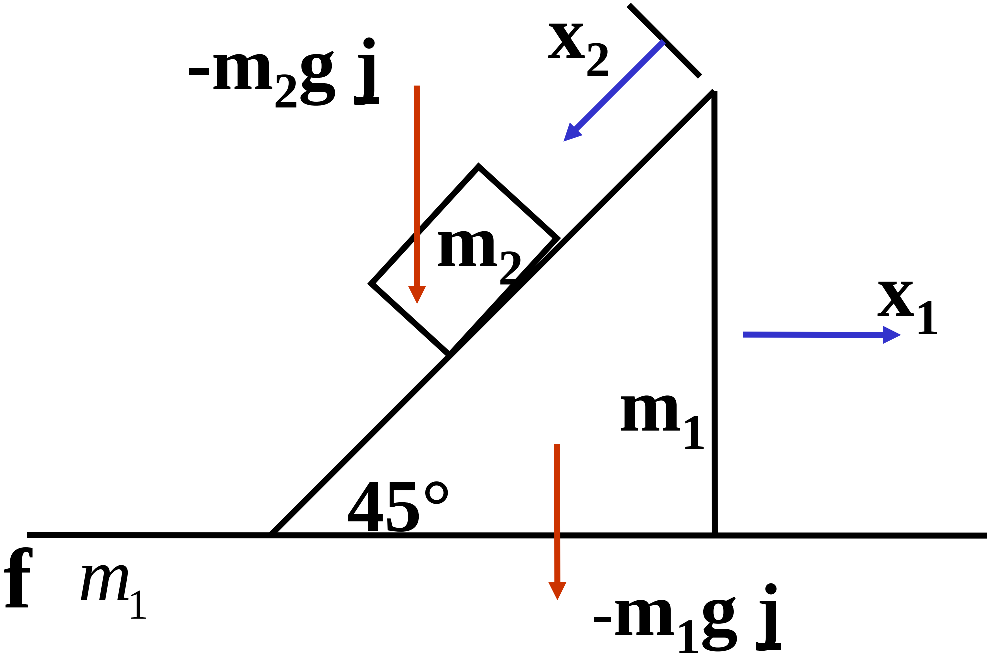
Ex 19 (text):



m_2 **slides on** m_1;

m_1 **slides on the horizontal surface:** $\mu = 0$.

All surfaces in contact are smooth.



$x_1 -$ **absolute position of** m_1

$x_2 -$ **position of** m_2 **relative to** m_1

Find: acceleration of m_1 **using Lagrange's equations.**

- **There are two generalized coordinates** x_1, x_2

n = degrees of freedom = 2. (no constraints on x_1, x_2**)**

We proceed step by step and develop the various quantities, starting with position vectors:

$$\underline{r}_1 = x_1 \underline{i} \;\; ; \;\; \underline{r}_2 = x_1 \underline{i} - (x_2 \underline{i} + x_2 \underline{j}) / \sqrt{2}$$

$$\underline{v}_1 = \dot{x}_1 \underline{i} \;\; ; \;\; \underline{v}_2 = (\dot{x}_1 - \dot{x}_2 / \sqrt{2}) \, \underline{i} - \dot{x}_2 \, \underline{j} / \sqrt{2}$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \{ (\dot{x}_1 - \dot{x}_2 / \sqrt{2})^2 + (\dot{x}_2 / \sqrt{2})^2 \}$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}_1^2 + \frac{1}{2} m_2 \{ \dot{x}_2^2 - \sqrt{2} \dot{x}_1 \dot{x}_2 \}$$

- **We need to find generalized forces** Q_1, Q_2 ?

$$\underline{F}_1 = -m_1 g\, \underline{j} \;\; ; \;\; \delta \underline{r}_1 = \delta x_1 \underline{i} \, ;$$

$$\underline{F}_2 = -m_2 g\, \underline{j} \;\; ; \;\; \delta \underline{r}_2 = (\delta x_1 - \frac{\delta x_1}{\sqrt{2}})\underline{i} - \frac{\delta x_2}{\sqrt{2}} \underline{j}$$

$$\delta W = \sum_{i=1}^{2} \underline{F}_i \bullet \delta \underline{r}_i = m_2 g \frac{\delta x_2}{\sqrt{2}} = Q_1 \delta x_1 + Q_2 \delta x_2$$

$$\Rightarrow \; Q_1 = 0 \;\; ; \;\; Q_2 = m_2 g / \sqrt{2}$$

- **Then, the equations of motion are:**

$$\underline{\underline{x}}_1 : \frac{\partial T}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1 - \frac{m_2}{\sqrt{2}} \dot{x}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = (m_1 + m_2)\ddot{x}_1 - \frac{m_2}{\sqrt{2}}\ddot{x}_2$$

$$\frac{\partial T}{\partial \dot{x}_1} = 0 \;\; ; \;\; Q_1 = 0$$

$$\Rightarrow \quad \boxed{(m_1 + m_2)\ddot{x}_1 - \frac{m_2}{\sqrt{2}}\ddot{x}_2 = 0}$$

$$\underline{\underline{x}}_2 : \quad \frac{\partial T}{\partial \dot{x}_2} = m_2\dot{x}_2 - \frac{m_2}{\sqrt{2}}\dot{x}_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2 - \frac{m_2}{\sqrt{2}}\ddot{x}_1 \;\; ; \quad \frac{\partial T}{\partial x_2} = 0$$

$$Q_2 = \frac{m_2 g}{\sqrt{2}}$$

$$\Rightarrow \boxed{m_2\ddot{x}_2 - \frac{m_2}{\sqrt{2}}\ddot{x}_1 = \frac{m_2 g}{\sqrt{2}}}$$

Solving the two equations for $\ddot{x}_1 \Rightarrow$

$$\boxed{\ddot{x}_1 = m_2 g/(2m_1 + m_2)}$$

Another form of Lagrange's equations:

Suppose that some forces acting on the system are conservative, i.e., the corresponding forces $\underline{F}_i$ (as well as the generalized forces) are derivable from a potential function $V(q_1,\ldots,q_n)$, then

$$Q_i(t,q,\dot{q}) \equiv \underbrace{-\frac{\partial V}{\partial q_i}}_{\text{potential part}} + \underbrace{Q_i'(t,q,\dot{q})}_{\text{nonconservative part}}$$

$\Rightarrow$**The equations of motion for an n degree of freedom holonomic system take the form:**

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} + Q_i' \,,$$

$$\boldsymbol{i = 1, 2, \ldots, n}$$

Let $L \equiv T - V$, **the Lagrangian function.**

Then

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i'(t, q, \dot{q})} \,, \quad i = 1, 2, \ldots, n$$

This is the standard form of Lagrange's equations.

- **Explicit form of the equations of motion**

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=Q_i' \ , \ i=1,2,\ldots,n$$

The idea here is to express all the quantities in terms of the generalized coordinates and their time-derivatives

Now:

$$T=\frac{1}{2}\sum_{i=1}^{N} m_i \,\dot{\underline{r}}_i \bullet \dot{\underline{r}}_i$$

where $\underline{r}_i \equiv \underline{r}_i(t,q_1,q_2,\ldots,q_n) \ , \ i=1,2,\ldots,n$

$$\dot{\underline{r}}_i=\sum_{j=1}^{n}\frac{\partial \underline{r}_i}{\partial q_j}\dot{q}_j+\frac{\partial \underline{r}_i}{\partial t}$$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{j=1}^{n} \frac{\partial \underline{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \underline{r}_i}{\partial t}\right) \bullet \left(\sum_{k=1}^{n} \frac{\partial \underline{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \underline{r}_i}{\partial t}\right)$$

$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{N} m_i \frac{\partial \underline{r}_i}{\partial q_j} \bullet \frac{\partial \underline{r}_i}{\partial q_k}\right)\dot{q}_j \dot{q}_k +$$

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{N} m_i \frac{\partial \underline{r}_i}{\partial q_j} \bullet \frac{\partial \underline{r}_i}{\partial t}\right)\dot{q}_j + \frac{1}{2}\sum_{i=1}^{N} m_i \frac{\partial \underline{r}_i}{\partial t} \bullet \frac{\partial \underline{r}_i}{\partial t}$$

or

$$T = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} m_{jk}(q,t)\dot{q}_j \dot{q}_k + \sum_{j=1}^{n} a_j(q,t)\dot{q}_j + T_0(q,t)$$

$$\Rightarrow T = \underbrace{T_2}_{\substack{\text{quadratic} \\ \text{in } \dot{q}}} + \underbrace{T_1}_{\substack{\text{linear in} \\ \dot{q}}} + \underbrace{T_0}_{\substack{\text{independent of} \\ \text{generalized velocity } \dot{q}}}$$

