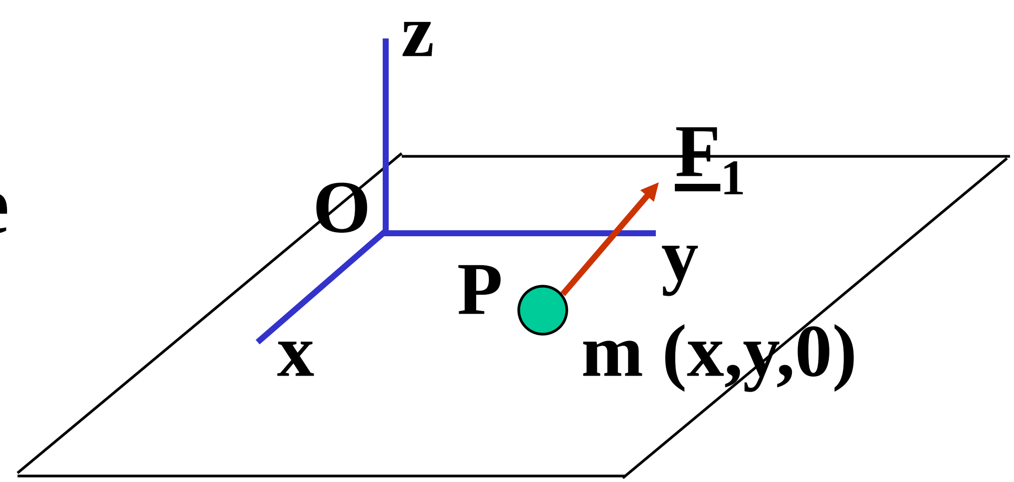
CHAPTER 6

LAGRANGE'S EQUATIONS

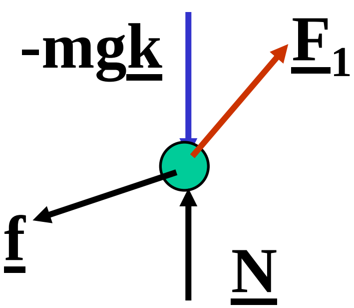
(Analytical Mechanics)

Ex. 1: Consider a particle moving on a fixed horizontal surface.



Let, $\underline{r}_P$ be the position and $\underline{F}$ be the total force on the particle.

The FBD is:



The equation of motion is $m\ddot{\underline{r}}_p = \underline{F}(\underline{r}, \dot{\underline{r}}, t)$

In component form, the equation of motion is

$$m\ddot{x}_P = F_x(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$$

$$m\ddot{y}_P = F_y(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$$

$$m\ddot{z}_P = F_z(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$$

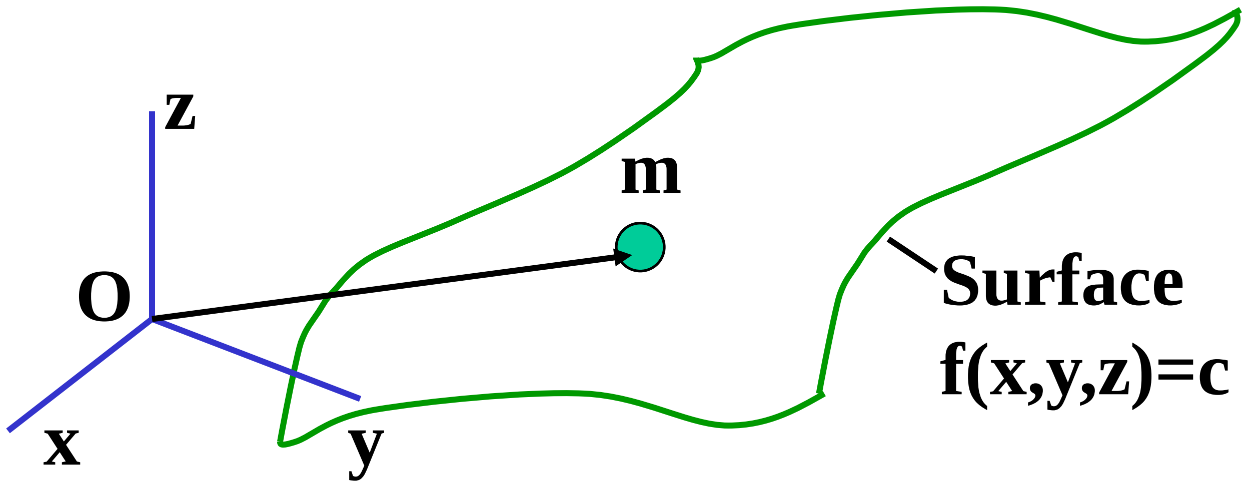
Also, motion is restricted to xy plane

→ <u>z = 0</u> - equation of constraint

→ It is a geometric restriction on where the particle can go in the 3-D space.

→ Clearly, there is a constraint reaction (force) that needs to be included in the total force <u>F</u>.

Ex. 2: Consider a particle moving on a surface.



Now, the motion is confined to a prespecified surface (e.g. a roller coaster). The surface is defined by the relation:

f (x, y, z) - c = 0 - equation of constraint.

The equation of motion will again be the same.

• **The constraints in the two examples are geometric or configuration constraints.**

They could be independent of time t, or could depend explicitly on it. For an N particle system, if the positions of particles are given by $\underline{r}_1, \underline{r}_2, \underline{r}_3, \ldots.,$ **the constraint can be written as:**

$$f(\underline{r}_1, \underline{r}_2, \cdots, \underline{r}_N, t) = 0$$

This is an equation of a finite or geometric or holonomic constraint.

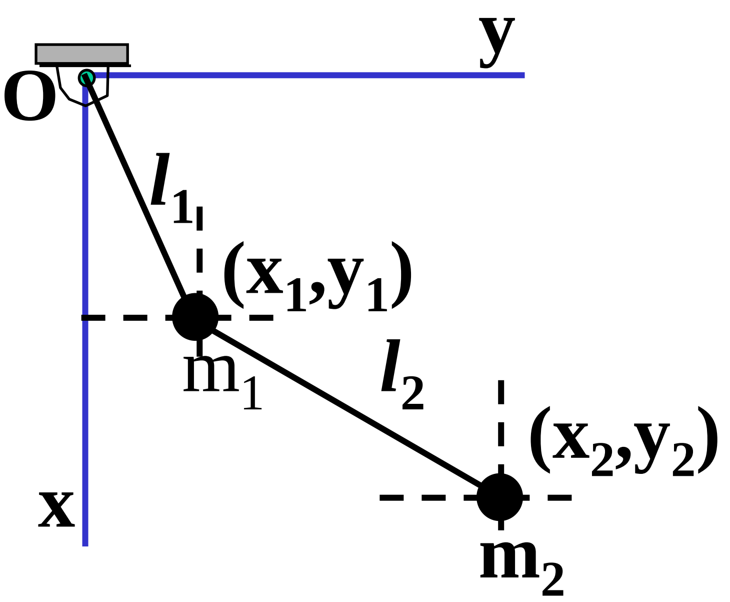
Ex. 3: Double pendulum: **it consists of two particles and two massless rigid rods**

The masses are

$m_1 : (x_1 , y_1)$

$m_2 : (x_2 , y_2)$

$(z_1 = z_2 = 0 : \text{planar motion})$



Number of coordinates required is 4 - used to define the configuration

- **There are certain constraints on motion:**

$$l_1^2 = (x_1^2 + y_1^2),\, l_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

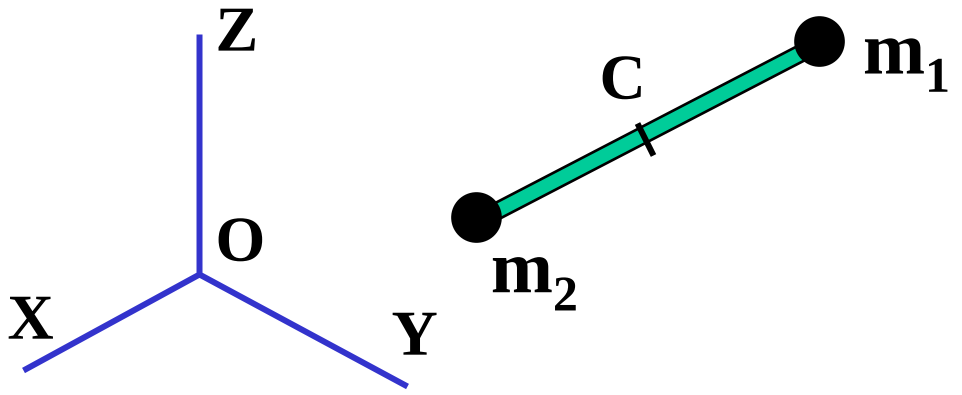
$\rightarrow$ 2 **equations of constraint (they are holonomic, geometric, finite etc.)**

- **Degrees-of-freedom: the number of independent coordinates needed to completely specify the configuration of the system (4 - 2) = 2.**

One could perhaps find another set of two coordinates (variables) that are independent: e.g., θ_1 , θ_2 **, the two angles with the vertical.**

Then, there are no constraints on θ_1 , θ_2 ,

Ex. 4: **A dumbbell moving in space**



- **one possible specification of position is:**

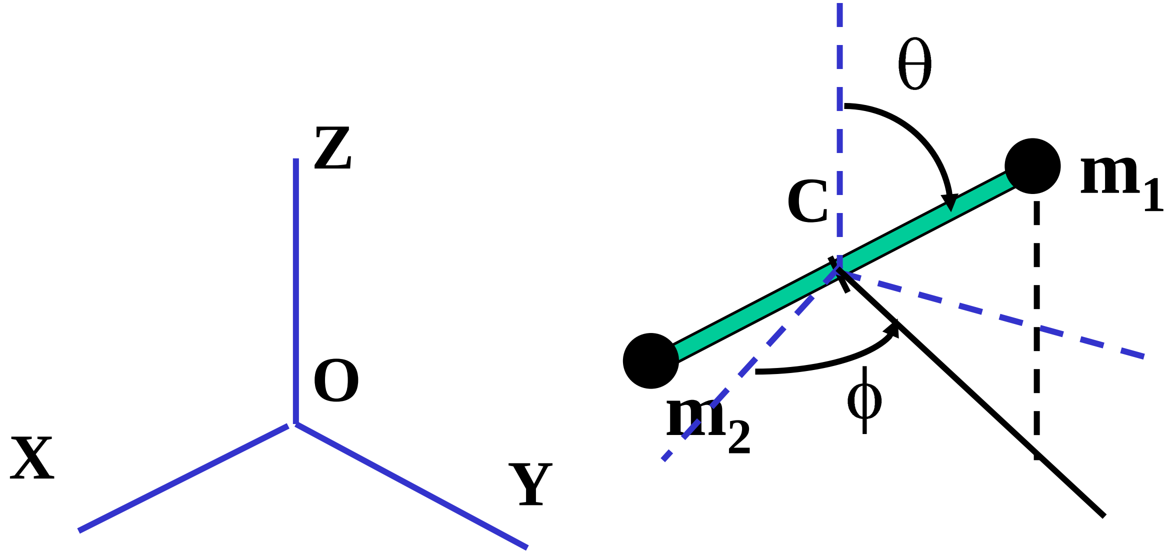
$$m_1 : \; x_1 , y_1 , z_1 \quad ; \quad m_2 : \; x_2 , y_2 , z_2$$

