CHAPTER 2
KINEMATICS OF A PARTICLE

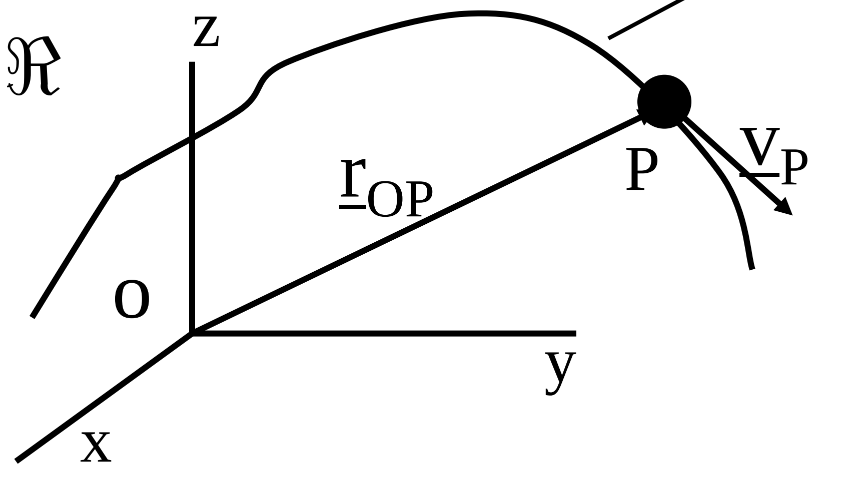
Kinematics: It is the study of the geometry of motion of particles, rigid bodies, etc., disregarding the forces associated with these motions.

Kinematics of a particle → motion of a point in space

- **Interest is on defining quantities such as position, velocity, and acceleration.**
- **Need to specify a reference frame (and a coordinate system in it to actually write the vector expressions).**
- **Velocity and acceleration depend on the choice of the reference frame.**
- **Only when we go to laws of motion, the reference frame needs to be the inertial frame.**

- **From the point of view of kinematics, no reference frame is more fundamental or absolute.**

2.1 Position, velocity, acceleration

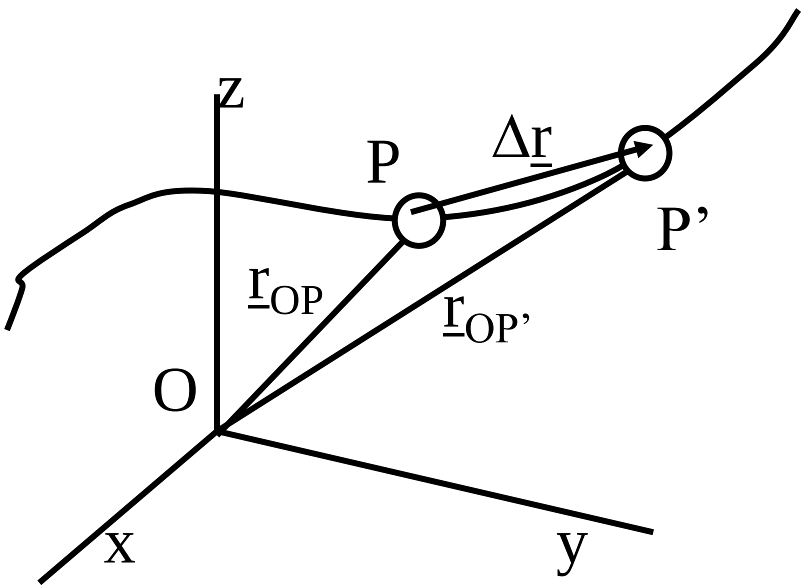


path followed by the object.

O - origin of a coordinate system in the reference frame.

$\underline{r}_{OP}$ - position vector (specifies position, given the choice of the origin O).