- **If the constraints are independent of time, (that is, the system is schleronomic),**

$$\underline{r}_i \equiv \underline{r}_i(q) \quad \Rightarrow \quad \frac{\partial \underline{r}_i}{\partial t} = 0 \quad \text{and}$$

$$T = T_2 = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} m_{jk}(q,t) \dot{q}_j \dot{q}_k$$

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(q,t)\,\dot{q}_i\dot{q}_j + \sum_{i=1}^{n} a_i(q,t)\,\dot{q}_i + T_0(q,t)$$

Then $$\frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} m_{ij}(q,t)\,\dot{q}_j + a_i(q,t)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_{j=1}^{n} \dot{m}_{ij}(q,t)\,\dot{q}_j + \sum_{j=1}^{n} m_{ij}(q,t)\,\ddot{q}_j + \dot{a}_i(q,t)$$

Now $$\dot{m}_{ij}(q,t) = \sum_{l=1}^{n}\frac{\partial m_{ij}}{\partial q_l}\,\dot{q}_l + \frac{\partial m_{ij}}{\partial t}$$

$$\dot{a}_i(q,t) = \sum_{j=1}^{n}\frac{\partial a_i}{\partial q_j}\,\dot{q}_j + \frac{\partial a_i}{\partial t}$$

$$\frac{\partial T}{\partial q_i} = \frac{\partial T_2}{\partial q_i} + \frac{\partial T_1}{\partial q_i} + \frac{\partial T_0}{\partial q_i}$$

$$\frac{\partial T_2}{\partial q_i} = \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{\partial m_{jl}}{\partial q_i} \dot{q}_j \dot{q}_l$$

$$\frac{\partial T_1}{\partial q_i} = \sum_{j=1}^{n} \frac{\partial a_j}{\partial q_i} \dot{q}_j$$

$$\frac{\partial T_0}{\partial q_i} = \frac{\partial T_0}{\partial q_i}$$

combining all these expressions $\Rightarrow$

$$\sum_{j=1}^{n} m_{ij}(q,t)\ddot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial m_{ij}}{\partial q_l} + \frac{\partial m_{il}}{\partial q_j} - \frac{\partial m_{jl}}{\partial q_i}\right)\dot{q}_j\dot{q}_l$$

$$+\sum_{j=1}^{n}\frac{\partial m_{ij}}{\partial t}\dot{q}_j + \sum_{j=1}^{n}\left(\frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i}\right)\dot{q}_j + \frac{\partial a_i}{\partial t}$$

$$-\frac{\partial T_0}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i'(t,q,\dot{q}), \quad i = 1,2,\ldots,n$$

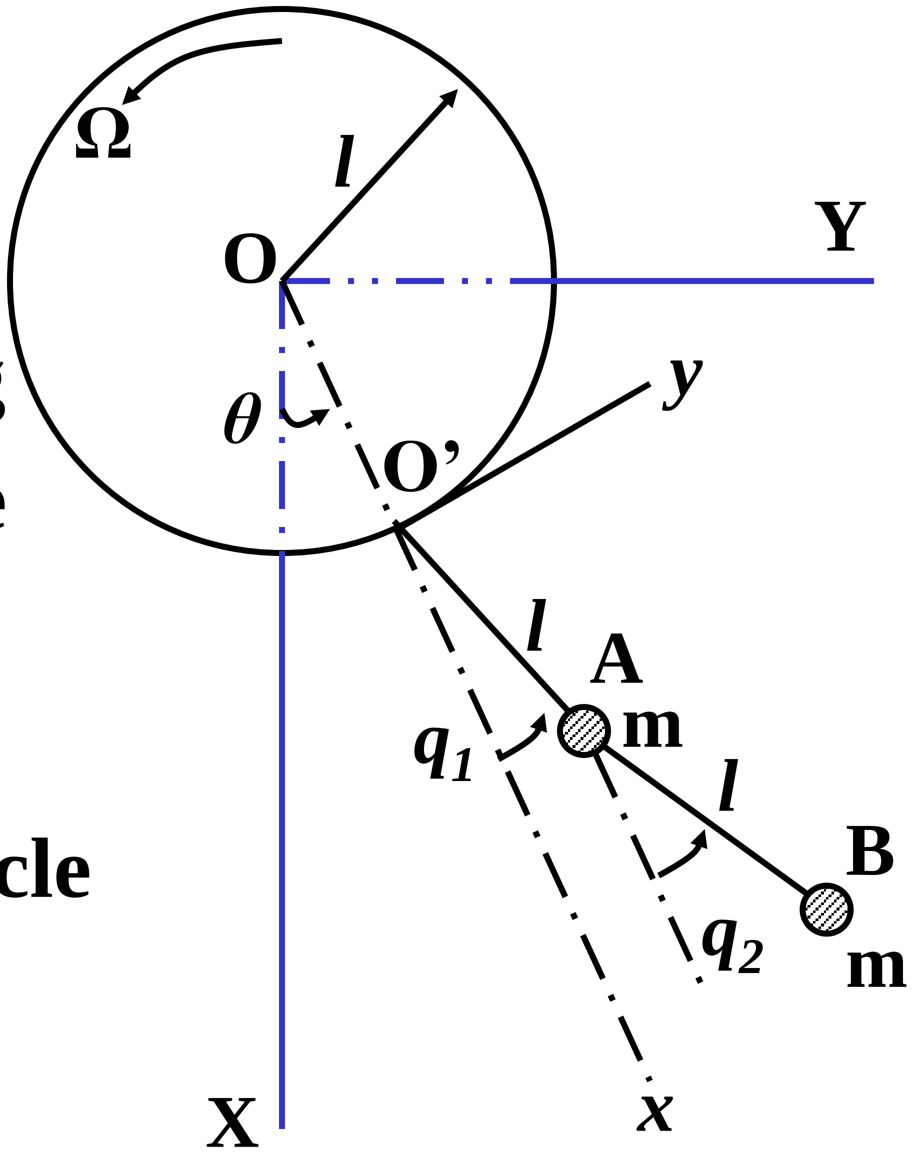
Note that

$$\gamma_{ij} \equiv \frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i} = -\gamma_{ji}$$

- **linear gyroscopic coefficients**

Ex 20:

Disk of radius *l* rotating with const. Ω. A double pendulum attached at P (O') on the disk. Rods are massless, each particle of mass m. Motion in horizontal plane.



Find: T – kinetic energy; identify terms of different type. $q_1, q_2; T_2, T_1, T_0$

XYZ – fixed frame

xyz – moving frame – attached to the disk at O' = P

$$\underline{\omega} = \Omega \underline{K} = \Omega \underline{k} \;\; ; \quad \dot{\underline{\omega}} = \underline{\alpha} = 0$$

$$\underline{v}_A = \underline{v}_{O'} + \underline{\omega} \times \underline{\rho} + (\dot{\underline{\rho}})_r$$

$$\underline{v}_{O'} = \underline{\omega} \times \underline{r}_{OO'} = \Omega \underline{k} \times l \underbrace{(\cos\theta \underline{I} + \sin\theta \underline{J}}_{\underline{i}}$$

$$= \Omega \underline{k} \times l \underline{i} = \Omega l \, \underline{j}$$

$$\underline{\rho}_A = l(\cos q_1 \underline{i} + \sin q_1 \underline{j})$$

$$(\dot{\underline{\rho}})_r = l\dot{q}_1(-\sin q_1 \underline{i} + \cos q_1 \underline{j})$$

$$\underline{\omega} \times \underline{\rho}_A = \Omega \underline{k} \times l(\cos q_1 \underline{i} + \sin q_1 \underline{j})$$

$$= \Omega l(\cos q_1 \underline{j} - \sin q_1 \underline{i})$$

$$\Rightarrow \underline{v}_A = -(\Omega l + \dot{q}_1 l)\sin q_1 \underline{i} + [(\Omega l + \dot{q}_1 l)\cos q_1 + \Omega l]\underline{j}$$

$$\underline{v}_B = \underline{v}_{O'} + \underline{\omega} \times \underline{\rho}_B + (\dot{\underline{\rho}}_B)_r$$

$$\underline{\rho}_B = l(\cos q_1 + \cos q_2)\underline{i} + l(\sin q_1 + \sin q_2)\underline{j}$$

$$(\dot{\underline{\rho}}_B)_r = l(-\sin q_1 \dot{q}_1 - \sin q_2 \dot{q}_2)\underline{i}$$

$$+ l(\cos q_1 \dot{q}_1 + \cos q_2 \dot{q}_2)\underline{j}$$

$$\Rightarrow \underline{v}_B = -[\Omega l(\sin q_1 + \sin q_2) + l(\sin q_1 \dot{q}_1 + \sin q_2 \dot{q}_2)]\underline{i} + [\Omega l(1 + \cos q_1 + \cos q_2) + l(\cos q_1 \dot{q}_1 + \cos q_2 \dot{q}_2)]\underline{j}$$