- these are 6 variables or coordinates, and there is one constraint

$$\ell^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

→ **degrees-of-freedom of the system   6 - 1 = 5**

- **another possible specification for the configuration of the system:**



Location of center of mass C: $(x_c\ , y_c\ , z_c)$ **;** **and orientation of the rod: (ϕ , θ). These are independent → no constraint relation for these variables.**

- **generalized coordinates - any number of variables needed to completely specify the configuration of a system.**

e.g., for the dumbbell in space motion:

$$\left.\begin{array}{c}(x_1, y_1, z_1, x_2, y_2, z_2), \\ (x_C, y_C, z_C, \phi, \theta)\end{array}\right\}$$

there are two sets of generalized coordinates

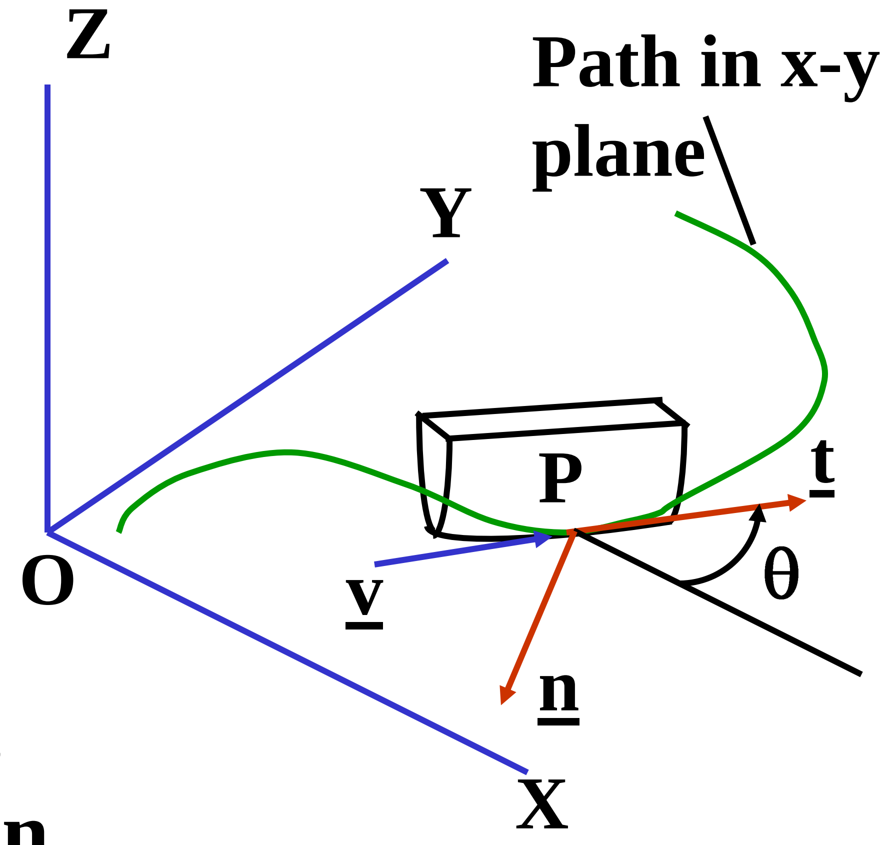
Important: some sets consist of independent coordinates (no constraints) where as others are not independent.

Ex. 5: Ice skate

Basic facts:

Configuration of the skate can be specified by the coordinates (x, y) and the angle θ.

The ice skates can only move along the plane of the skate, i.e., in the tangent direction specified by angle θ. (a constraint)



Let $\underline{t}$ - tangent to the path, $\underline{n}$ - normal to the path. Then $\underline{v}\cdot\underline{n}=0$ **for the skate, or**

$$(\dot{x}\underline{i}+\dot{y}\underline{j})\cdot(-\cos\theta\underline{j}+\sin\theta\underline{i})=0$$

or $\boxed{\dot{x}\sin\theta-\dot{y}\cos\theta=0}$ **a constraint which depends both, on coordinates and their time derivatives.**

- **In general**

$$\phi(\underline{r}_1,\cdots,\underline{r}_N,\dot{\underline{r}}_1,\cdots,\dot{\underline{r}}_N,t)=0$$

Such a constraint is called a kinematical, differential, nonholonomic constraint.

We have seen then that, in general:

Holonomic constraints are of the form

$$\phi_j(q_1,\ldots,q_N,t)=0,\quad j=1,2,3,\ldots,g$$

→ equality constraints involving only generalized coordinates and time

Nonholonomic constraints are of the form

$$\phi_j(q_1,\ldots,q_N,\dot{q}_1,\ldots,\dot{q}_N,t)=0,\quad j=1,2,3,\ldots,d$$

→ they depend on generalized coordinates, velocities, as well as time.

Fundamental difference:

- **A geometric constraint restricts the configurations that can be achieved during motion. Certain regions (positions) are inaccessible**
- **A kinematic constraint only restricts the velocities that can be acquired at a given position. The system can, however, occupy any position desired (e.g.: one can reach any point in the skating rink - it is just that one cannot move in arbitrary direction).**

We can also write the constraints in the form: (in differential form)

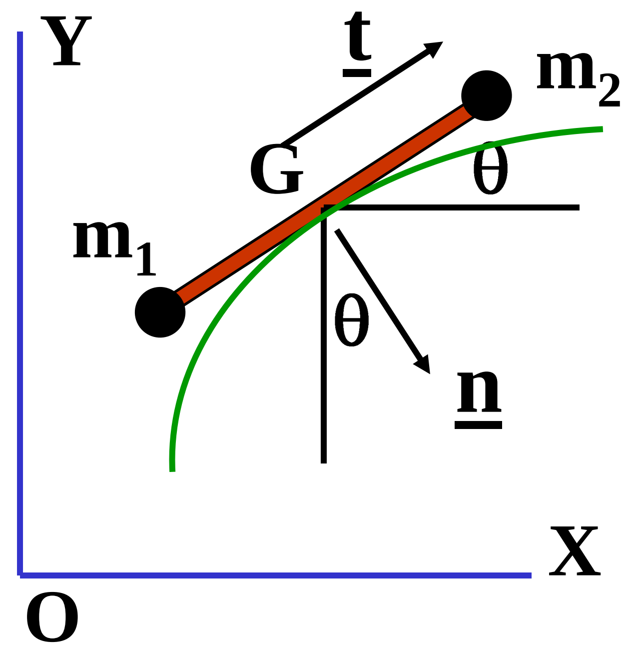
$$\sum_{i=1}^{n} a_{ji}(q_1,\dots.,q_n,t)dq_i + a_{jt}(q_1,\dots.,q_n,t)dt = 0,$$

$$j = 1, 2, \dots., d$$

- **Whether a constraint is holonomic or nonholonomic depends on whether the differential form is integrable or nonintegrable.**

Ex 6: Particle model of a skate: two equal masses are connected by a massless rigid rod.

They slide on the XY plane. G is the centroid of the system.



- $z_1 = z_2 = 0$

  The other constraints on motion are:

- **length is constant**

  $$(x_2 - x_1)^2 + (y_2 - y_1)^2 = \ell^2$$

  (holonomic)

- **Skate cannot move along $\underline{n}$ direction (nonholonomic)** $\underline{v}_G \bullet \underline{n} = 0$

We now define these constraints in terms of the physical coordinates, and then the generalized coordinates q_i:

The CG has $\underline{v}_G = [(\dot{x}_1 + \dot{x}_2)\underline{i} + (\dot{y}_1 + \dot{y}_2)\underline{j}]/2$

Now $\underline{n} = -\cos\theta\,\underline{j} + \sin\theta\,\underline{i}$

$$\cos\theta = (x_2 - x_1)/l, \quad \sin\theta = (y_2 - y_1)/l$$

$$\rightarrow \quad \underline{n} = [(y_2 - y_1)\underline{i} - (x_2 - x_1)\underline{j}]/l$$

The nonholonomic constraint is

$$\boxed{(\dot{x}_1 + \dot{x}_2)(y_2 - y_1) - (\dot{y}_1 + \dot{y}_2)(x_2 - x_1) = 0}$$

The coordinates are: $(x_1, y_1, z_1), (x_2, y_2, z_2)$

Generalized coordinates:

$$x_1 = q_1, y_1 = q_2, z_1 = q_3, x_2 = q_4, y_2 = q_5, z_2 = q_6$$

Then, the constraints have to be written in terms of q's:

$$z_1 = 0, z_2 = 0$$

constraints #1 and #2

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - \ell^2 = 0$$

constraint #3

$$(\dot{x}_1 + \dot{x}_2)(y_2 - y_1) - (\dot{y}_1 + \dot{y}_2)(x_2 - x_1) = 0$$

constraint #4

Constraint #1: $z_1 = 0 \;\Rightarrow\; \dot{z}_1 = 0$

In differential form

$$\frac{dz_1}{dt}dt = 0 \qquad \rightarrow dz_1 = 0 \;\text{ or }\; \boxed{d(q_3) = 0}$$

In general form, we have

$$\sum_{i=1}^{6} a_{ji} dq_i + a_{jt} dt = 0,\; j = 1$$

or

$$a_{11}dq_1 + a_{12}dq_2 + \cdots + a_{16}dq_6 + a_{1t}dt = 0$$

$$\rightarrow a_{11} = 0, a_{12} = 0, a_{13} = 1,\; a_{14} = 0,$$

$$a_{15} = 0,\; a_{16} = 0,\; a_{1t} = 0$$

Constraint #2: $z_2 = 0 \Rightarrow \dot{z}_2 = 0$

In differential form

$$\frac{dz_2}{dt}dt = 0 \quad \rightarrow dz_2 = 0 \text{ or } \boxed{d(q_6) = 0}$$

In general form, we have

$$\sum_{i=1}^{6} a_{ji} dq_i + a_{jt} dt = 0, \ j = 2$$

or

$$a_{21}dq_1 + a_{22}dq_2 + \cdots + a_{26}dq_6 + a_{2t}dt = 0$$

$$\rightarrow a_{21} = 0, a_{22} = 0, a_{23} = 0, \ a_{24} = 0,$$

$$a_{25} = 0, \ a_{26} = 1, \ a_{2t} = 0$$

constraint #3:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - \ell^2 = 0$$