Clearly, $\underline{r}_{OP}$ changes with time → $\underline{r}_{OP}(t)$



velocity vector:

$$ {}^{\Re}\underline{v}_{P} = \frac{d}{dt}\underline{r}_{OP}(t) = \lim_{\Delta t \to 0} \frac{\Delta \underline{r}_{OP}}{\Delta t}. $$

acceleration vector:

$$ {}^{\Re}\underline{a}_{P} = \frac{d}{dt}\underline{v}_{P}(t) = \frac{d^2}{dt^2}\underline{r}_{OP}(t). $$

- **speed:** $$v_P \equiv \left| \underline{v}_P \right| = \sqrt{\underline{v}_P \bullet \underline{v}_P}$$
- **magnitude of acceleration:**

$$a_P \equiv \left| \underline{a}_P \right| = \sqrt{\underline{a}_P \bullet \underline{a}_P}$$

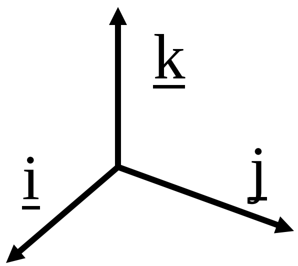
Important: the time derivatives or changes in time have been considered relative to (or with respect to) a reference frame.

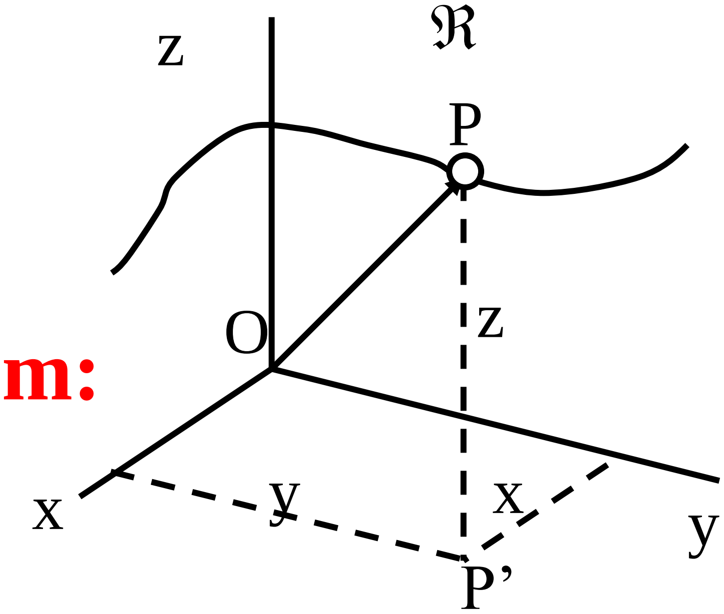
<u>Description in various coordinate systems</u>

(slightly different from the text)

- **Cartesian coordinates, cylindrical coordinates etc.**

Let $\underline{i}$, $\underline{j}$, $\underline{k}$ be the unit vectors





Cartesian coordinate system:

The reference frame is

ℜ- it is fixed.

$\underline{i} \times \underline{j} = \underline{k}$, $\underline{j} \times \underline{k} = \underline{i}$, $\underline{k} \times \underline{i} = \underline{j}$ etc.

$\rightarrow$ $\underline{i}$, $\underline{j}$, $\underline{k}$ **are an <u>orthogonal set</u>. Then,**

position of P is: $\underline{\mathbf{r}}_{OP} = x(t)\,\underline{\mathbf{i}} + y(t)\,\underline{\mathbf{j}} + z(t)\,\underline{\mathbf{k}}$

The time derivative of position is velocity:

$$\underline{v}_P = \frac{d\underline{r}_{OP}}{dt} = \frac{dx(t)}{dt}\underline{i} + \frac{dy(t)}{dt}\underline{j} + \frac{dz(t)}{dt}\underline{k}$$

$$+ x(t)\frac{d\underline{i}}{dt} + y(t)\frac{d\underline{j}}{dt} + z(t)\frac{d\underline{k}}{dt}$$

If considering rate of change in a frame in which $\underline{i}$, $\underline{j}$, $\underline{k}$ **are fixed,** $\frac{{}^{\Re}d\underline{i}}{dt} = \frac{{}^{\Re}d\underline{j}}{dt} = \frac{{}^{\Re}d\underline{k}}{dt} = 0$