Now $T = \frac{1}{2} m(\underline{v}_A \bullet \underline{v}_A + \underline{v}_B \bullet \underline{v}_B)$

$$= \frac{1}{2} ml^2 [2\dot{q}_1^2 + 2\dot{q}_1 \dot{q}_2 \cos(q_1 - q_2) + \dot{q}_2^2]$$

$$+ml^2 \Omega [2\dot{q}_1 (1 + \cos q_1) + \dot{q}_2 (1 + \cos q_2) + (\dot{q}_1 + \dot{q}_2)\cos(q_1 - q_2)]$$

$$+ \frac{1}{2} ml^2 \Omega^2 [5 + 4\cos q_1 + 2\cos q_2 + 2\cos(q_1 - q_2)]$$

Lagrange's equations:

$$q_1: \quad \frac{\partial T}{\partial \dot{q}_1} = ml^2[2\dot{q}_1 + \dot{q}_2\cos(q_1 - q_2)]$$

$$+ml^2\Omega[2(1+\cos q_1) + \cos(q_1 - q_2)]$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_1}) = ml^2[2\ddot{q}_1 + \ddot{q}_2\cos(q_1 - q_2)$$

$$+\dot{q}_2(-\dot{q}_1 + \dot{q}_2)\sin(q_1 - q_2)]$$

$$+ml^2\Omega[-2\dot{q}_1\sin q_1 - (\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2)]$$

$$\frac{\partial T}{\partial q_1} = ml^2 \dot{q}_1 \dot{q}_2 (-1) \sin(q_1 - q_2)$$

$$+ml^2\Omega[-2\,\dot{q}_1 \sin q_1 - (\dot{q}_1 + \dot{q}_2)\sin(q_1 - q_2)]$$

$$+\frac{1}{2} ml^2[-2\Omega^2 \sin q_1 - 2\Omega^2 \sin q_1 - 2\Omega^2 \sin(q_1 - q_2)]$$

The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1(q, \dot{q}, t)$$

$$
\begin{aligned}
\Rightarrow \quad & ml^2[2\ddot{q}_1 + \ddot{q}_2\cos(q_1 - q_2) \\
& +\dot{q}_2(-\dot{q}_1 + \dot{q}_2)\sin(q_1 - q_2)] \\
& +ml^2\Omega[-2\dot{q}_1\sin q_1 - (\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2)] \\
& +ml^2\dot{q}_1\dot{q}_2\sin(q_1 - q_2) + ml^2\Omega[2\dot{q}_1\sin q_1 \\
& +(\dot{q}_1 + \dot{q}_2)\sin(q_1 - q_2)] \\
& +ml^2[\Omega^2\sin q_1 + \Omega^2\sin q_1 + \Omega^2\sin(q_1 - q_2)] \\
& \qquad = Q_1(q, \dot{q}, t)
\end{aligned}
$$

Simplifying $\Rightarrow$

$$ml^2[2\ddot{q}_1 + \ddot{q}_2\cos(q_1 - q_2) + \dot{q}_2^2\sin(q_1 - q_2)]$$
$$+2ml^2\Omega\dot{q}_2\sin(q_1 - q_2)$$
$$+ml^2[2\Omega^2\sin q_1 + \Omega^2\sin(q_1 - q_2)] = Q_1$$

There is a similar equation for q_2 **:**

$$\underline{\underline{q_2}}: \quad \frac{\partial T}{\partial \dot{q}_2} = ml^2[\dot{q}_1 \cos(q_1 - q_2) + \dot{q}_2]$$

$$+ml^2\Omega[(1+\cos q_2) + \cos(q_1 - q_2)]$$

$$\frac{d[}{dt}(\frac{\partial T}{\partial \dot{q}_2}) = ml^2[\ddot{q}_1 \cos(q_1 - q_2) - \dot{q}_1(\dot{q}_1 - \dot{q}_2) +$$

$$\sin(q_1 - q_2) + \ddot{q}_2] + ml^2\Omega[-\dot{q}_2 \sin q_2$$

$$-(\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2)]$$

$$\frac{\partial T}{\partial q_2} = ml^2 \dot{q}_1 \dot{q}_2 \sin(q_1 - q_2)$$

$$+ml^2 \Omega[-\dot{q}_2 \sin q_2 + (\dot{q}_1 + \dot{q}_2) \sin(q_1 - q_2)]$$

$$+\frac{1}{2} ml^2 [-2\Omega^2 \sin q_2 + 2\Omega^2 \sin(q_1 - q_2)]$$

The final equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2(q, \dot{q}, t)$$

$$\Rightarrow \quad ml^2[\ddot{q}_1 \cos(q_1 - q_2) - \dot{q}_1(\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2) + \ddot{q}_2]$$

$$+ml^2\Omega[-\dot{q}_2 \sin q_2 - (\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2)]$$

$$-ml^2\dot{q}_1\dot{q}_2 \sin(q_1 - q_2) + ml^2\Omega[\dot{q}_2 \sin q_2$$

$$-(\dot{q}_1 + \dot{q}_2)\sin(q_1 - q_2)]$$

$$+ml^2\Omega^2[\sin q_2 - \sin(q_1 - q_2)] = Q_2$$

Simplifying $\Rightarrow$

$$\boxed{\begin{aligned}&ml^2[\ddot{q}_1\cos(q_1-q_2)-\dot{q}_1^2\sin(q_1-q_2)+\ddot{q}_2]\\&-2ml^2\Omega\dot{q}_1\sin(q_1-q_2)+ml^2\Omega^2[\sin q_2\\&-\sin(q_1-q_2)]=Q_2(q,\dot{q},t)\end{aligned}}$$

In vector form

$$ml^2\begin{bmatrix}2 & \cos(q_1-q_2)\\ \cos(q_1-q_2) & 1\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix}+$$

$$ml^2\begin{bmatrix} 0 & \sin(q_1-q_2) \\ -\sin(q_1-q_2) & 0 \end{bmatrix}\begin{Bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{Bmatrix}$$

$$+2m\Omega l^2\begin{bmatrix} 0 & \sin(q_1-q_2) \\ -\sin(q_1-q_2) & 0 \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

$$+ml^2\Omega^2\begin{Bmatrix} 2\sin q_1+\sin(q_1-q_2) \\ \sin q_1-\sin(q_1-q_2) \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

or, in compact notation

$$M \ddot{q} + G_1 \dot{q}^2 + G_2 \dot{q}$$

$$+ F(q) = Q(q)$$

Note: $$\left.\begin{array}{l} G_1 = -G_1^T \\ G_2 = -G_2^T \end{array}\right\} \text{skew-symmetric matrices}$$

Reading Assignment: Examples 6.3 – 6.7

Imp: Note the discussion in Ex. 6.6 on page 277 (begins at the bottom of p. 276) and page 278. Specially, note the redefined T' and V' **(the fictitious kinetic and potential energies). We will see this when we study linearization and stability in the last week of the course.**

6.7 Lagrange Multipliers:

(nonholonomic systems or systems with $n > n_o$

Recall the equations for dynamics of an N particle system

$$\sum_{i=1}^{N}(\underline{F}_i - m_i \ddot{\underline{r}}_i) \bullet \delta\underline{r}_i = 0$$

Subject to finite constraints:

$$f_i(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N, t) = 0, \; i = 1, 2, \ldots, d$$

kinematic constraints:

$$\sum_{j=1}^{N} \underline{l}_{ij}(\underline{r}_1, \ldots, \underline{r}_N, t) \bullet \dot{\underline{r}}_j + D_i = 0, \; i = 1, 2, \ldots, g$$

Note that there are only d geometric constraints

$\Rightarrow$ **let** $\bar{n} > n_0 = (3N - d)$ **be the number of generalized coordinates.**

Note: degrees of freedom of the system are

$$n = 3N - (d + g).$$

In terms of generalized coordinates and virtual displacements in the generalized coordinates, the constraints can be written as $\sum_{j=1}^{\bar{n}} a_{ij}(t,q)\,\delta q_j = 0,\ i = 1,2,\ldots,(d+g)$

Note that if $\bar{n} = n_0$, then the d geometric constraints are automatically satisfied and only g constraint expressions remain.