In differential form

$$(x_2 - x_1)(dx_2 - dx_1) + (y_2 - y_1)(dy_2 - dy_1)$$
$$+ (z_2 - z_1)(dz_2 - dz_1) = 0$$

or $(x_1 - x_2)dx_1 + (x_2 - x_1)dx_2 + (y_1 - y_2)dy_1 + (y_2 - y_1)dy_2$

$$+ (z_1 - z_2)dz_1 + (z_2 - z_1)dz_2 = 0$$

or

$$\boxed{\begin{aligned}&(q_1 - q_4)dq_1 + (q_4 - q_1)dq_4 + (q_2 - q_5)dq_2 \\ &+ (q_5 - q_2)dq_5 + (q_3 - q_6)dq_3 + (q_6 - q_3)dq_6 = 0\end{aligned}}$$

constraint #4:

$$(\dot{x}_1 + \dot{x}_2)(y_2 - y_1) - (\dot{y}_1 + \dot{y}_2)(x_2 - x_1) = 0$$

In Differential form:

$$(q_5 - q_2)(dq_1 + dq_4) - (q_4 - q_1)(dq_2 + dq_5) = 0$$

or

$$\boxed{\begin{aligned}&(q_5 - q_2)dq_1 - (q_4 - q_1)dq_2 + (q_5 - q_2)dq_4 \\ &-(q_4 - q_1)dq_5 + (0)dq_5 + (0)dq_6 = 0\end{aligned}}$$

- **differential form of constraints (in general):** $\sum_{i=1}^{n} a_{ji} dq_i + a_{jt} dt = 0, \; j = 1, 2, 3, \ldots.. m$

A constraint (or differential form) is integrable if

$$\partial a_{ji} / \partial q_k = \partial a_{jk} / \partial q_i$$

$$\partial a_{ji} / \partial t = \partial a_{jt} / \partial q_i \,, \quad i, k = 1, 2, \ldots, n$$

These are conditions for exactness (of a differential form)

Ex 7: Consider a constraint $a_{11}\dot{x}_1 + a_{12}\dot{x}_2 + a_{1t} = 0$

In differential form, it is: $a_{11}dx_1 + a_{12}dx_2 + a_{1t}dt = 0$

Suppose that a_{11}, a_{12}, a_{1t} are constants.

Clearly, the constraint is integrable:

The integrated form is: $\boxed{a_{11}x_1 + a_{12}x_2 + a_{1t}t = c}$

Mathematically, if integrable, there is a function ϕ such that $d\phi/dt = 0$

or $\dfrac{\partial\phi}{\partial x_1}dx_1 + \dfrac{\partial\phi}{\partial x_2}dx_2 + \dfrac{\partial\phi}{\partial t}dt = 0$

$\rightarrow \boxed{a_{11} = \partial\phi/\partial x_1 \ , \ a_{12} = \partial\phi/\partial x_2 \ , \ a_{1t} = \partial\phi/\partial t}$

Clearly, then $$\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \frac{\partial a_{12}}{\partial x_2} = \frac{\partial}{\partial x_2}\frac{\partial \phi}{\partial x_1} = \frac{\partial a_{11}}{\partial x_2} = 0$$

or $$\frac{\partial a_{11}}{\partial x_2} = \frac{\partial a_{12}}{\partial x_1} \leftarrow$$

Similarily $$\frac{\partial a_{11}}{\partial t} = \frac{\partial a_{1t}}{\partial x_1} = 0 \ , \quad \frac{\partial a_{12}}{\partial t} = \frac{\partial a_{1t}}{\partial x_2} = 0$$

- **These are sufficient conditions for the constraint to be integrable.**

e.g.: consider the constraint #3 (j = 3)

$$(q_1 - q_4)dq_1 + (q_4 - q_1)dq_4 + (q_2 - q_5)dq_2 + (q_5 - q_2)dq_5 = 0$$

Here $a_{j1} = q_1 - q_4, a_{j2} = q_2 - q_5, a_{j3} = 0$

Thus $\partial a_{j1} / \partial q_2 = 0 = \partial a_{j2} / \partial q_1$

Similarly $a_{j4} = q_4 - q_1, a_{j5} = q_5 - q_2, a_{j6} = 0$

and $\partial a_{j1} / \partial q_4 = -1 = \partial a_{j4} / \partial q_1$, etc.

$\Rightarrow$ **This constraint is integrable.**

Also, $\partial a_{j1} / \partial q_3 = 0 = \partial a_{j3} / \partial q_1$

$$\partial a_{j1} / \partial q_5 = 0 = \frac{\partial a_{j5}}{\partial q_1} \;;\; a_{j5} = q_5 - q_2$$

$$\partial a_{j1} / \partial q_6 = 0 = \partial a_{j6} / \partial q_1 \;;\; \partial a_{jt} / \partial q_1 = \partial a_{j1} / \partial t = 0$$

$$\partial a_{j2} / \partial q_3 = 0 = \partial a_{j3} / \partial q_2$$

.

.

Now, consider constraint #4:

$$(q_5 - q_2)dq_1 - (q_4 - q_1)dq_2 + (q_5 - q_2)dq_4 - (q_4 - q_1)dq_5 = 0$$

Then $a_{41} = q_5 - q_2$, $a_{42} = -(q_4 - q_1)$, $a_{43} = 0$,

or, $\partial a_{41} / \partial q_2 = -1 \neq \partial a_{42} / \partial q_1 = 1$

$\Rightarrow$ **not an exact differential; i.e., it is not an integrable constraint.**

Classification: an N particle system is said to be:

- **Holonomic** - **if all** constraints are geometric, or if kinematic - are integrable (reducible to geometric).
- **Nonholonomic** - if there is a constraint which is kinematic and not integrable.
- **Schleronomic** - all the constraints, geometric as well as kinetic, are independent of time t explicitly.
- **Rheonomic** - if at least one constraint depends explicitly on time t.

Possible and Virtual Displacements

Suppose that a system of N particles, with position vectors $\underline{r}_1, \underline{r}_2, \cdots, \underline{r}_N$ **has *d***
geometric constraints

$$\phi_i(\underline{r}_1, \underline{r}_2, \underline{r}_3, \ldots., \underline{r}_N, t) = 0, \;\; i = 1, 2, 3, \ldots., d,$$

and g kinematic constraints

$$\sum_{i=1}^{N} \underline{l}_{ji} \bullet \dot{\underline{r}}_j + D_j = 0, \;\; i = 1, 2, \ldots, g$$

Here $\underline{l}_{ji} \equiv \underline{l}_{ji}(\underline{r}_1, \underline{r}_2, \underline{r}_3, \ldots., \underline{r}_N, t), \, etc.$

In differential form

$$\sum_{j=1}^{N} \nabla \phi_{ij} \bullet \dot{\underline{r}}_j + \partial \phi_i / \partial t = 0 \,, \ i = 1, 2, \ldots, d \qquad (1)$$

and

$$\sum_{i=1}^{N} \underline{l}_{ji} \bullet \dot{\underline{r}}_j + D_j = 0 \,, \ i = 1, 2, \ldots, g \qquad (2)$$

$\Rightarrow$ **For the given system at time t, with position fixed by the values of $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N$, the velocities cannot be arbitrary. They must satisfy d + g equations.**

Possible velocities: the set of all velocities which satisfy the (d + g) linear equations of constraints.

3N > (d + g) – infinity of possible velocities.

One of these is realized in an actual motion of the system. Let

$$d\underline{r}_i \equiv \dot{\underline{r}}_i dt\ ,\ \ i = 1, 2, ..., N$$

These are the possible (infinitesimal) displacements. They satisfy

$$\sum_{j=1}^{N} \nabla \phi_{ij} \bullet d\underline{r}_j + \frac{\partial \phi_i}{\partial t} dt = 0\ ,\ \ i = 1, 2, ..., d \qquad (3)$$

and $\sum_{i=1}^{N} \underline{l}_{ji} \bullet d\underline{r}_i + D_j dt = 0\,, \quad j = 1, 2, \ldots., g$ **(4)**

Again, there are d + g equations in 3N possible (scalar) displacements $d\underline{r}_i\,,\ i = 1, 2, \cdots, N.$

- **Consider two sets of possible displacements at the same instant at a given position of the system:**

$$d\underline{r}_i' = \underline{v}_i' dt \quad \text{and} \quad d\underline{r}_i'' = \underline{v}_i'' dt\,,\ i = 1, 2, \ldots, N$$

Both these displacements satisfy the above equations.

Taking their differences $\Rightarrow$

$$\sum_{i=1}^{N} \nabla \phi_{ij} \bullet (d\underline{r}_j' - d\underline{r}_j'') = 0\,,\;\; i = 1, 2, \ldots, d$$

and

$$\sum_{i=1}^{N} \underline{l}_{ji} \bullet (d\underline{r}_j' - d\underline{r}_j'') = 0\,,\;\; i = 1, 2, \ldots, g$$

These are homogeneous relations not involving (*dt*).

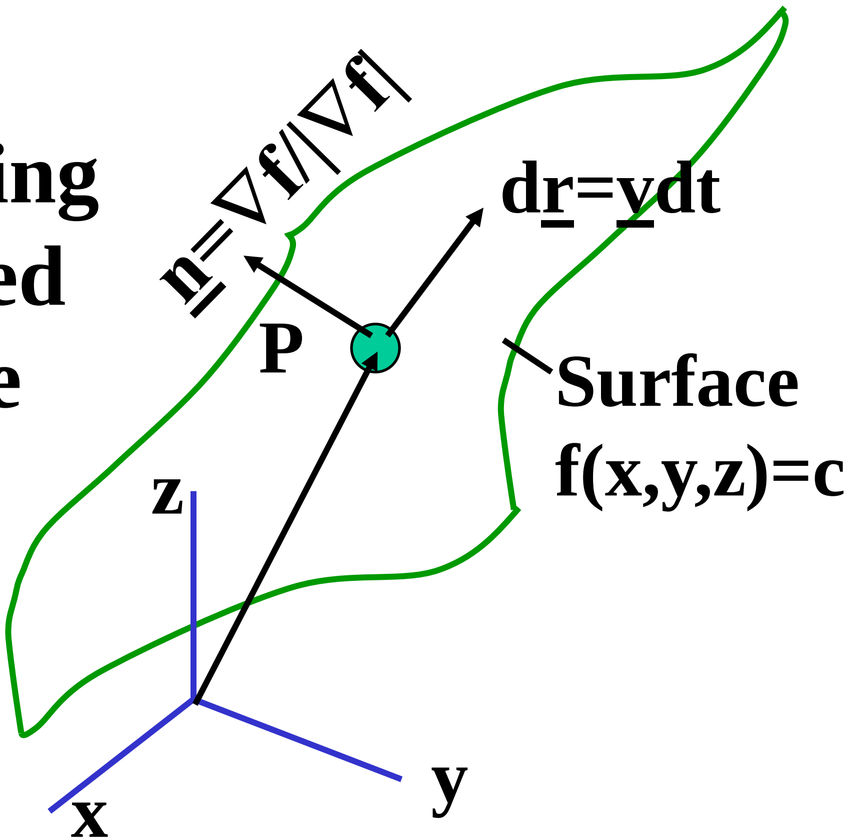
<u>Def</u>: $\delta \underline{r}_i \equiv d\underline{r}' - d\underline{r}''$ **- virtual displacement**

Virtual displacement ≡ a possible displacement with frozen time. (*dt* set to 0).