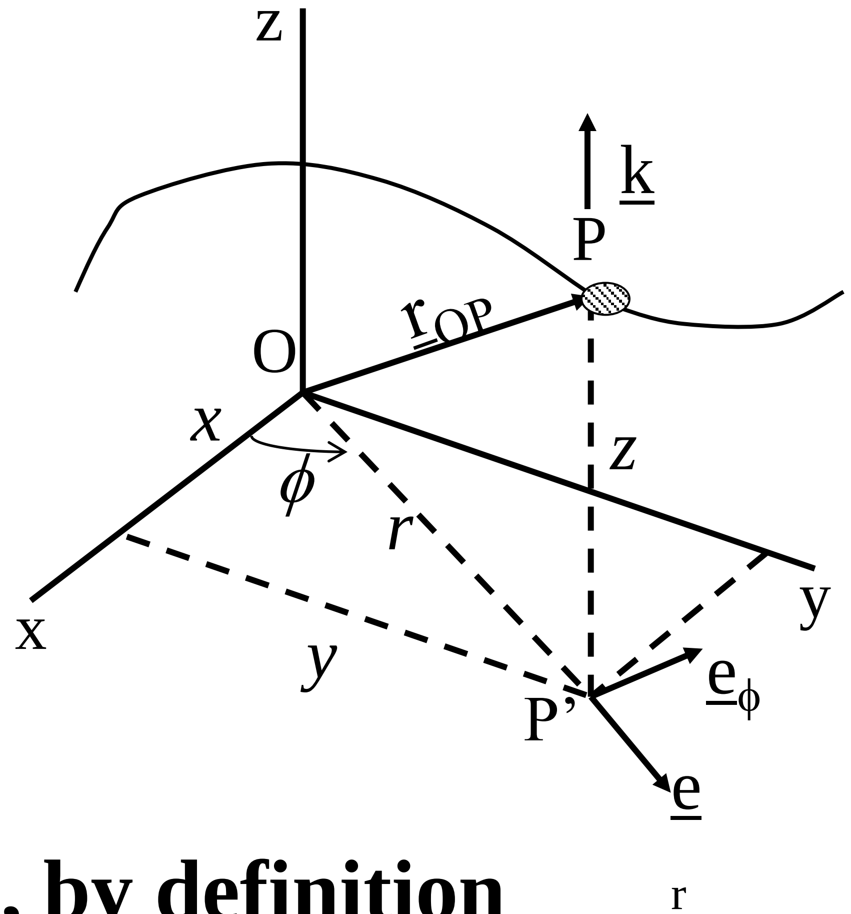
$$\rightarrow \boxed{{}^{\Re}\underline{v}_P = \frac{dx(t)}{dt}\underline{i} + \frac{dy(t)}{dt}\underline{j} + \frac{dz(t)}{dt}\underline{k}}$$ **velocity vector**

Similarily,

$$\boxed{{}^{\Re}\underline{a}_P = \ddot{x}(t)\underline{i} + \ddot{y}(t)\underline{j} + \ddot{z}(t)\underline{k}}$$ **acceleration vector**

Cylindrical Coordinates:

$\underline{e}_r$- unit vector in xy plane in radial direction.



$\underline{e}_\phi$- unit vector in xy plane $\underline{\perp}^{r}$ to $\underline{e}_r$ in the direction of increasing φ

$\underline{k}$- unit vector in z. Then, by definition

$$r = (x^2 + y^2)^{1/2} \;\; ; \quad \phi = \tan^{-1}(y/x).$$

The position is: $\underline{\mathbf{r}}_{OP} = x(t)\,\underline{\mathbf{i}} + y(t)\,\underline{\mathbf{j}} + z(t)\,\underline{\mathbf{k}}$