- **We now assume that the constraints are workless in any virtual displacements permitted by the constraints.**

  Let $C_j, \;\; j = 1, 2, ..., \bar{n}$ **be the constraint forces corresponding to the generalized coordinates** $q_j, \;\; j = 1, 2, ..., \bar{n}$

- **Then, the virtual work done by constraint forces in any virtual displacement is**

$$\delta W = \sum_{i=1}^{N} \underline{R}_i \bullet \delta \underline{r}_i = \sum_{j=1}^{\bar{n}} C_j \delta q_j = 0$$

(*this implies that* $C_j \equiv \sum_{i=1}^{N} \underline{R}_i \bullet \dfrac{\partial \underline{r}_i}{\partial q_j}$**)**

- **Now, *if the virtual displacements*** $\delta \underline{q}_j$ ***are all independent*, ** $C_j = 0, \;\; j = 1, 2, \ldots, n.$

(Note: *this does not mean that* $\underline{R}_i$ ***are zero*).**

In the present case, we have constraint relations involving δq_i's and, hence,

$$C_j \neq 0, \quad j = 1, 2, \ldots, \bar{n}$$

even though

$$\delta W = \sum_{j=1}^{\bar{n}} C_j \, \delta q_j = 0.$$

Summarizing: the D'Alembert's principle written in terms of the generalized coordinates is

$$\sum_{j=1}^{\bar{n}} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right\} \delta q_j = 0 \qquad \textbf{(1)}$$

subjected to the associated constraint relations:

$$\sum_{j=1}^{\bar{n}} a_{ij}(t,q)\delta q_j = 0,\ i = 1,2,...,(d+g) \qquad (2)$$

We need some way of making $\delta q's$ **independent.**

Let $\lambda_i,\ i = 1,2,...,(d+g)$ **be new parameters (equal in number to the constraint relations) that will be utilized to accomplish this task (making** $\delta q's$ **independent)**

λ_i – **Lagrange multipliers**

The idea is to manipulate the constraints and the workless constraint forces in some way.

The constraint equations are

$$\lambda_i \sum_{j=1}^{\bar{n}} a_{ij}(t,q)\delta q_j = 0, \;\; i = 1,2,\ldots,(d+g)$$

Adding these $\Rightarrow$

$$\sum_{i=1}^{d+g} \lambda_i \sum_{j=1}^{\bar{n}} a_{ij}(t,q)\,\delta q_j = 0 \qquad (3)$$

Additionally, the virtual work relation as

$$\delta W = \sum_{j=1}^{\bar{n}} C_j \,\delta q_j = 0$$

Adding these two $\Rightarrow$

$$\sum_{j=1}^{\bar{n}} C_j - \sum_{i=1}^{d+g} \lambda_i a_{ij} \quad \delta q_j = 0 \qquad (4)$$

Note: at this point we have (d + g) unknown multipliers λ_i; $\bar{n}$ unknowns C_j, the generalized constraint forces; and the virtual displacements $\delta q's$ are not independent – they satisfy (d + g) constraints.

- **The trick now is to choose $\lambda's$ such that $\delta q's$ become independent $\Rightarrow$ assume that $\lambda's$ have values such that**

$$C_j = \sum_{i=1}^{d+g} \lambda_i a_{ij}, \quad j = 1, 2, \ldots, \bar{n} \qquad (5)$$

- **Add the constraint relations (3) to (1)**

$$\sum_{i=1}^{\bar{n}} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} - Q_i - \sum_{j=1}^{d+g} \lambda_j a_{ji} \right\} \delta q_i = 0 \qquad (6)$$

- **Since $\delta q's$ are independent with the $\lambda_j 's$ related to $C_j 's$ by (5),**

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}-Q_i=\sum_{j=1}^{d+g}\lambda_j a_{ji}},\quad i=1,2,\ldots,\bar{n} \quad \Rightarrow \quad (6)$$

We thus have $\bar{n}$ equations in $\bar{n}+(d+g)$ variables q_i, λ_i.

The additional (d + g) equations are the constraint relations

$$\sum_{j=1}^{\bar{n}} a_{ij}(t,q)\,\dot{q}_j + d_i = 0,\quad i=1,2,\ldots,(d+g)$$