Note: **If the constraints are independent of time (<u>schleronomic</u>), a possible displacement = virtual displacement.**

<u>Ex</u> 8: **A particle is moving on a fixed surface defined by f (x, y, z) – c = 0. The velocity $\underline{v}$ is always tangent to the surface**

$$\Rightarrow d\underline{r} \bullet \underline{n} = \delta \underline{r} \bullet \underline{n} = 0$$



where

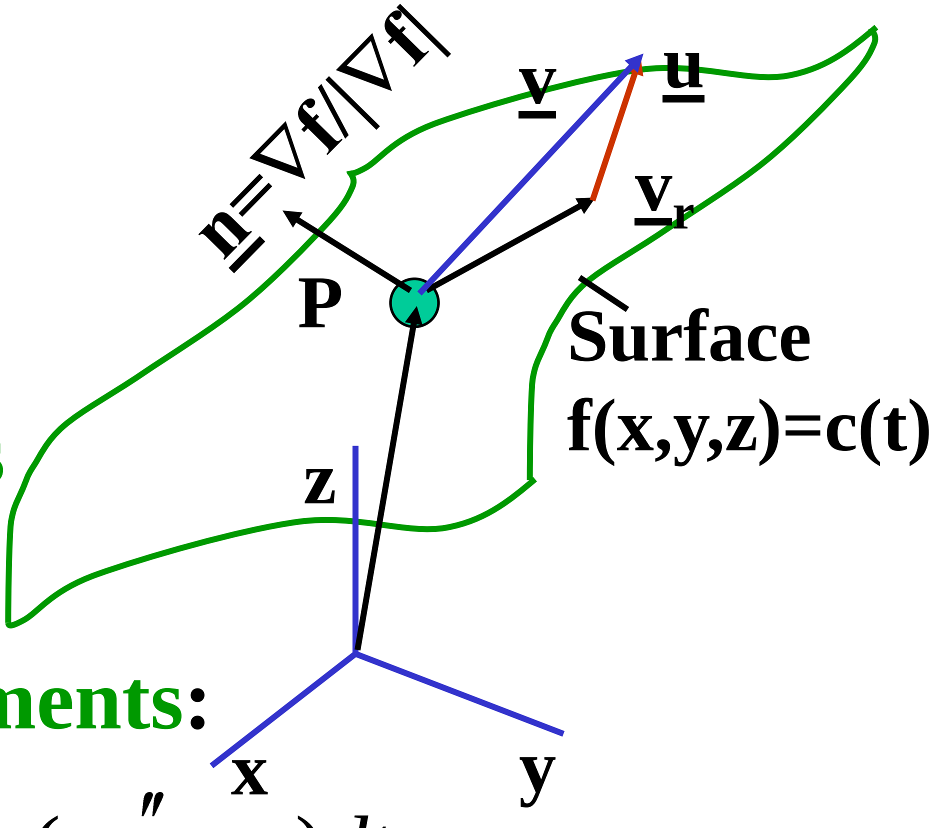
$$\underline{n} = \nabla f / |\nabla f|, \nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$$

Ex 9: A particle is moving on a surface which itself moves to the right with velocity $\underline{u}$.

Possible velocities

$$\underline{v} = \underline{v}_R + \underline{u}$$

($\underline{v}_R$ − **relative velocity)**



Possible displacements

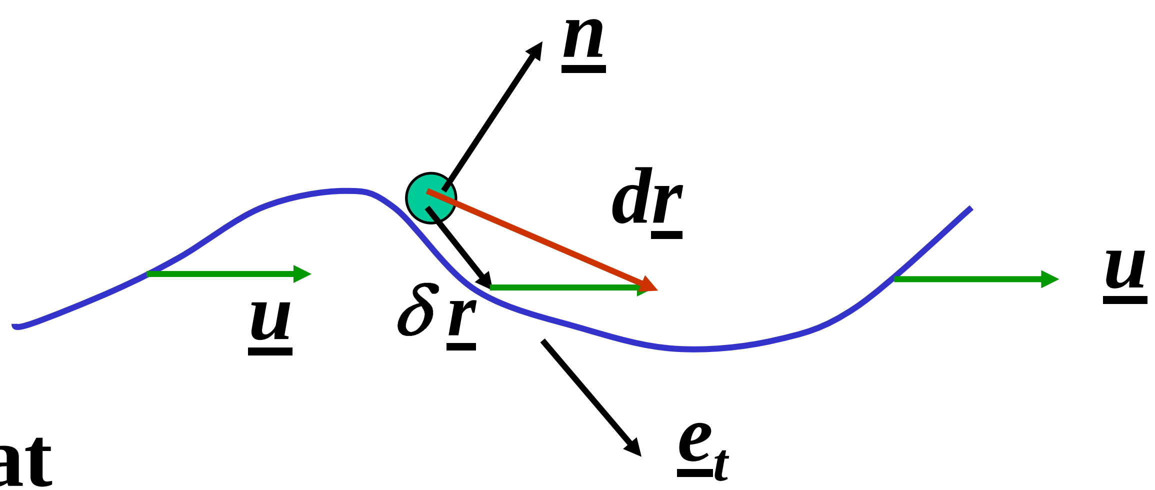
$$d\underline{r} = \underline{v}\,dt = (\underline{v}_R + \underline{u})dt$$

Two possible displacements:

$$\Rightarrow\ d\underline{r}' = (\underline{v}_R{}' + \underline{u})dt,\ d\underline{r}'' = (\underline{v}_R{}'' + \underline{u})dt$$

- **Thus, a virtual displacement is**

$$\delta \underline{r} = d\underline{r}' - d\underline{r}'' = (\underline{v}_R{}' - \underline{v}_R{}'')dt = \delta \underline{r}_R{}' - \delta \underline{r}_R{}''$$



Note that

$d\underline{r}$ – **is along absolute velocity direction,**

whereas $\delta\underline{r}$ – **is along relative velocity or tangent to the surface (frozen constraint) (set dt = 0).**

Degrees-of-freedom:

N - number of particles

(d + g) - geometric + kinematic constraints

→ there are n = 3N - (d + g) independent virtual displacements

Problem of Dynamics:

Given a system with - external forces

$$\underline{F}_i \equiv \underline{F}_i(\underline{r}, \dot{\underline{r}}, t), i = 1, 2, ..., N;$$

Initial positions $\underline{r}_{io}$ **, and initial velocities** $\underline{v}_{io}$ **compatible with constraints; we need to**

determine the motion of the system of particles, i.e., the positions $(\underline{r}_i(t))$, the velocities $\dot{\underline{r}}_i$, and the constraint or reaction forces $\underline{R}_i$, $i = 1, 2, \cdots, N$.

- $m_i \ddot{\underline{r}}_i = \underline{F}_i + \underline{R}_i, \;\; i = 1, 2, \ldots, N$

  (3N equations)

- $\sum_{j=1}^{N} \nabla \phi_{ij} \bullet \dot{\underline{r}}_j + \partial \phi_i / \partial t = 0, \;\; i = 1, 2, \ldots, d$

  (d equations)

- $\sum_{i=1}^{N} \underline{l}_{ji} \bullet d\underline{r}_i + D_j dt = 0, \;\; j = 1, 2, \ldots, g$

  (g equations)

In these equations, the unknowns are: $\underline{r}_i$, $\underline{R}_i$ **- 6N unknowns**

Thus, additional relations required:

$$6N - (3N + d + g) = 3N - (d + g) \equiv n$$

(equal to the number of degrees-of-freedom)

Need to define concept of workless constraints.

6.4 Virtual Work

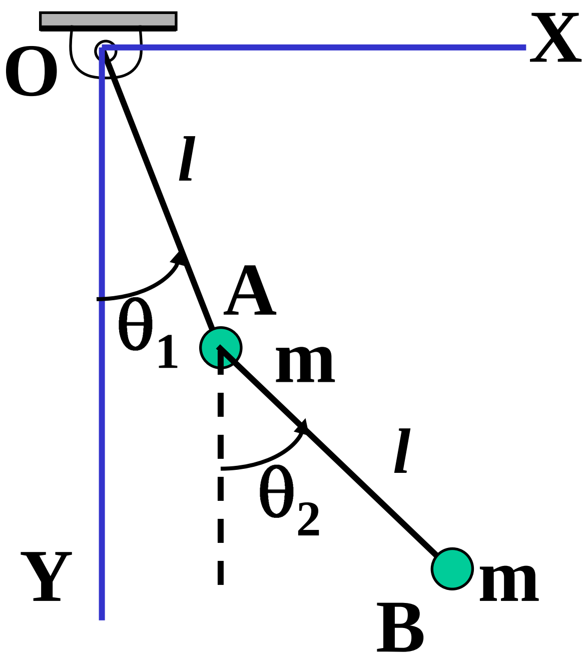
Definition: A workless constraint is any constraint such that the virtual work (work done in a virtual displacement) of the constraint forces acting on the system is zero for any reversible virtual displacement.

Ex 10: Consider a double pendulum.