Also, $\underline{\mathbf{r}}_{OP} = r(\phi)\,\underline{\mathbf{e}}_r + z(t)\,\underline{\mathbf{k}}$

and $\underline{\mathbf{r}}_{OP} = r(\phi)\cos\phi\,\underline{\mathbf{i}} + r(\phi)\sin\phi\,\underline{\mathbf{j}} + z(t)\,\underline{\mathbf{k}}$

$$\rightarrow \quad \underline{e}_r = \cos\phi\,\underline{i} + \sin\phi\,\underline{j} \equiv \frac{\partial \underline{r}}{\partial r} \Bigg/ \left|\frac{\partial \underline{r}}{\partial r}\right|$$

Also,

$$\underline{e}_\phi \equiv \frac{\partial \underline{r}}{\partial \phi} \Bigg/ \left|\frac{\partial \underline{r}}{\partial \phi}\right| \quad \text{but} \quad \frac{\partial \underline{r}}{\partial \phi} = r\left(-\sin\phi\,\underline{i} + \cos\phi\,\underline{j}\right)$$

$$\text{and} \quad \left|\frac{\partial \underline{r}}{\partial \phi}\right| = r \quad \rightarrow \quad \underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}$$

- **Imp. to Note**: $\underline{e}_r$ and $\underline{e}_\phi$ **change with position (ϕ).**

- **position**: $\underline{\mathbf{r}}_{OP} = r(\phi)\,\underline{\mathbf{e}}_r + z(t)\,\underline{\mathbf{k}}$

or $\underline{\mathbf{r}}_{OP} = r(\phi)\cos\phi\,\underline{\mathbf{i}} + r(\phi)\sin\phi\,\underline{\mathbf{j}} + z(t)\,\underline{\mathbf{k}}$

- **velocity**:

$$\underline{v}_P = d\underline{r}_{OP}/dt = \dot{r}\underline{e}_r + r\,d\underline{e}_r/dt + \dot{z}\underline{k} + z\,d\underline{k}/dt$$

z- direction$(\underline{k})$ **fixed** $\rightarrow d\underline{k}/dt = 0$

Thus

$$d\underline{e}_r/dt = (d\underline{e}_r/d\phi)(d\phi/dt) = (d\underline{e}_r/d\phi)\dot{\phi} = \underline{e}_\phi\,\dot{\phi}$$

or

$$\boxed{\underline{v}_P = \dot{r}\underline{e}_r + r\dot{\phi}\underline{e}_\phi + \dot{z}\underline{k}}$$

=radial comp+transverse comp+axial comp

•**acceleration**:

$$\underline{a}_P = \frac{d\underline{v}_P}{dt} = \frac{d}{dt}(\dot{r}\underline{e}_r + r\dot{\phi}\underline{e}_\phi + \dot{z}\underline{k})$$

$$= \ddot{r}\underline{e}_r + \dot{r}\dot{\underline{e}}_r + \dot{r}\dot{\phi}\underline{e}_\phi + r\ddot{\phi}\underline{e}_\phi + r\dot{\phi}\dot{\underline{e}}_\phi + \ddot{z}\underline{k}$$