Reading Assignment: Ex: 6-8, 6-9

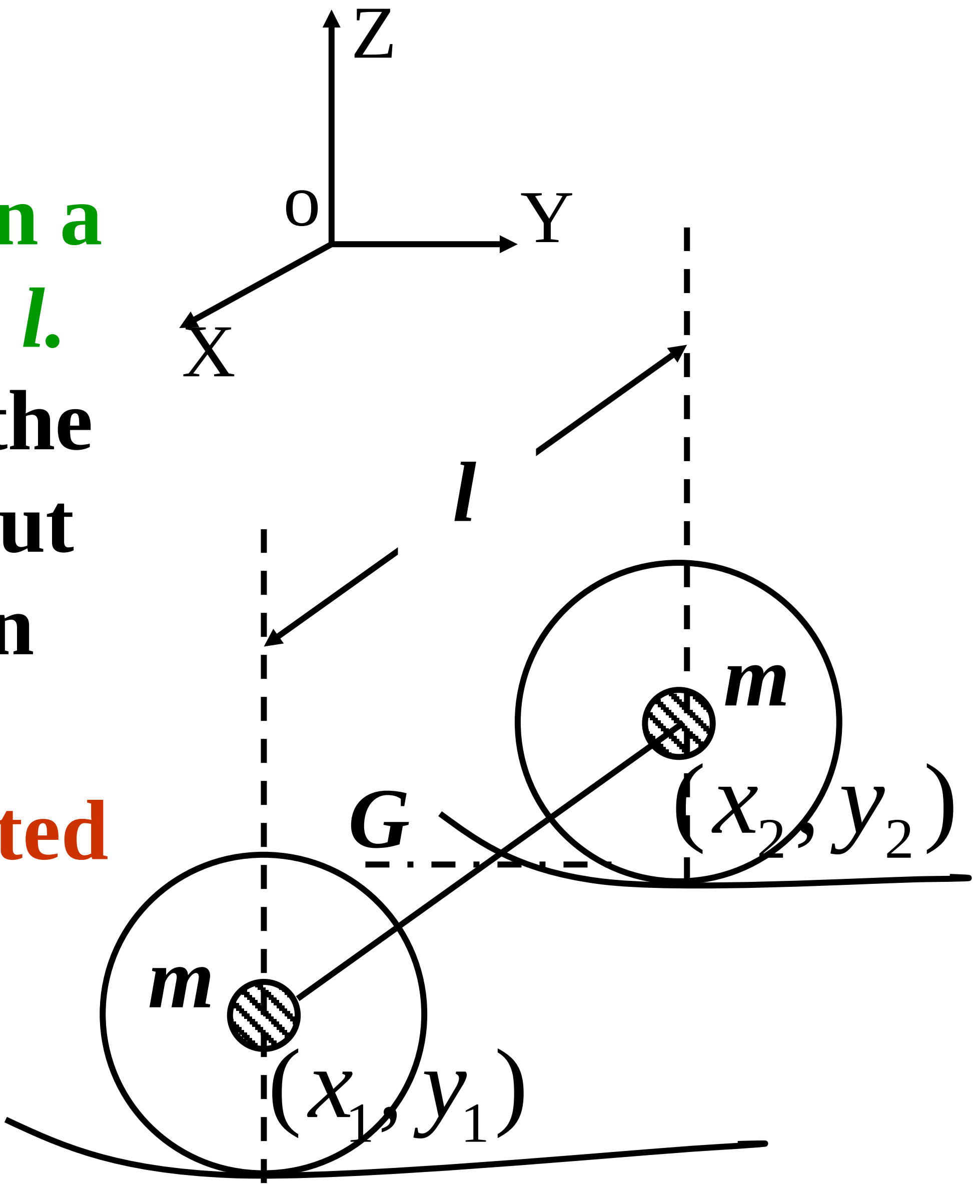
Ex. 6-9

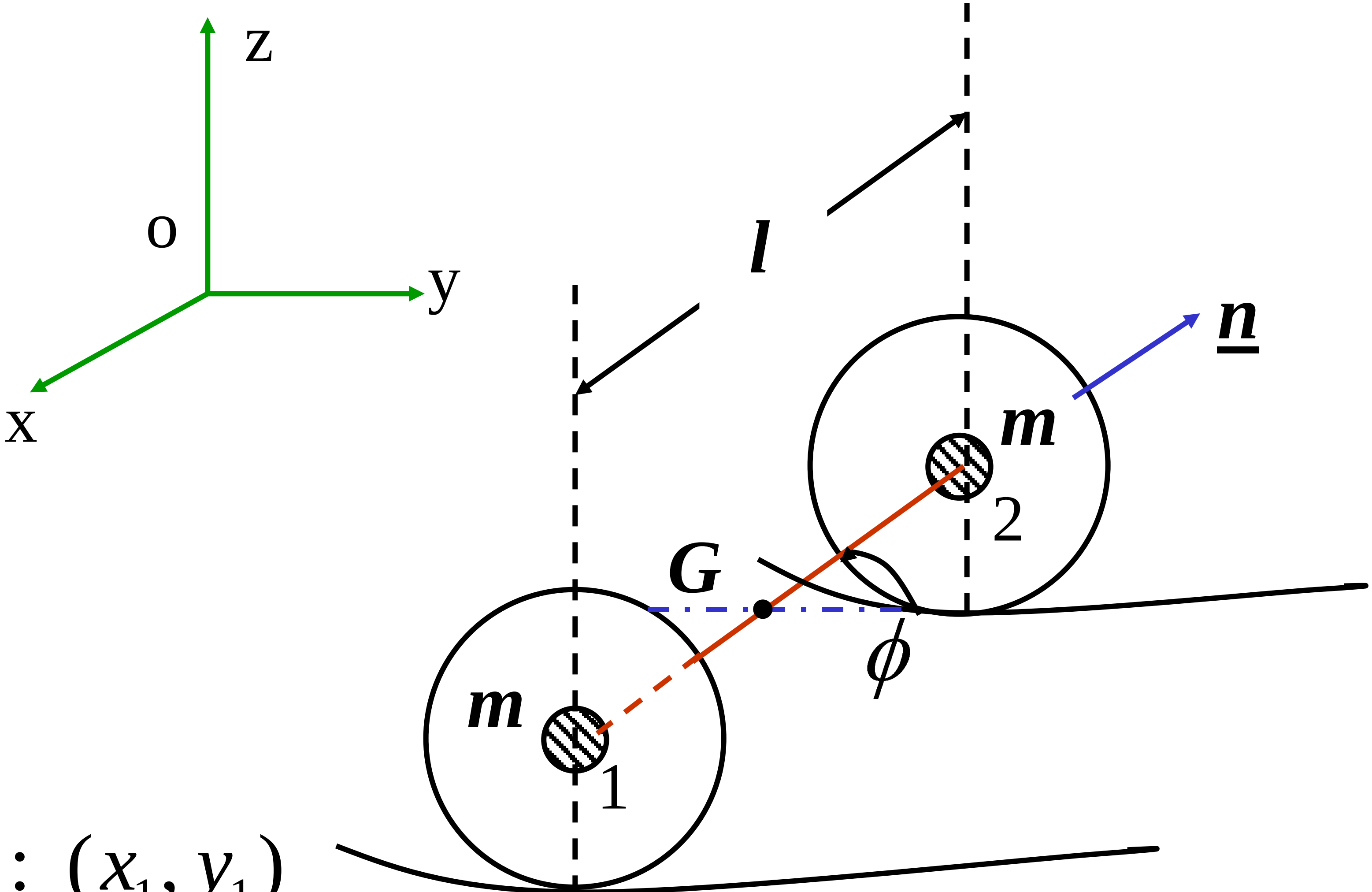
Consider two wheels on a common axle of length *l*.

Wheels are rolling on the horizontal plane without slipping, wheels remain normal to the ground.

Wheel mass concentrated at the hubs.

System defined by the positions (x_i, y_i)





$$1:\ (x_1, y_1)$$

$$2:\ (x_2, y_2)$$

$$\underline{v}_G \bullet \underline{n} = 0$$

$$\underline{v}_G = \frac{1}{2}\left[\left(\dot{x}_1 + \dot{x}_2\right)\underline{i} + \left(\dot{y}_1 + \dot{y}_2\right)\underline{j}\right]$$

$$\underline{n} = (\underline{r}_2 - \underline{r}_1)/\left|\underline{r}_2 - \underline{r}_1\right| = \frac{(x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}}{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}$$

$$\underline{v}_G \bullet \underline{n} = 0$$

$$\Rightarrow (\dot{x}_1 + \dot{x}_2)(x_2 - x_1) + (\dot{y}_1 + \dot{y}_2)(y_2 - y_1) = 0$$

Wheels roll without slipping $\Rightarrow$ **G moves** $\perp^{r}$ **to the axle.**

Constraints:

- **finite** $(x_1 - x_2)^2 + (y_1 - y_2)^2 - l^2 = 0$ (1)

- **kinematic** $\dfrac{\dot{y}_1 + \dot{y}_2}{\dot{x}_1 + \dot{x}_2} = -\dfrac{(x_2 - x_1)}{(y_2 - y_1)}$

or $(\dot{y}_1 + \dot{y}_2)(y_2 - y_1) + (\dot{x}_1 + \dot{x}_2)(x_2 - x_1) = 0$ (2)

4 general coordinates, 2 constraints

n = degrees of freedom = 4 – 2 = 2.

$n_0 = 2N - (d) = 4 - 1 = 3,\ \bar{n} = 4$

- **We now use the generalized coordinates (x, y, ϕ):** $x_1 = x - (\ell \sin\phi)/2,$

$$y_1 = y - (\ell \cos\phi)/2 \;\; , \;\; x_2 = x + (\ell \sin\phi)/2,$$

$$y_2 = y + (\ell \cos\phi)/2 \quad \Rightarrow$$ **this definition allows the finite constraint to be satisfied automatically**

$\Rightarrow$ **there are 3 generalized coordinates,** $\bar{n} = 3,$

There is only 1 constraint, d = 0, g = 1.

$$\boxed{\dot{x}\sin\phi - \dot{y}\cos\phi = 0} \qquad (3)$$

(*another form of* $\underline{v}_G \bullet \underline{n} = 0$)

Let $q_1 = x,\ \ q_2 = y,\ \ q_3 = \phi.$ **Then, the constraint can be written in standard form as**

$$\text{(3)} \Rightarrow \sum_{j=1}^{\bar{n}=3} a_{1j}\dot{q}_j + d_1 = 0 \quad - \quad \text{constraint}$$

$$a_{11} = \sin\phi,\ a_{12} = -\cos\phi,\ a_{13} = 0,\ d_1 = 0.$$

- **The equations of motion for the constrained system are:**

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i = \sum_{j=1}^{d+g} \lambda_j a_{ji}$$

i = 1, 2, 3.

$$T = \frac{1}{2} m \quad \dot{x}_1 + \dot{x}_2 \quad {}^2 + (\dot{y}_1 + \dot{y}_2)^2$$

$$x_1 = x - (\ell \sin\phi)/2 \Rightarrow \dot{x}_1 = \dot{x} - \ell\dot{\phi}\cos\phi/2$$

$$y_1 = y - (\ell \cos\phi)/2 \Rightarrow \dot{y}_1 = \dot{y} + \ell\dot{\phi}\sin\phi/2$$

$$x_2 = x + (\ell \sin\phi)/2 \Rightarrow \dot{x}_2 = \dot{x} + \ell\dot{\phi}\cos\phi/2$$

$$y_2 = y + (\ell \cos\phi)/2 \Rightarrow \dot{y}_2 = \dot{y} - \ell\dot{\phi}\sin\phi/2$$

etc.