The positions are:

$$\underline{r}_1 = x_1\underline{i} + y_1\underline{j}$$

$$\underline{r}_2 = x_2\underline{i} + y_2\underline{j}$$



Constraints are:

$$(x_1^2 + y_1^2) - \ell^2 = 0 \qquad (1)$$

or $x_1\dot{x}_1 + y_1\dot{y}_1 = 0$

(differential form)

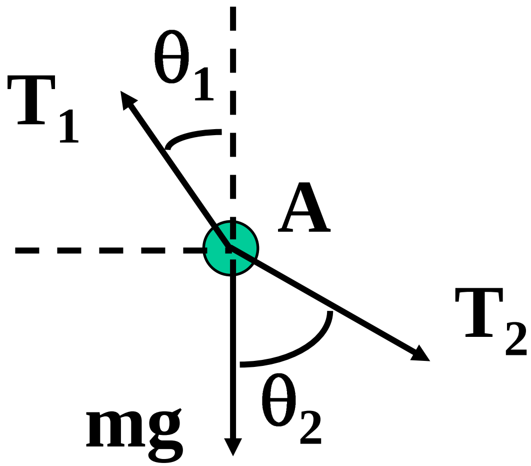
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - \ell^2 = 0 \qquad (2)$$

or $(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + (y_2 - y_1)(\dot{y}_2 - \dot{y}_1) = 0$

(differential form)

Consider FBD's

Then,



$$\theta_1 = \tan^{-1}(x_1 / y_1)$$

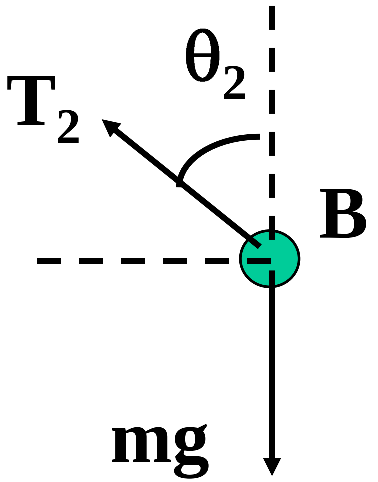
$$\theta_2 = \tan^{-1}\frac{(x_2 - x)}{y_2 - y_1}$$

The equations of motion for A are:

$\underline{\underline{x}}:\quad m\ddot{x}_1 = T_2 \sin\theta_2 - T_1 \sin\theta_1 \qquad (3)$

$\underline{\underline{y}}:\quad m\ddot{y}_1 = T_2 \cos\theta_2 - T_1 \cos\theta_1 + mg \qquad (4)$

The equations of motion for B are:



$$\underline{\underline{x}}: m\ddot{x}_2 = -T_2 \sin\theta_2 \qquad (5)$$

$$\underline{\underline{y}}: \quad m\ddot{y}_2 = mg - T_2 \cos\theta_2 \qquad (6)$$

2 N – differential equations of motion

2 – equations of constraint

variables (unknowns): $x_1(t),\ y_1(t)$

$x_2(t), y_2(t) \qquad T_1(t), T_2(t)$

Ex 11: Consider the motion of an ideal pendulum

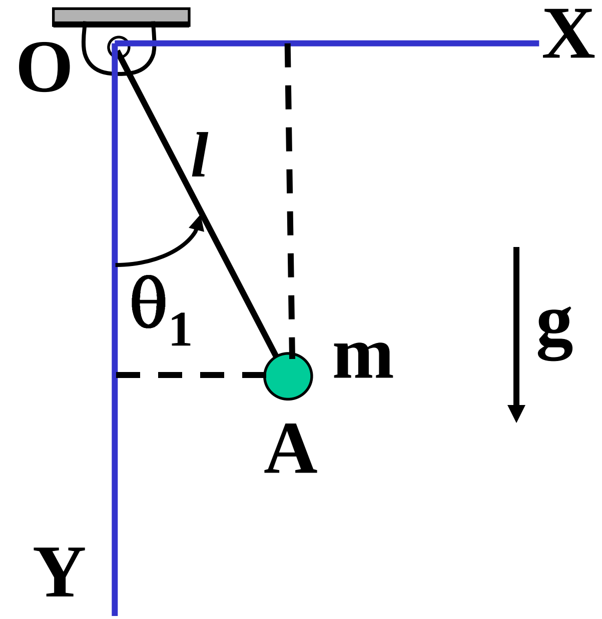
The position is

$$\underline{r} = x\underline{i} + y\,\underline{j}$$

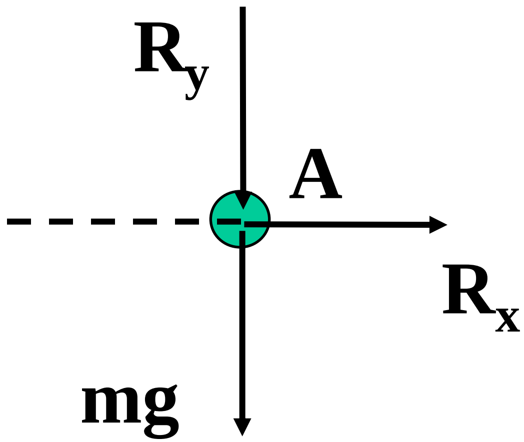
Newton's Law:

$$\ddot{\underline{r}} + \ddot{x}\underline{i} + \ddot{y}\,\underline{j}$$

$$\sum \underline{F} = m\ddot{\underline{r}}$$



FBD:



The reaction force is:

$\underline{R} = R_x\,\underline{i} + R_y\,\underline{i}$

Newton's 2nd law gives

$$\underline{\underline{x}}: \quad R_x = m\ddot{x} \qquad (1)$$

$$\underline{\underline{y}}: \quad R_y + mg = m\ddot{y} \qquad (2)$$

Constraint on motion is:

$$x^2 + y^2 - l^2 = 0 \qquad (3)$$

or $$x\dot{x} + y\dot{y} = 0 \qquad (\underline{r} \bullet \underline{\dot{r}} = 0)$$

(differentiated form)

or **x dx + y dy = 0**

(differential form)

Counting: **3 equations**

4 variables – x, y, R_x, R_y

Need one more relation:

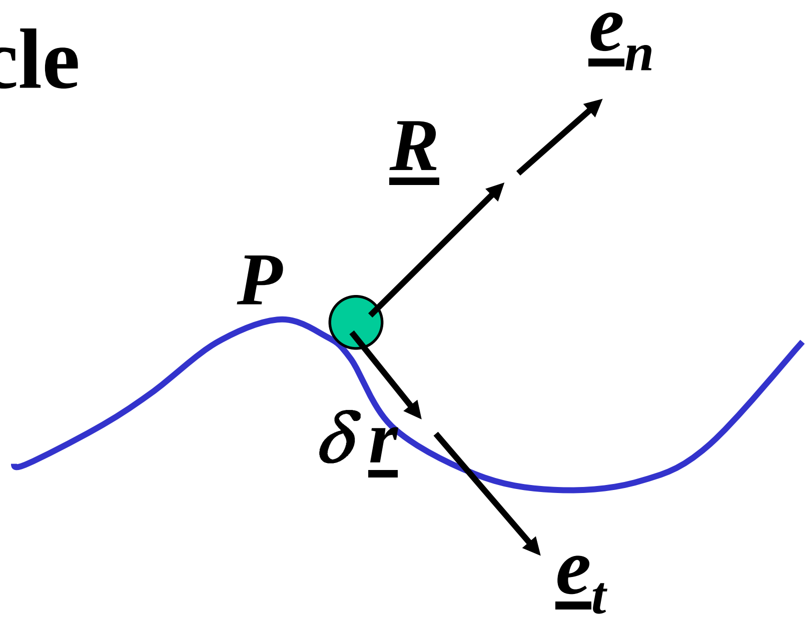
- something about the nature of the constraint force $\underline{R} = R_x \underline{i} + R_y \underline{j}$

Hindsight: We know $\underline{R}$ **along the rod – normal to the direction of velocity – does no work in motion of the particle (motion that is consistent with the constraint).**

Work done in a virtual displacement of the system

$$\delta W = \sum_{i=1}^{N} \underline{R}_i \bullet \delta \underline{r}_i$$

Ex 12: Consider a particle moving on a smooth surface. Then the work done by the constraint force $\underline{R}$ in a virtual displacement $\delta \underline{r}$ (consistent with constraint) is



$$\delta W = \underline{R} \bullet \delta \underline{r} = R\underline{n} \bullet \delta \underline{r} = 0$$

Ex 13: **Consider the same situation, with the particle now moving on a moving surface**:

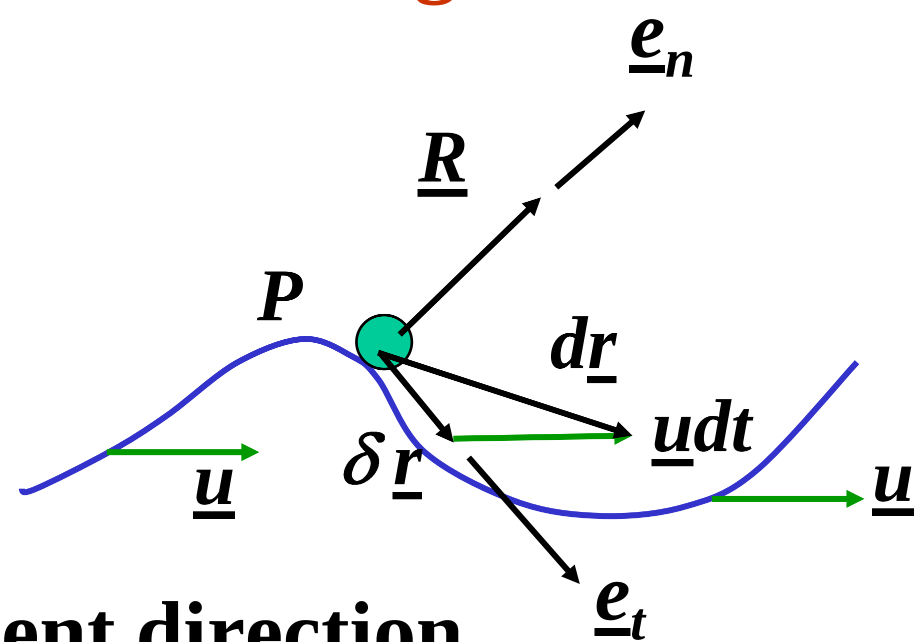
Here again

$$\delta W = R\underline{n}\bullet\delta\underline{r} = 0$$

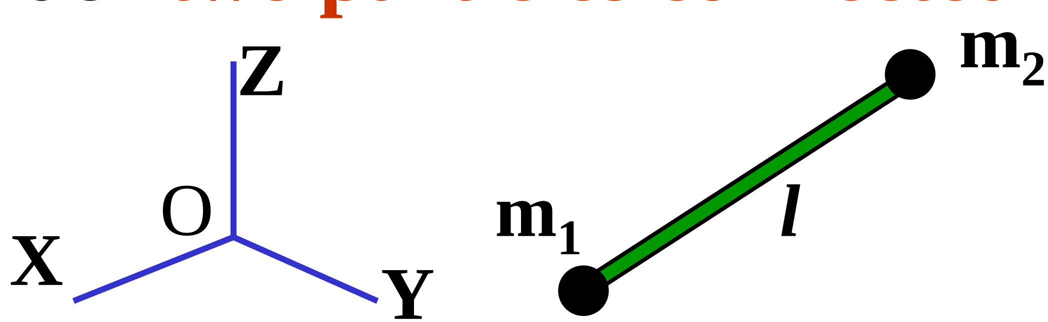
Note however that

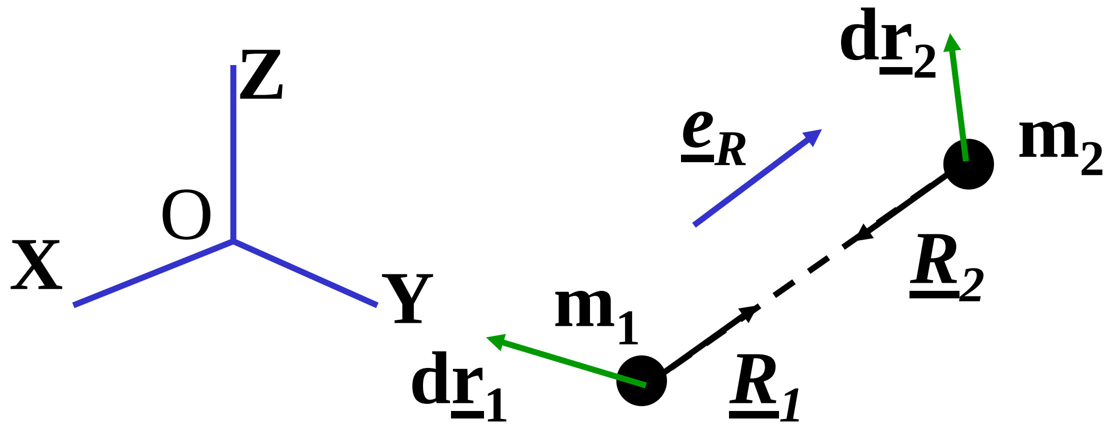
$\underline{R}\bullet d\underline{r} \neq 0$ **since**

$d\underline{r}$ **is not in the tangent direction.**



Ex 14: **Consider two particles connected by a rigid rod**:





$$\underline{R}_1 = -\underline{R}_2 = +R_2\,\underline{e}_R$$ **where** $$|\underline{R}_2| = R_2 = |\underline{R}_1|$$

$\underline{e}_R$ – **unit vector from** m_1 to m_2

The length constraint is: $(\underline{r}_1 - \underline{r}_2)\bullet(\underline{r}_1 - \underline{r}_2) - l^2 = 0$

Differentiating, the constraint on possible displacements is: $(\underline{r}_1 - \underline{r}_2)\bullet(d\underline{r}_1 - d\underline{r}_2) = 0$

Thus, $(\underline{r}_1 - \underline{r}_2)\bullet(\delta\underline{r}_1 - \delta\underline{r}_2) = 0 = \underline{e}_R\bullet(\delta\underline{r}_1 - \delta\underline{r}_2)$

or $$\underline{e}_R\bullet\delta\underline{r}_1 = \underline{e}_R\bullet\delta\underline{r}_2$$

$\delta W =$ **virtual work done on the system (the two particles)**

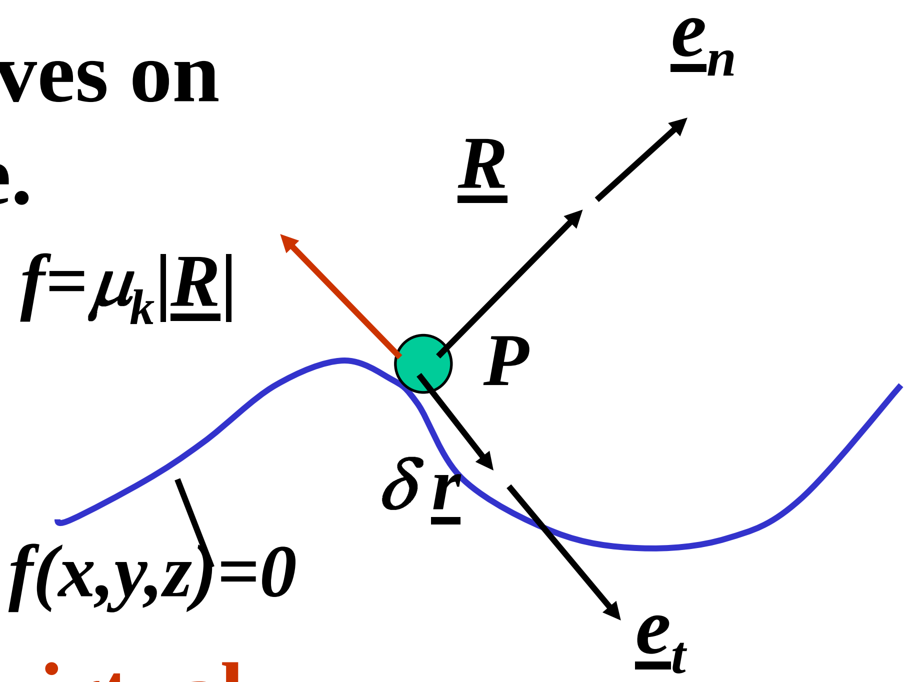
$$= \underline{R}_1 \bullet \delta \underline{r}_1 + \underline{R}_2 \bullet \delta \underline{r}_2 = R_2 \underline{e}_R \bullet \delta \underline{r}_1 - R_2 \underline{e}_R \bullet \delta \underline{r}_2$$

$$= R_2 (\underline{e}_R \bullet \delta \underline{r}_1 - \underline{e}_R \bullet \delta \underline{r}_2) = 0$$

Other examples of workless constraints: hinged constraints; sliding on smooth surfaces; rolling without slipping, etc.

Remark: Reaction forces corresponding to workless constraints may do work on individual components of the system.

Ex 15: A particle moves on a fixed rough surface.



Clearly, the work done by the normal force $\underline{R}$ in a virtual displacement is $\underline{R}\bullet\delta\underline{r}=0$

Note that the friction force does do work – it <u>can be accounted</u> for by treating $f=\mu_k\left|\underline{R}\right|$ as an external force.

The Principle of Virtual Work:

Consider a system of N particles, with positions

$$\underline{r}_i, \;\; i = 1, 2, \ldots, N$$

Forces acting on the ith particle of mass

$$m_i : \underbrace{\underline{F}_i}_{\uparrow} + \underline{R}_i \} \leftarrow$$ **workless constraint forces**

external as well
as constraint forces
not accounted for in workless constraint forces.

Static equilibrium for the ith particle $\Rightarrow$

$$\underline{F}_i + \underline{R}_i = 0, \; i = 1, 2, \ldots, N$$

Suppose that the system also satisfies some constraints:

$$\sum_{j=1}^{N} \nabla \phi_{ij} \bullet \delta \underline{r}_j = 0, \; i = 1, 2, \ldots, d$$ **geometric**

$$\sum_{i=1}^{N} \underline{l}_{ji} \bullet \delta \underline{r}_i = 0, \; j = 1, 2, \ldots, g$$ **kinematic**

(these are requirements written in terms of the virtual displacements)

Now, virtual work done by all the forces acting on the system as a result of an arbitrary virtual displacement $\delta \underline{r}_i$ at a given system configuration is

$$\delta W = \sum_{i=1}^{N} (\underline{F}_i + \underline{R}_i) \bullet \delta \underline{r}_i$$

(Note: $\delta \underline{r}_i$ are required to satisfy the constraint relations, i.e., are the possible infinite displacement with frozen time).

- **Assume workless constraints:**

$$\sum_{i=1}^{N} \underline{R}_i \bullet \delta \underline{r}_i = 0$$

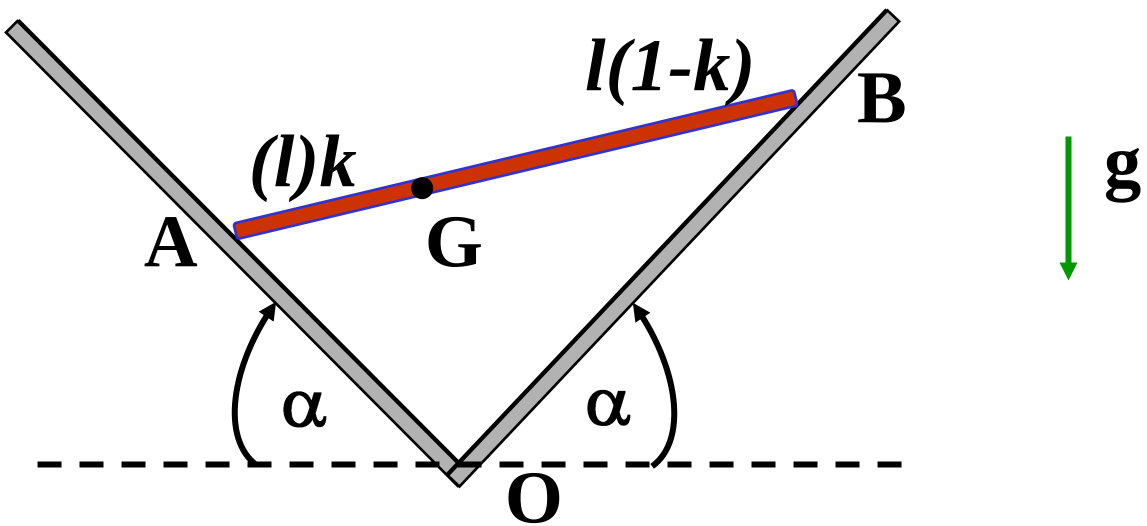
$$\Rightarrow \sum_{i=1}^{N} \underline{F}_i \bullet \delta \underline{r}_i = \delta W = 0 \qquad \text{(**scalar eqn**.)}$$

If a system of particles with workless constraints is in static equilibrium, the virtual work of the applied forces is zero for any virtual displacement consistent with constraints.

- **Also, if the work done at a given configuration is zero in any arbitrary virtual displacement from that configuration, the system must be in static equilibrium.**
- **Principle of virtual work.**

Ex 16: **A inhomogeneous rod AB is resting on two smooth planes. The rod is nonuniform with its center of mass located at G: AG:GB = k:(1–k).**

Find: The equilibrium position of the rod.



The constraints are: the ends must remain in contact with respective surfaces.