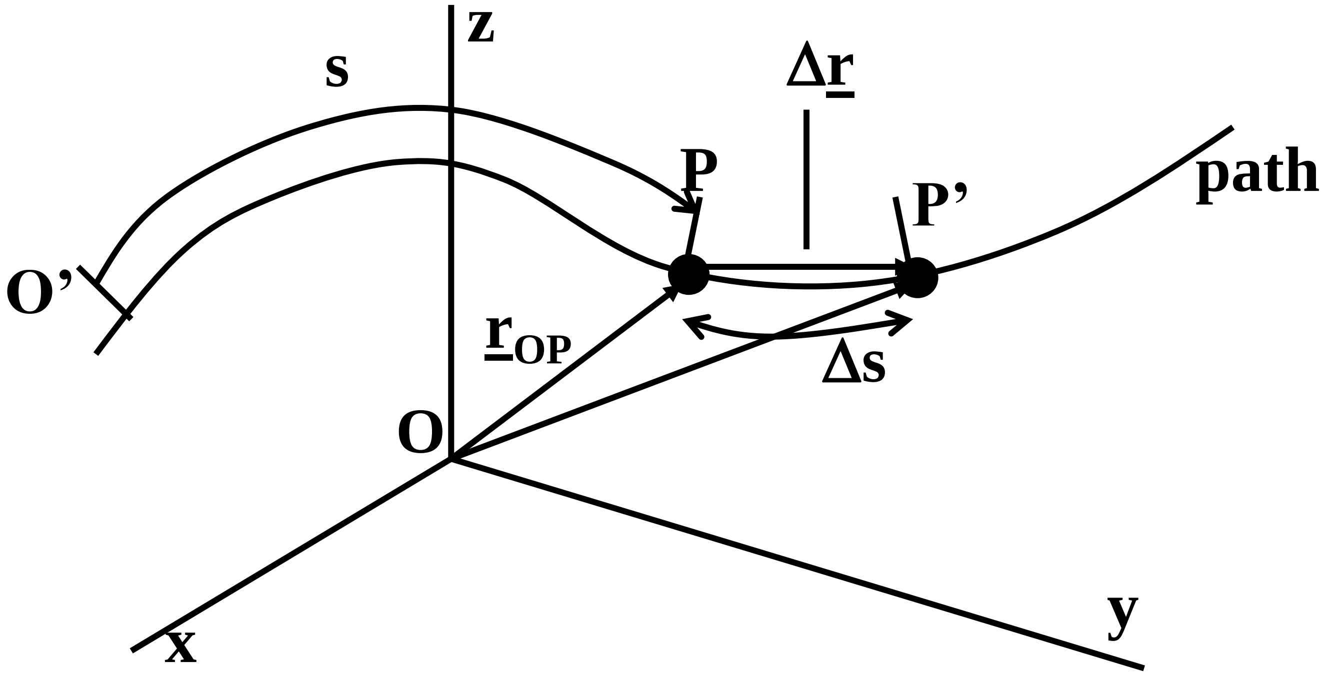
Now, $\dot{\underline{e}}_r = \dot{\phi}\underline{e}_\phi; \quad \dot{\underline{e}}_\phi = -\dot{\phi}\underline{e}_r$

$$\rightarrow \quad \boxed{\underline{a}_P = (\ddot{r} - r\dot{\phi}^2)\underline{e}_r + (2\dot{r}\dot{\phi} + r\ddot{\phi})\underline{e}_\phi + \ddot{z}\underline{k}}$$

=radial comp+transverse comp+axial comp

Spherical coordinates: reading suggested later

Tangential and Normal Components:
(intrinsic description)



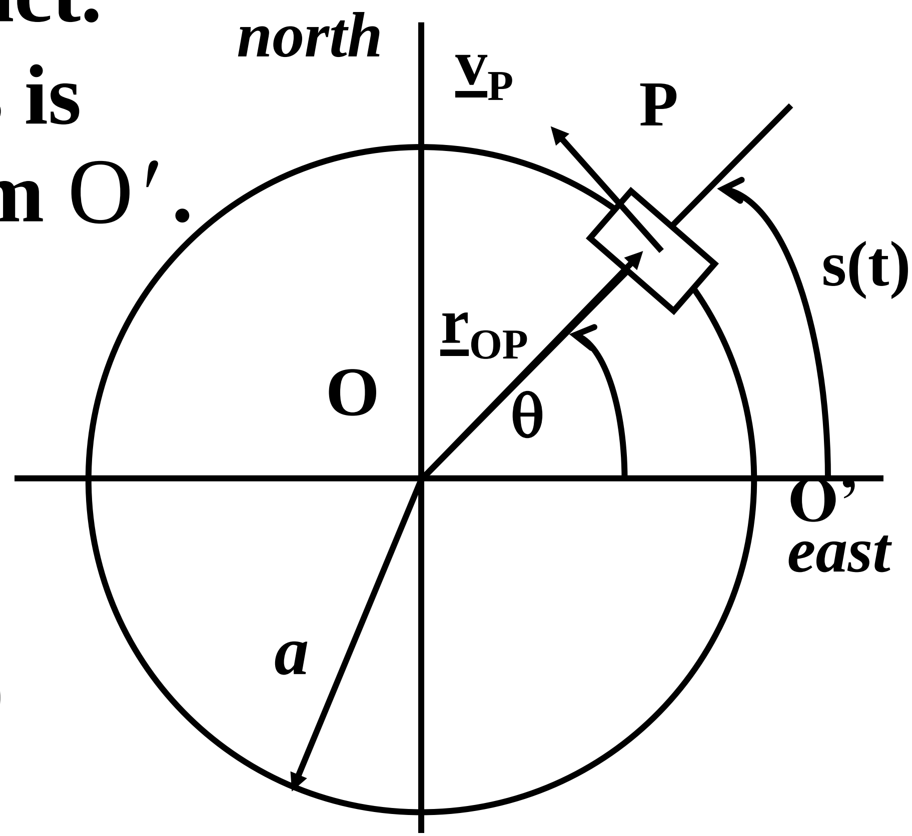
s - scalar parameter defining distance along the path from some landmark O′.