The kinetic energy (K.E.) is

$$T = \underbrace{\frac{1}{2}(2m)(\dot{x}^2 + \dot{y}^2)}_{\text{K.E. of the C.M.}} + \underbrace{\frac{1}{4} m\ell^2 \dot{\phi}^2}_{\substack{\text{K.E. of rotation} \\ \text{about the C.M.}}}$$

$$\underline{\underline{q_1 = x}}: \quad \partial T / \partial \dot{x} = 2m\dot{x}\,; \quad \frac{d}{dt}(\partial T / \partial \dot{x}) = 2m\ddot{x}$$

$$\partial T / \partial x = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} - Q_1 = \sum_{j=1}^{d+g} \lambda_j a_{ji}\,; \quad d + g = 1$$

Note that there is only one constraint $\Rightarrow$ **there is only one Lagrange multiplier.**

$$\lambda_1 = \lambda \ , \quad a_{11} = \sin\phi \ , \quad Q_1 = 0$$

(there is no external effective force)

$$\Rightarrow \quad \boxed{2m\ddot{x} = \lambda \sin\phi} \qquad \textbf{(4)}$$

$$\underline{\underline{q_2 = y}}: \quad \partial T / \partial\dot{y} = 2m\dot{y} \ ; \ \frac{d}{dt}(\partial T / \partial\dot{y}) = 2m\ddot{y}$$

$$\partial T / \partial y = 0, \ \ a_{12} = -\cos\phi$$

$$\Rightarrow \quad \boxed{2m\ddot{y} = -\lambda \cos\phi} \qquad (5)$$

$$\underline{\underline{q_3 = \phi}}: \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} - Q_3 = \lambda a_{13}$$

$$\partial T / \partial \dot{\phi} = ml^2 \dot{\phi} / 12; \quad \frac{d}{dt}(\partial T / \partial \dot{\phi}) = ml^2 \ddot{\phi} / 12$$

$$\partial T / \partial \phi = 0, \quad a_{13} = 0$$

$$\Rightarrow \quad \boxed{\frac{1}{12} ml^2 \ddot{\phi} = 0} \qquad (6)$$

There are 3 equations of motion + 1 constraint equation for the four variables x, y, ϕ and λ.

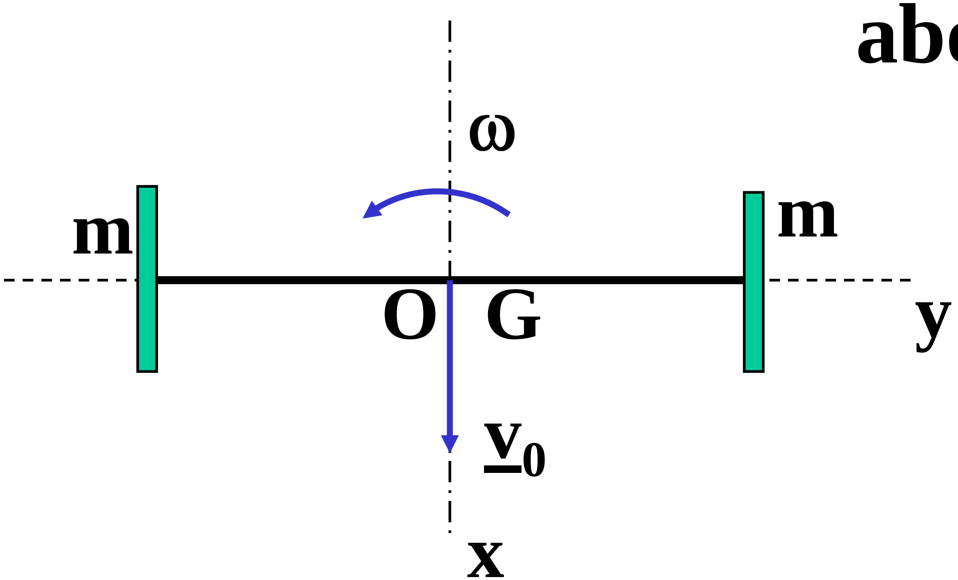
Need initial conditions to determine the motion.

example of a motion:

Initial conditions: $(x_0, y_0) = 0 -$ **mass center at the origin;**

$(\dot{x}_0, \dot{y}_0) = (v_0, 0)$ **- mass center given a velocity in x-direction;**

$(\phi_0 = 0, \dot{\phi}_0 = \omega)$ **- given angular velocity about the z-axis**

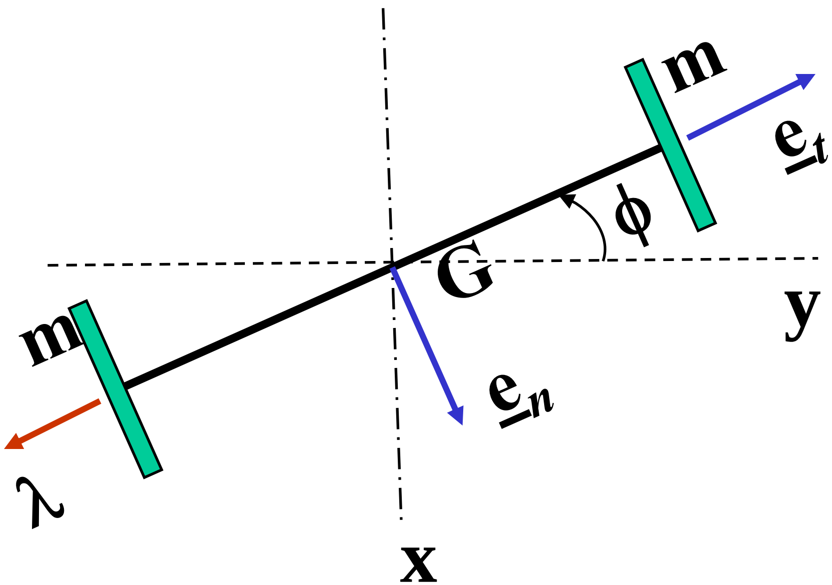


initial conditions

$$(6) \Rightarrow \quad ml^2\ddot{\phi}/12 = 0$$

$$\Rightarrow \ddot{\phi} = 0 \Rightarrow \boxed{\dot{\phi} = \omega, \phi = \omega t} \qquad (7)$$

$$(4) \Rightarrow 2m\ddot{x} = \lambda \sin\phi, \quad (5) \Rightarrow 2m\ddot{y} = -\lambda\cos\phi$$



Now, note that

$a_t =$ **tangent acceler**

$$= -\frac{\lambda}{2m}\cos^2\phi - \frac{\lambda}{2m}\sin^2\phi$$

$$= -\lambda/2m$$

$\Rightarrow$ *λ is the force along $\underline{e}_t$*

Now $\ddot{x} = (\lambda \sin \omega t) / 2m, \ddot{y} = -(\lambda \cos \omega t) / 2m$

Thus, the acceleration in $\underline{e}_n$ **direction is**

$$a_n = \ddot{x} \cos \phi + \ddot{y} \sin \phi$$

$$= \left(\frac{\lambda \sin \phi \cos \phi}{2m} - \frac{\lambda \cos \phi \sin \phi}{2m} \right) = 0$$

$\Rightarrow$ **Integrating** $v_n = \text{constant} = v_n(t = 0) = v_0$

$$\Rightarrow \quad \left. \begin{aligned} \dot{x} &= v_0 \cos \phi = v_0 \cos \omega t \\ \dot{y} &= v_0 \sin \phi = v_0 \sin \omega t \end{aligned} \right\} \text{ velocity components}$$

Integrating again, we get

$$\boxed{x(t) = \frac{v_0}{\omega} \sin \omega t} \qquad \textbf{(8)}$$

$$y(t) = -\frac{v_0}{\omega} \cos \omega t + \underbrace{\text{constant}}_{v_0/\omega} \qquad (\phi = \omega t)$$

or $$\boxed{y(t) = \frac{v_0}{\omega}(1 - \cos \omega t)} \qquad \textbf{(9)}$$

Then (4) $\Rightarrow \lambda \sin \omega t = 2m\ddot{x} = 2m(-v_0 \omega \sin \omega t)$