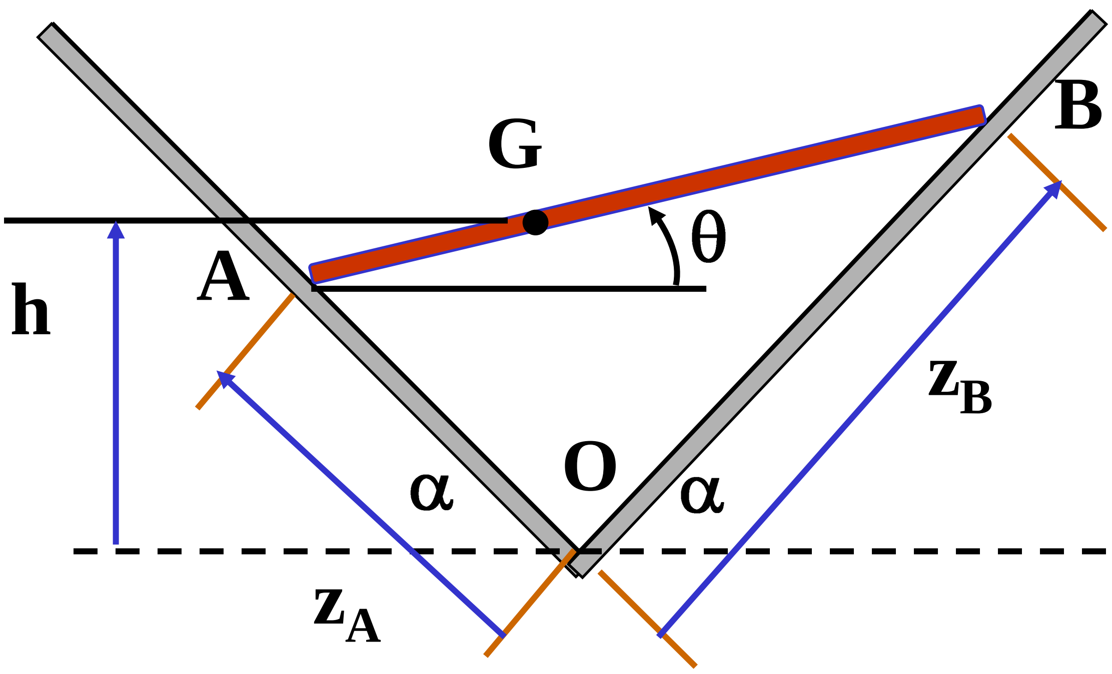
To properly set up the problem, we need to first define a coordinate system so that the appropriate position vectors can be defined. Then, we can define the constraints and the virtual displacements.

Let z_A and z_B**- positions of A and B along the inclined surfaces. Also, θ - the angle of inclination of the rod. The constraints are:**

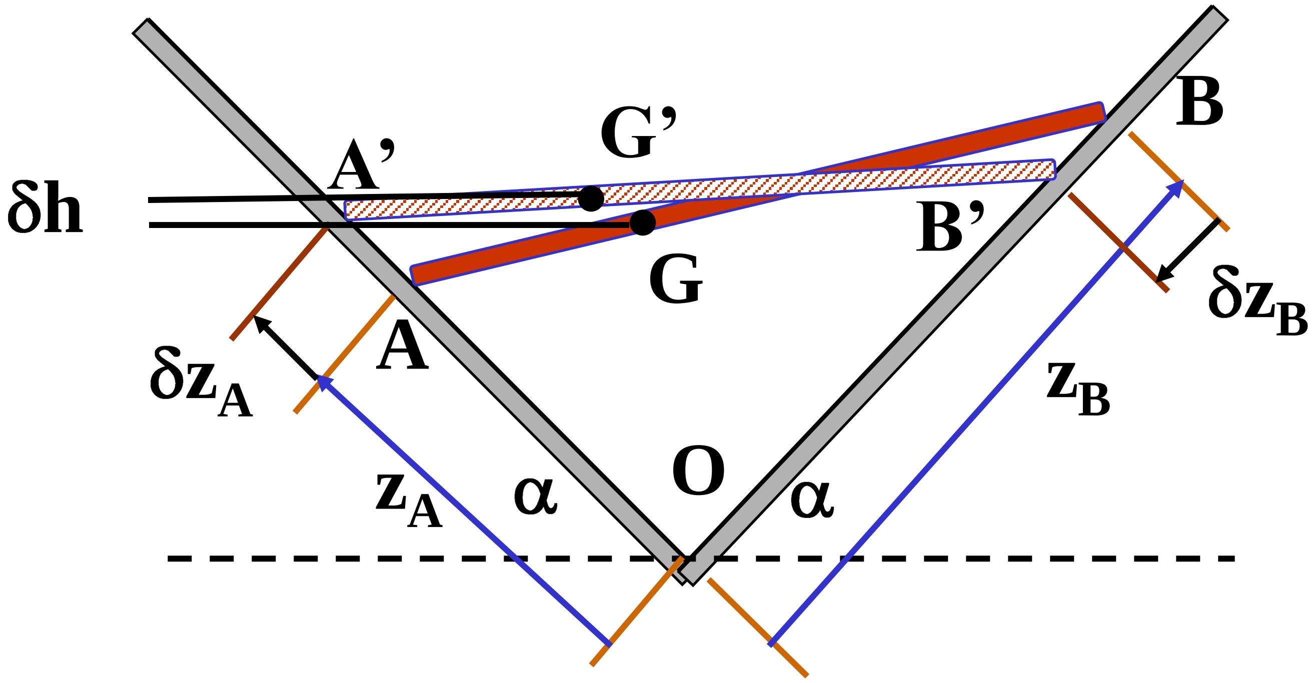
$$\ell \cos\theta = (Z_A + Z_B)\cos\alpha$$

$$\ell \sin\theta = (Z_B - Z_A)\sin\alpha$$

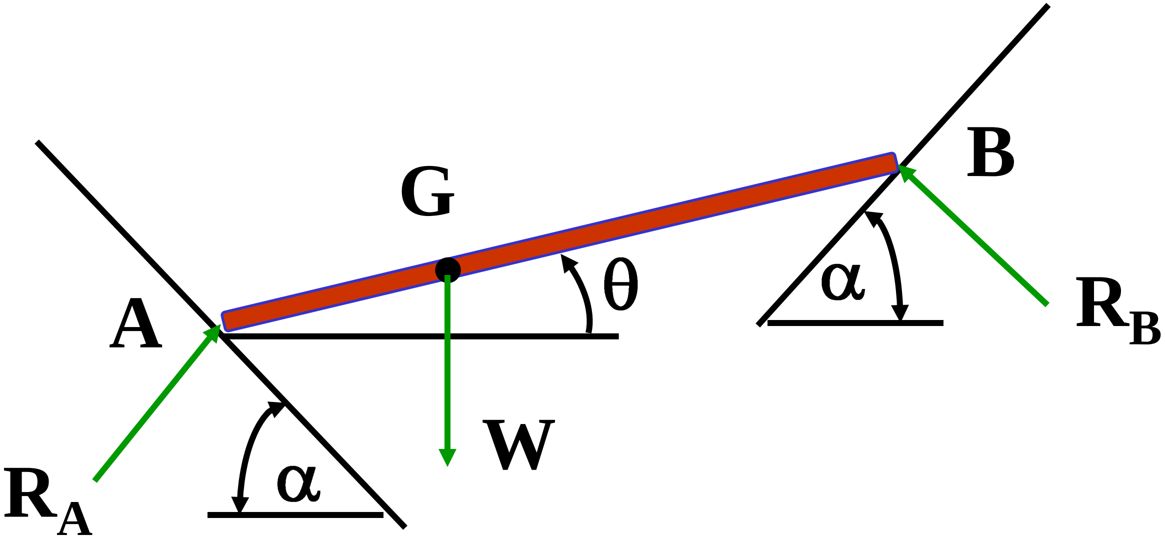
The variables are described here on the picture more clearly:



A possible set of virtual displacements consistent with constraints are shown here:



FBD:



Principle of virtual work:

$$\underbrace{\underline{R}_A \bullet \delta \underline{r}_A + \underline{R}_B \bullet \delta \underline{r}_B}_{} + \underline{W} \bullet \delta \underline{r}_G = 0$$

= 0 (workless constraints)

$$\Rightarrow \underline{W} \bullet \delta \underline{r} = 0 \quad \text{or} \quad W \delta h = 0 \Rightarrow \delta h = 0$$

$$-W \underline{j} \bullet (\delta x \underline{i} + \delta h \underline{j}) = 0$$

Now,

$$h = z_A \sin\alpha + k\ell \sin\theta$$

$$\Rightarrow \; \delta h = \delta z_A \sin\alpha + k\ell \cos\theta \, \delta\theta$$

- **constraints:**

$$z_A + z_B = l\cos\theta/\cos\alpha, \; z_B - z_A = l\sin\theta/\sin\alpha$$

$$\Rightarrow \; z_A = \left\{ \frac{\cos\theta}{\cos\alpha} - \frac{\sin\theta}{\sin\alpha} \right\} \frac{1}{2}$$

$$= \frac{1}{2} \frac{\cos\theta\sin\alpha - \sin\theta\cos\alpha}{\cos\alpha\sin\alpha}$$

or $\; z_A = l\sin(\alpha - \theta)/\sin 2\alpha$

differentiating, we get

$$\delta z_A = -\{l\cos(\alpha-\theta)/\sin 2\alpha\}\delta\theta.$$

Thus,

$$\delta h = -\{l\cos(\alpha-\theta)\sin\alpha/\sin 2\alpha\}\delta\theta$$
$$+kl\cos\theta\ \delta\theta$$

$$= \{-\frac{l\cos(\alpha-\theta)}{\sin 2\alpha}\sin\alpha + kl\cos\theta\}\delta\theta = 0$$

$$\delta\theta - \text{arbitrary virtual displacement} \ \Rightarrow$$

$$\{-\sin\alpha\cos(\alpha-\theta)/\sin 2\alpha + k\cos\theta\} = 0$$

$$\text{or} \ \boxed{\tan\theta = (2k-1)/\tan\alpha}$$

D'Alembert's Principle:

Consider a system with

- **N particles, the masses are given by** m_i
- **The external force on ith particle** $\underline{F}_i(t,\underline{r},\underline{\dot{r}})$
- **geometric constraints:**

$$f_i(\underline{r}_1,\underline{r}_2,....,\underline{r}_N,t)=0,\ \ i=1,2,....,d,$$

- **kinematic constraints:**

$$\sum_{j=1}^{N}\underline{l}_{ij}\bullet\underline{\dot{r}}_j+D_i(\underline{r},t)=0, i=1,2,....,g$$

constraints → reaction forces

$$\underline{R}_i,(\underline{r},\underline{\dot{r}},t), i=1,2,...,N$$

- **The equations of motion are:**

$$m_i \ddot{\underline{r}}_i = \underline{F}_i + \underline{R}_i, \ i = 1, 2, \ldots., N$$

These are subject to the constraints: (in differential form)

$$\sum_{j=1}^{N} (\partial f_i / \partial \underline{r}_j) \bullet d\underline{r}_j + (\partial f_i / \partial t) dt = 0, i = 1, 2, \ldots., d$$

$$\sum_{j=1}^{N} \underline{l}_{ij} \bullet d\underline{r}_j + D_i(\underline{r}, t) dt = 0, i = 1, 2, \ldots., g$$