- called the path variable.

Note: $s \equiv s(t)$ (it depends on time). Suppose that the path is fixed, a given highway for example. Then, $\underline{r}_{OP} = \underline{r}_{OP}(s)$ is known. Different vehicles travel at different rates - speeds, changes in speeds. Properties of the highway, a planar or space curve are distinct from the motion s(t).

Ex: automobile traveling along a circular race track.

- **O and O' are distinct. Position is from O, s is being measured from O'.**

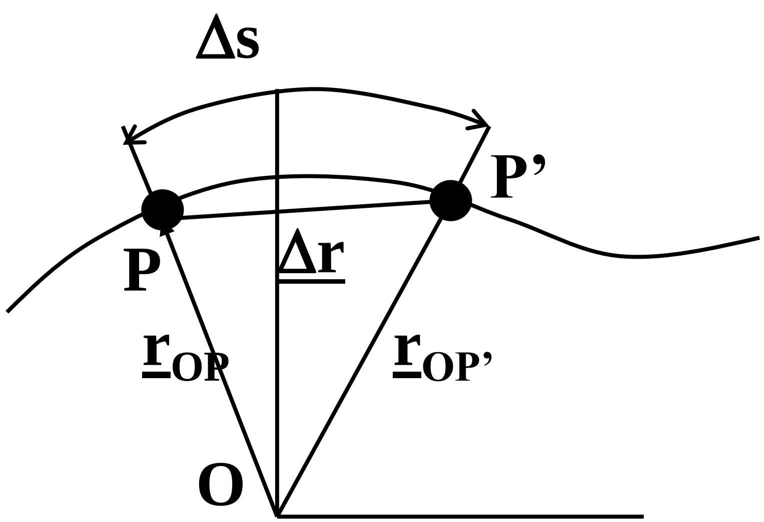


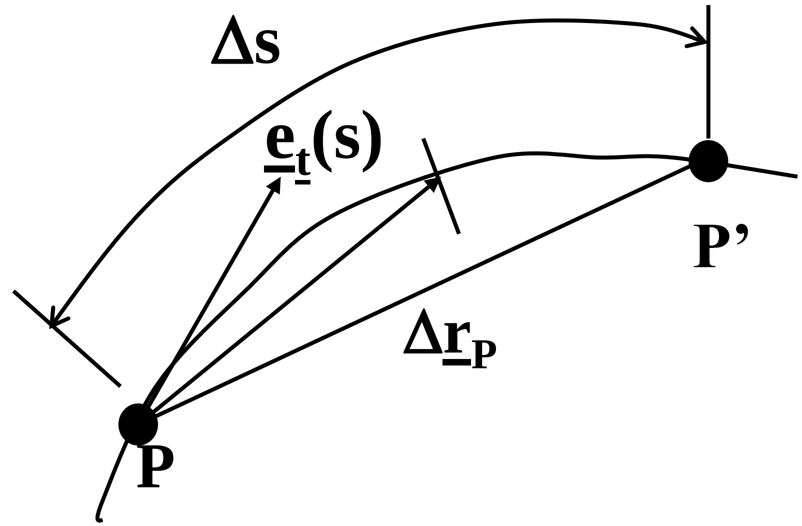
<u>Now</u>:

$$\underline{r}_{OP} = \underline{r}_{OP}(s); \quad s = s(t)$$

Then

$$\underline{v}_P = \frac{d\underline{r}_{OP}}{dt} = \frac{d\underline{r}_{OP}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\lim_{\Delta s \to 0}\frac{\Delta \underline{r}_P}{\Delta s}$$