or $\boxed{\lambda = -2mv_0\omega}$ **(10)**

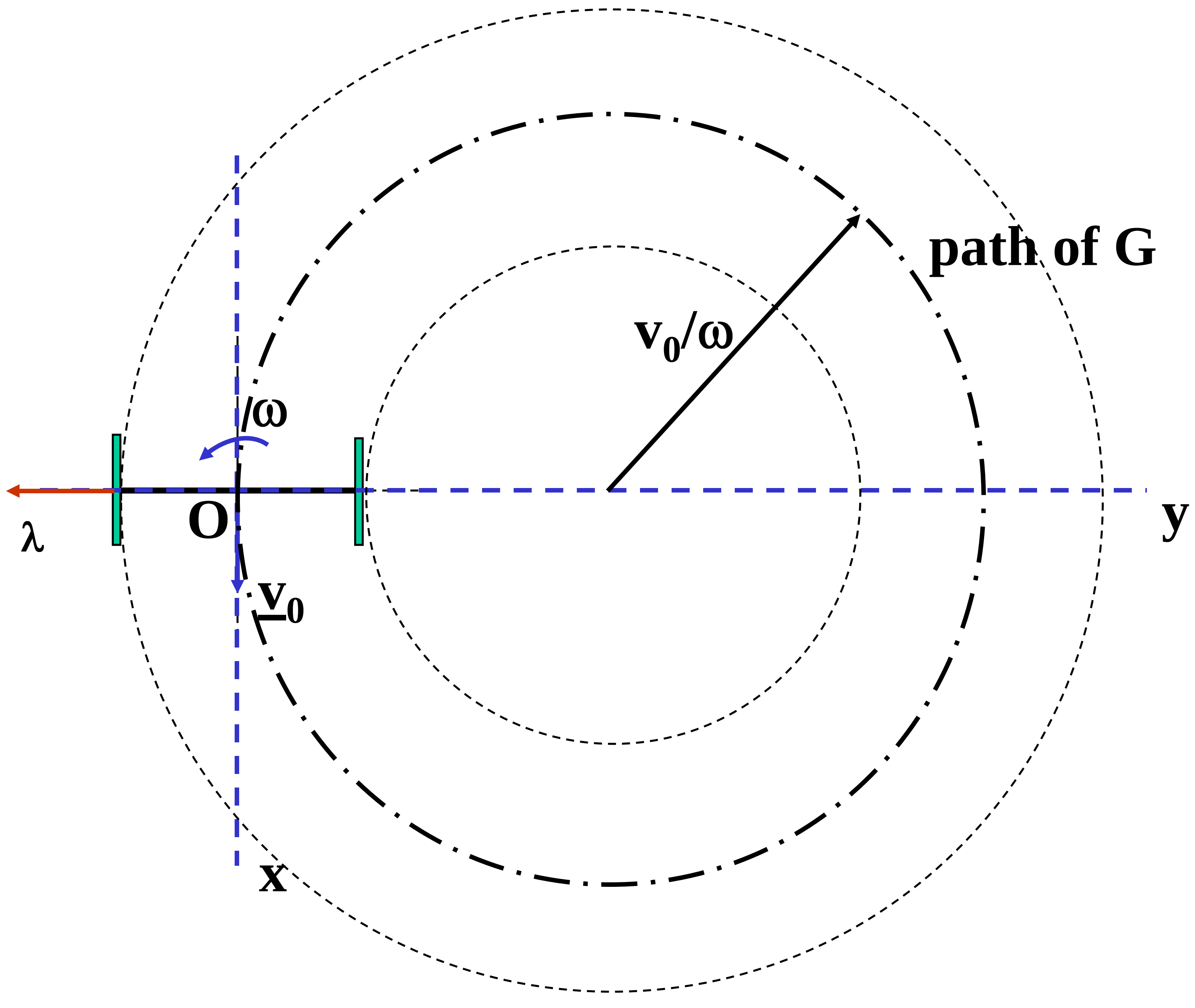
Note: λ turned out to be constant here.

Clearly

$$x^2 + (y - \frac{v_0}{\omega})^2 = (\frac{v_0}{\omega})^2 \qquad \textbf{(11)}$$

with $\phi = \omega t$

$\Rightarrow$ **Path of G is a circle with center at (x, y) =** $(0, v_0 / \omega)$ and radius v_0 / ω


path of G
v_0/ω
$/\omega$
λ
O
$\underline{v}_0$
y
x

6.8 Conservation Laws:

The Lagrange's equations for a holonomic system with n degrees of freedom and n generalized coordinates are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i' \ , \ \ i = 1, 2, \ldots, n$$

These equations can be put in first-order form by defining *generalized momenta:*

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}, \ \ i = 1, 2, \ldots, n$$

Then, the equations can be written as:

$$\frac{d}{dt}(p_i) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i',$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \;\; i = 1, 2, ..., n$$

Suppose that there is a system for which a generalized coordinate, say q_s, is absent from the Lagrangian L although its time derivative does appear, i.e., $L = T(q_1, ..., q_{s-1}, q_{s+1}, .., q_{n,} \dot{q}_1, \dot{q}_2, ... \dot{q}_n)$

$$- V(q_1, ..., q_{s-1}, q_{s+1}, .., q_n)$$

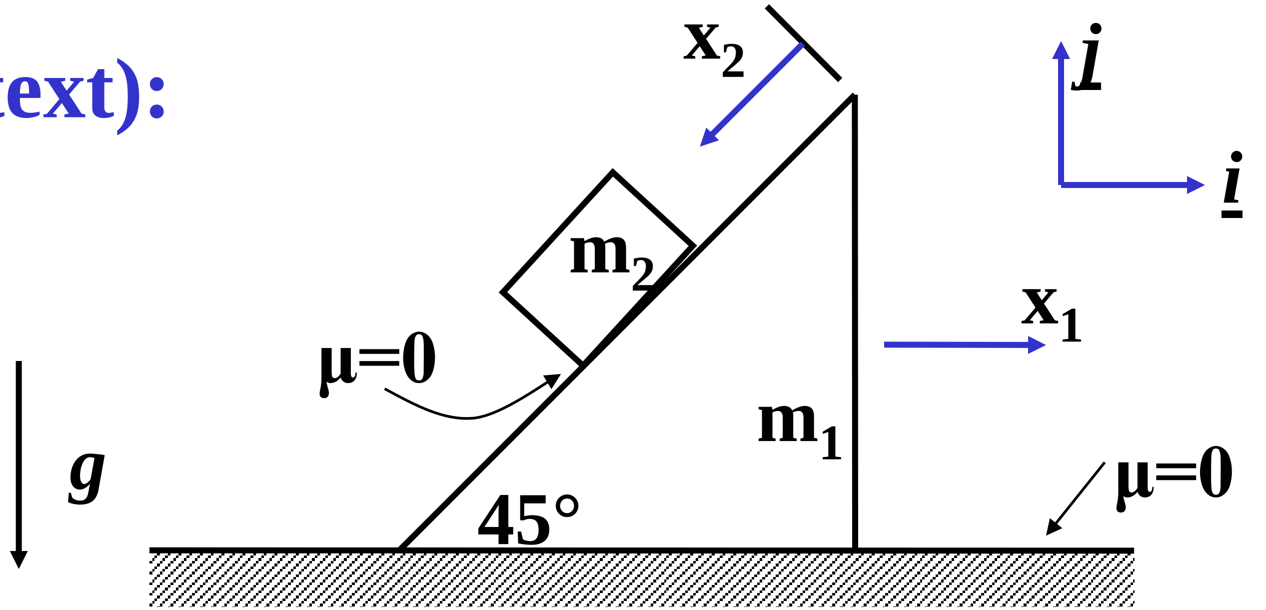
Further, suppose that the generalized force $Q_s' = 0.$

Then Lagrange's equations give for the generalized coordinate q_s: $\frac{d}{dt}(p_s) = 0 \quad or \quad p_s = \frac{\partial L}{\partial \dot{q}_s} = constant$

This says that the ***generalized momentum associated with the*** coordinate q_s ***is conserved – remains constant throughout motion.*** q_s ***is called an <u>ignorable coordinate</u>.***

The term 'ignorable' refers to the fact that the degree of freedom corresponding to the coordinate q_s can be ignored from the formulation of the problem.