- **Then, the relations satisfied by virtual displacements are**

$$\sum_{j=1}^{N} (\partial f_i / \partial \underline{r}_j) \bullet \delta \underline{r}_j = 0, i = 1, 2, ..., d$$

and

$$\sum_{j=1}^{N} \underline{l}_{ij} \bullet \delta \underline{r}_j = 0, i = 1, 2, ..., g$$

- **The condition on constraint forces for the constraints to be workless is:**

$$\sum_{i=1}^{N} \underline{R}_i \bullet \delta \underline{r}_i = 0$$

Newton's law $\Rightarrow$

$$\sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \bullet \delta \underline{r}_i = \sum_{i=1}^{N} (\underline{F}_i + \underline{R}_i) \bullet \delta \underline{r}_i$$

Workless constraints $\Rightarrow$

$$\boxed{\sum_{i=1}^{N} (m_i \ddot{\underline{r}}_i - \underline{F}_i) \bullet \delta \underline{r}_i = 0}$$

D'Alembert's Principle

(a single scalar equation)

(Note: $\delta \underline{r}_i$ **are not independent. They satisfy the differential constraints).**

Ex 17: Spherical pendulum with variable length

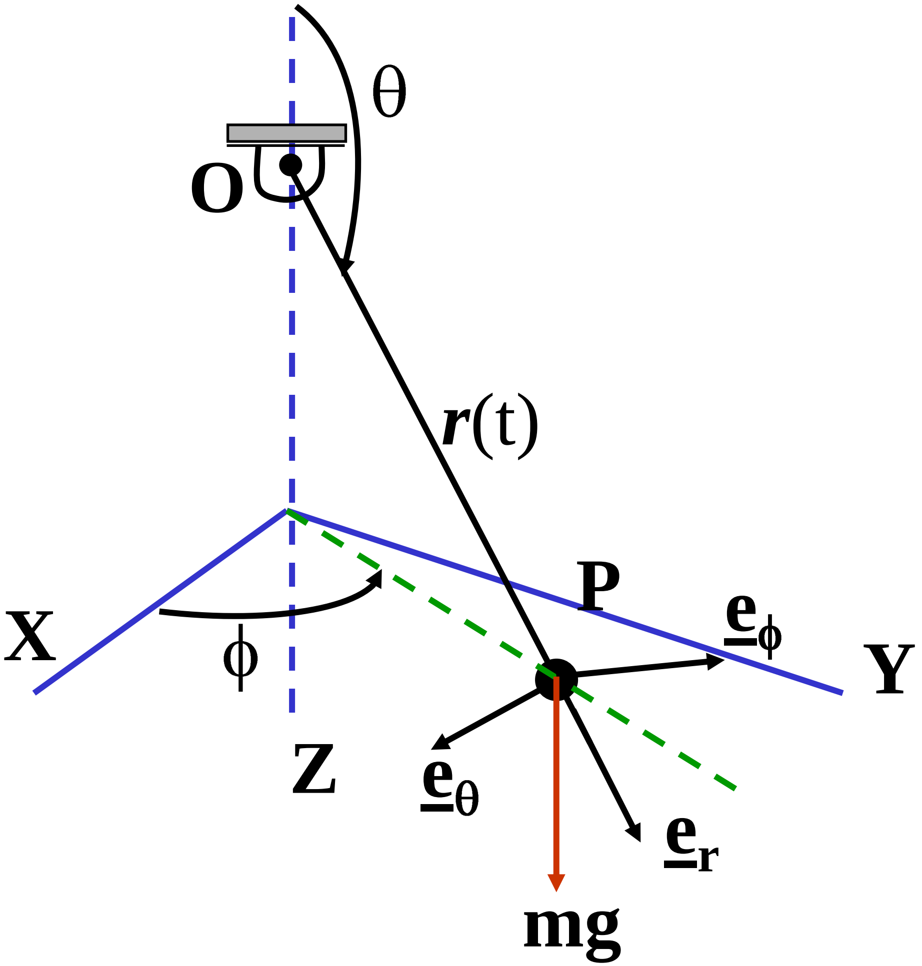
particle of mass m

$r = (a + b\cos\omega t),$

$a > b > 0.$

$\underline{e}_r, \underline{e}_\theta$ **in OPZ plane;**

$\underline{e}_\phi \perp^r$ **to OPZ plane**



position of the ball: $\underline{r}_P = r\underline{e}_r$

velocity: $\dot{\underline{r}}_P = \dot{r}\underline{e}_r + r\dot{\underline{e}}_r$

or $\dot{\underline{r}}_P = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta + r\dot{\phi}\sin\theta\underline{e}_\phi$

acceleration: $\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\underline{e}_r +$

$(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\underline{e}_\theta + (r\ddot{\phi}\sin\theta +$

$2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\underline{e}_\phi$

virtual displacement:

possible velocity $\dot{\underline{r}}_P = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta + r\dot{\phi}\sin\theta\underline{e}_\phi$

possible displ. $d\underline{r} = dr\underline{e}_r + rd\theta\underline{e}_\theta + rd\phi\sin\theta\underline{e}_\phi$

constraint: $r = a + b\cos\omega t$

$or \quad dr = -(b\omega \sin \omega t)dt$

$\text{since constraint frozen} \rightarrow dt = 0 \rightarrow dr = 0$

virtual displacement: $\delta \underline{r} = r\delta\theta \underline{e}_\theta + r\delta\phi \sin\theta \underline{e}_\phi$

External force acting:

$$\underline{F} = -mg\cos\theta \underline{e}_r + mg\sin\theta \underline{e}_\theta = -mg\underline{K}$$

D'Alembert's Principle

$$(m\ddot{\underline{r}} - \underline{F}) \bullet \delta \underline{r} = 0$$

Note: on the FBD of the particle, tension force also acts along the rod - a workless constraint force

$$\Rightarrow mr[g\sin\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta)]\delta\theta$$
$$-mr\sin\theta[r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta +$$
$$2r\dot{\theta}\dot{\phi}\cos\theta]\delta\phi = 0$$

$\delta\theta, \delta\phi -$ **independent virtual displacements**

$$\Rightarrow (\quad)\delta\theta + (\quad)\delta\phi = 0$$

$$\Rightarrow \boxed{(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta) - g\sin\theta = 0};$$

$$\boxed{r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta = 0}$$

(**equations of motion**)

Here $r = a + b\cos\omega t \neq 0$

6.5 Generalized Coordinates and Forces

Ex 18: consider the double pendulum:

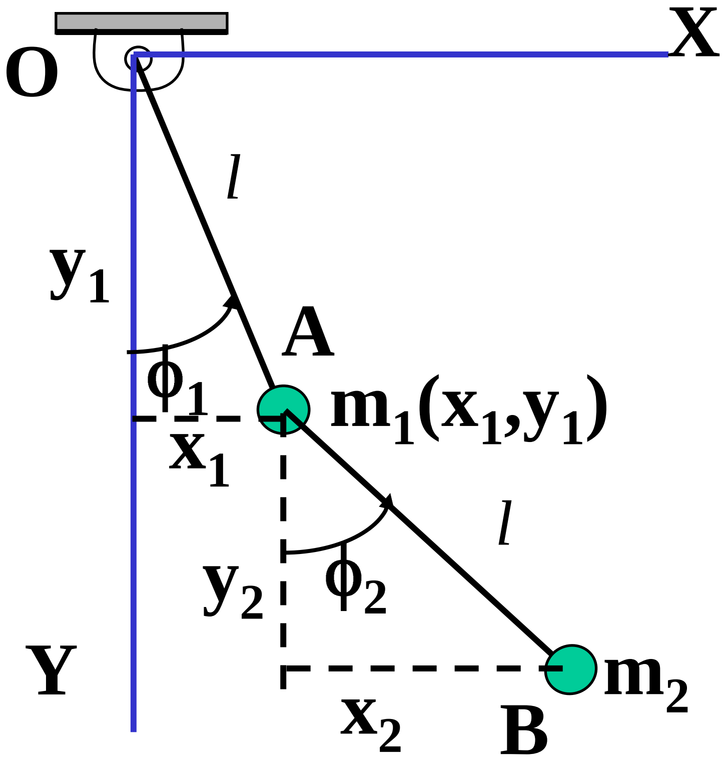
the position vectors are

$$\underline{r}_1 = x_1 \underline{i} + y_1 \underline{j}$$

$$\underline{r}_2 = (x_1 + x_2)\underline{i} + (y_1 + y_2)\underline{j}$$

the constraints are

- $x_1^2 + y_1^2 - l_1^2 = 0$
- $x_2^2 + y_2^2 - l_2^2 = 0$



Thus, there are:

4 (or 6 counting z's) variables or generalized coordinates

- **2 (or 4 if z included) constraints** $(z_1 = 0,\ z_2 = 0)$
- **n = degrees of freedom = 2.**

$\Rightarrow$ **Need only 2 independent variables (for geometric constraints case) to specify the configuration at any given time**

e.g.: let y_1, y_2 **be the two chosen independent coordinates. Then, we can write**

$$x_1 = \pm\sqrt{(l_1^2 - y_1^2)},\ \ x_2 = \pm\sqrt{(l_2^2 - y_2^2)}$$

$$\underline{r}_1 = \pm\sqrt{l_1^2 - y_1^2)}\,\underline{i} + y_1\,\underline{j} \equiv \underline{r}_1(y_1)$$

$$\underline{r}_2 = \{\pm\sqrt{l_1^2 - y_1^2} \pm \sqrt{(l_2^2 - y_2^2)}\}\underline{i} + (y_1 + y_2)\,\underline{j}$$

$$\equiv \underline{r}_2(y_1, y_2)$$

- **<u>Note</u>: geometric constraint now automatically satisfied.**
- **Another possible choice of generalized coordinates are:**

$\phi_1, \phi_2 -$ **angles with y axis**

Then

$$x_1 = l_1 \sin\phi_1, \quad y_1 = l_1 \cos\phi_1$$

$$x_2 = l_2 \sin\phi_2, \quad y_2 = l_2 \cos\phi_2$$

$$\Rightarrow$$

$$\underline{r}_1 = l_1(\sin\phi_1 \underline{i} + \cos\phi_1 \underline{j}) \equiv \underline{r}_1(\phi_1, \phi_2)$$

$$\underline{r}_2 = l_1(\sin\phi_1 + l_2 \sin\phi_2)\underline{i}$$

$$+(l_1 \cos\phi_1 + l_2 \cos\phi_2)\underline{j} \equiv \underline{r}_2(\phi_1, \phi_2)$$

Again: geometric constraints automatically satisfied.