Consider

$$\lim_{\Delta s \to 0} \frac{\Delta \underline{r}_P}{\Delta s} = \lim \frac{\Delta \underline{r}_P}{\left|\Delta \underline{r}_P\right|} . \lim \frac{\left|\Delta \underline{r}_P\right|}{\Delta s} = 1.\underline{e}_t(s) = \underline{e}_t(s)$$

- $\underline{e}_t$ **depends in orientation on s (location), its magnitude is always one.**

→ $\boxed{\underline{v}_P = \dot{s}\underline{e}_t(s)}$ **velocity vector**

- **velocity is always tangent to path with magnitude (speed)** $= v_P = \left|\dot{s}\underline{e}_t(s)\right| = \left|\dot{s}\right|$

To find expression for acceleration:

$$\underline{a}_P = \frac{d}{dt}(\dot{s}\underline{e}_t(s)) = \ddot{s}\underline{e}_t(s) + \dot{s}\underline{\dot{e}}_t(s)$$

$$= \ddot{s}\underline{e}_t(s) + \dot{s}\frac{d}{ds}(\underline{e}_t(s))\frac{ds}{dt} = \ddot{s}\underline{e}_t(s) + \dot{s}^2\frac{d\underline{e}_t(s)}{ds}$$

To find $\frac{\mathrm{d}\underline{\mathrm{e}}_\mathrm{t}}{\mathrm{ds}}$, **consider** $\underline{e}_t(s)\cdot\underline{e}_t(s) = 1$

(unit vector at every s)

$$\frac{d}{ds}\left\{\underline{e}_t(s)\cdot\underline{e}_t(s)\right\} = 0 \rightarrow 2\underline{e}_t(s)\cdot\frac{d\underline{e}_t(s)}{ds} = 0$$

$\rightarrow \quad \underline{e}_t(s)$ is $\perp^r$ to $\dfrac{d\underline{e}_t(s)}{ds}$

Let: $\dfrac{d\underline{e}_t(s)}{ds} = \kappa \underline{e}_n \equiv \dfrac{1}{\rho}\underline{e}_n(s)$

where

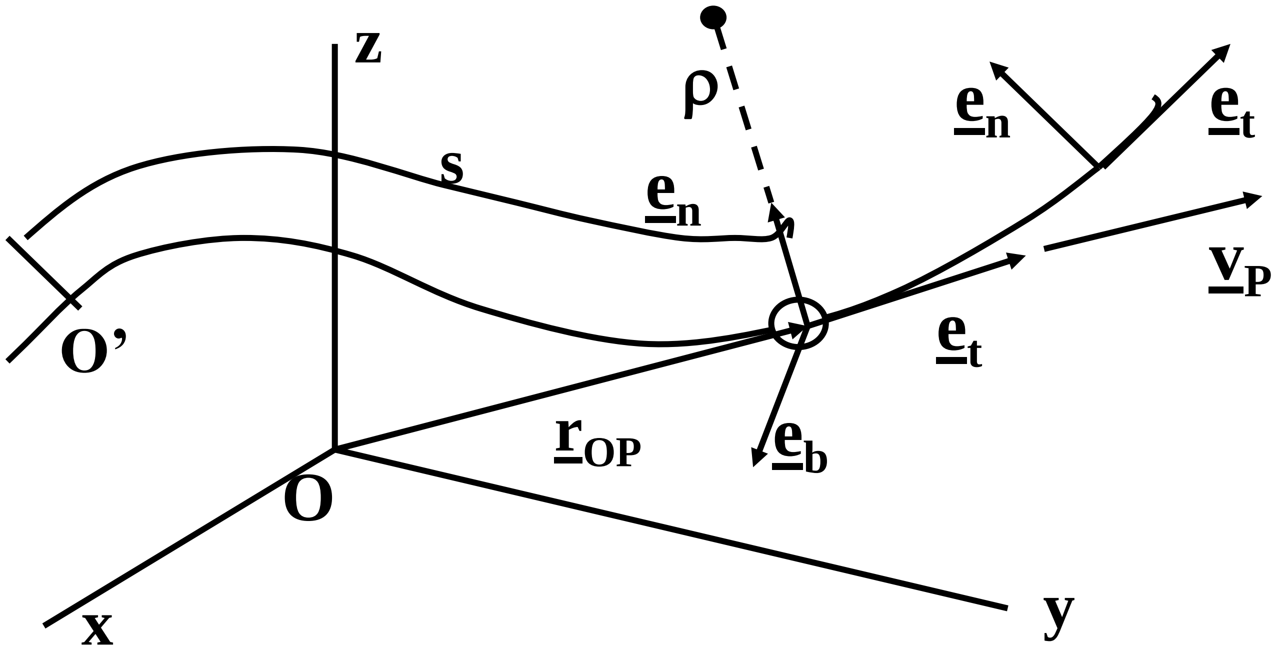
$\underline{e}_n(s) = \dfrac{d\underline{e}_t(s)}{ds} / \left|\dfrac{d\underline{e}_t(s)}{ds}\right|$ is a normal vector