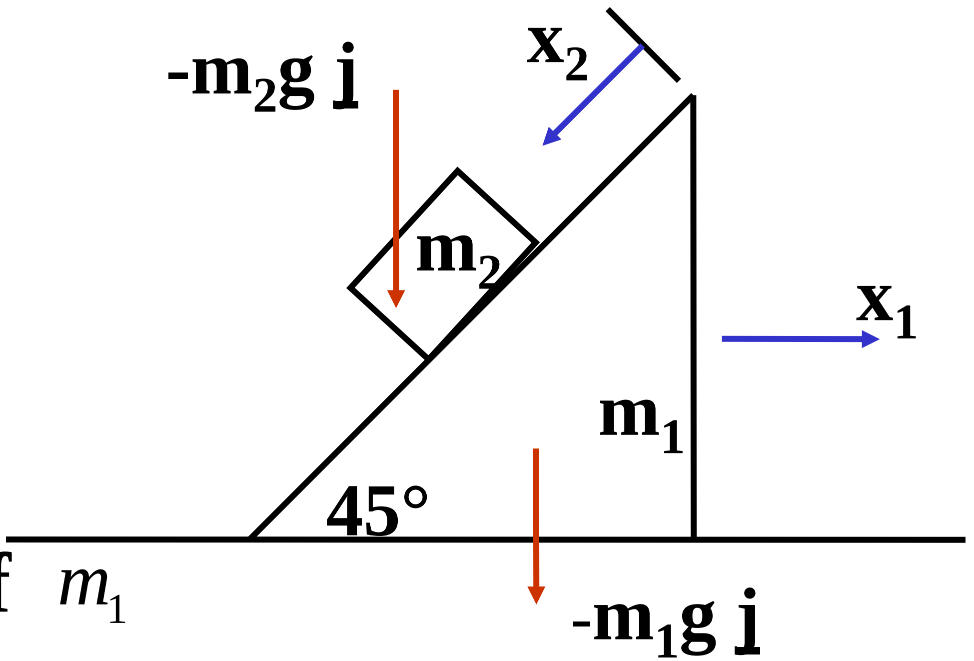
Ex 19 (text):



m_2 **slides on** m_1;

m_1 **slides on the horizontal surface:**

All surfaces in contact are smooth. $\mu = 0$.



x_1 – **absolute position of** m_1

x_2 – **position of** m_2 **relative to** m_1

Find: Equations of motion of the system, ignorable coordinates, and the conserved quantities.

- **There are two generalized coordinates** x_1, x_2

n = degrees of freedom = 2. (no constraints on x_1, x_2 **)**

We proceed as before and develop the various quantities, starting with position vectors:

$$\underline{r}_1 = x_1 \underline{i} \;\; ; \;\; \underline{r}_2 = x_1 \underline{i} - (x_2 \underline{i} + x_2 \underline{j}) / \sqrt{2}$$

$$\underline{v}_1 = \dot{x}_1 \underline{i} \;\; ; \;\; \underline{v}_2 = (\dot{x}_1 - \dot{x}_2 / \sqrt{2})\, \underline{i} - \dot{x}_2 \underline{j} / \sqrt{2}$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \{ (\dot{x}_1 - \dot{x}_2 / \sqrt{2})^2 + (\dot{x}_2 / \sqrt{2})^2 \}$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}_1^2 + \frac{1}{2} m_2 \{ \dot{x}_2^2 - \sqrt{2} \dot{x}_1 \dot{x}_2 \}$$

- **We need to find generalized forces** Q_1, Q_2 ?

$$\underline{F}_1 = -m_1 g\, \underline{j} \;\; ; \;\; \delta \underline{r}_1 = \delta x_1 \underline{i} \; ;$$

$$\underline{F}_2 = -m_2 g\, \underline{j} \;\; ; \;\; \delta \underline{r}_2 = (\delta x_1 - \frac{\delta x_1}{\sqrt{2}}) \underline{i} - \frac{\delta x_2}{\sqrt{2}} \underline{j}$$

$$\delta W = \sum_{i=1}^{2} \underline{F}_i \bullet \delta \underline{r}_i = m_2 g \frac{\delta x_2}{\sqrt{2}} = Q_1 \delta x_1 + Q_2 \delta x_2$$

$$\Rightarrow \; Q_1 = 0 \;\; ; \;\; Q_2 = m_2 g / \sqrt{2}$$

- **Then, the equations of motion are:**

$$\underline{\underline{x}}_1 : \frac{\partial T}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1 - \frac{m_2}{\sqrt{2}} \dot{x}_2$$

Note that T does not depend on x_1. In addition, $Q_1 = 0$.

So, x_1 is an 'ignorable' coordinate, and the corrosponding generalized momentum p_{x_1} is conserved.

$$\Rightarrow \left(\frac{\partial T}{\partial \dot{x}_1}\right) = \boxed{(m_1 + m_2)\dot{x}_1 - \frac{m_2}{\sqrt{2}}\dot{x}_2 = constant = p_{x_1}}$$

$$\underline{\underline{x}}_2 : \quad \frac{\partial T}{\partial \dot{x}_2} = m_2 \dot{x}_2 - \frac{m_2}{\sqrt{2}}\dot{x}_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 - \frac{m_2}{\sqrt{2}}\ddot{x}_1 \quad ; \quad \frac{\partial T}{\partial x_2} = 0$$

$$Q_2 = \frac{m_2 g}{\sqrt{2}} \quad \Rightarrow \boxed{m_2 \ddot{x}_2 - \frac{m_2}{\sqrt{2}}\ddot{x}_1 = \frac{m_2 g}{\sqrt{2}}}$$

Lagrangian independent of time:

Suppose that L does not depend explicitly on time. Then,

$$\frac{\partial L}{\partial t} = 0$$

Now, the Lagrangian is $L=T\text{-}V$, and it depends both on generalized coordinates and generalized velocities. The total derivative of Lagrangian is:

$$\frac{dL}{dt} = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k$$

Now, Lagrange's equations give

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \ \ (\,for\ the\ case\ of\ \ Q_k\,'s = 0)$$

So, we can write

$$\frac{dL}{dt} = \sum_{k=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k = \sum_{k=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right)$$

$$or \quad \sum_{k=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) - \frac{dL}{dt} = 0 \Rightarrow \boxed{\frac{d}{dt}\left(\sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\right) = 0}$$

This shows that the quantity $\left(\sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\right)$

remains constant during motion, i.e.,

$$\boxed{\left(\sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\right) = h} - \ Jacobi's\ Integral$$

We can further manipulate this as follows.

Consider the expression for a Lagrangian:

$$L = T_2 + T_1 + T_0 - V = \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\alpha_{rs}\dot{q}_r\dot{q}_s + \sum_{s=1}^{n}\beta_s\dot{q}_s + \gamma - V$$

Since the Lagrangian is independent of time, the coefficients α_{rs}, β_s, and γ depend on generalized coordinates only, and thus

$$h = (\sum_{k=1}^{n}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L) = 2T_2 + T_1 - (T_2 + T_1 + T_0 - V) = T_2 - T_0 + V$$

When the kinetic energy is a homogeneous quadratic function of the generalized velocities, we have $T = T_2 = \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\alpha_{rs}\dot{q}_r\dot{q}_s$

and thus $h = (\sum_{k=1}^{n}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L) = 2T - (T - V) = T + V$

$$= E \ (total \ energy)$$

In other words, ***in a natural system for which the Lagrangian does not depend explicitly on time, the total energy of the system E is conserved.***

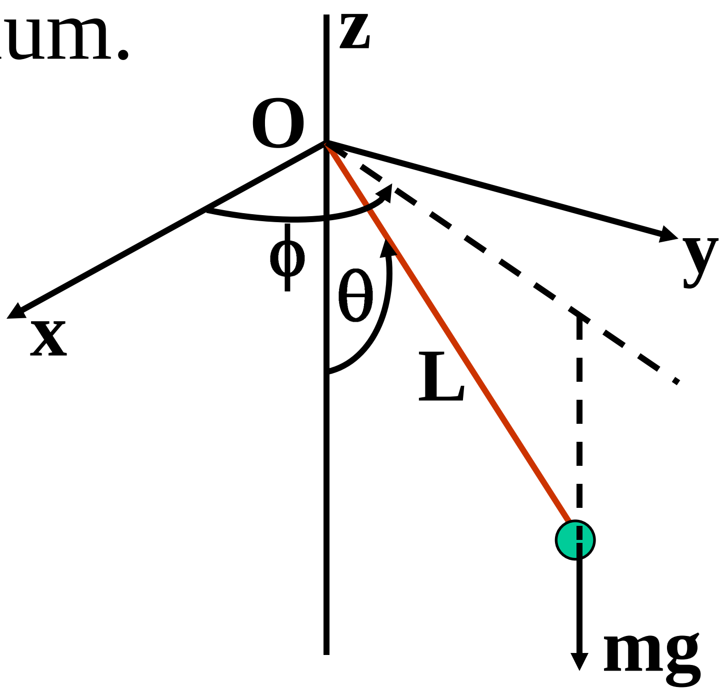
Example: Spherical pendulum

Consider the spherical pendulum.

The kinetic and potential energies are:

$$T = \frac{1}{2} m[(L\dot{\theta})^2 + (\dot{\phi} L \sin\theta)^2]$$

$$V = mgL(1 - \cos\theta)$$



The Lagrangian is

$$L = \frac{1}{2} m[(L\dot{\theta})^2 + (\dot{\phi} L \sin\theta)^2] - mgL(1 - \cos\theta)$$