κ - curvature of the path at the location 's'.

ρ - radius of curvature at P (at location 's').

$$\rightarrow \quad \boxed{\underline{a}_P = \ddot{s}\underline{e}_t(s) + \frac{\dot{s}^2}{\rho}\underline{e}_n(s) = \dot{v}_P\underline{e}_t(s) + \frac{v_P^{\ 2}}{\rho}\underline{e}_n}$$

or $\boxed{\underline{a}_P = a_t\underline{e}_t(s) + a_n\underline{e}_n(s)}$



Now define: $\underline{e}_b \equiv \underline{e}_t(s) \times \underline{e}_n(s)$ **binormal vector**

Note that the vectors $\underline{e}_t$, $\underline{e}_n$, and $\underline{e}_b$ satisfy

$$\underline{e}_t \cdot \underline{e}_t = \underline{e}_n \cdot \underline{e}_n = \underline{e}_b \cdot \underline{e}_b = 1;$$

$$\underline{e}_t \cdot \underline{e}_n = \underline{e}_t \cdot \underline{e}_b = \underline{e}_n \cdot \underline{e}_b = 0.$$

Rate of change of unit vectors along the path:

One can show that:

$$\frac{d\underline{e}_t}{ds} = \frac{\underline{e}_n}{\rho} \equiv \kappa \underline{e}_n \qquad \textbf{(1)}$$

$$\frac{d\underline{e}_b}{ds} = -\frac{\underline{e}_n}{\tau} \qquad \textbf{(2)}$$ **τ - torsion**

$$\frac{d\underline{e}_n}{ds} = -\frac{\underline{e}_t}{\rho} + \frac{\underline{e}_b}{\tau} \qquad \textbf{(3)}$$

Frenet's formulas (in differential geometry)

Ex (2): **Suppose we want to show:** $\dfrac{d\underline{e}_b}{ds} = -\dfrac{\underline{e}_n}{\tau}$

Consider $\dfrac{d}{ds}(\underline{e}_t \cdot \underline{e}_b) = 0 \rightarrow \dfrac{d\underline{e}_t}{ds} \cdot \underline{e}_b + \underline{e}_t \cdot \dfrac{d\underline{e}_b}{ds} = 0$

Now $\dfrac{d\underline{e}_t}{ds} = \dfrac{\underline{e}_n}{\rho} \rightarrow \underline{e}_t \cdot \dfrac{d\underline{e}_b}{ds} = -\dfrac{\underline{e}_n}{\rho} \cdot \underline{e}_b = 0$

$$\rightarrow \underline{e}_t \perp^r \frac{d\underline{e}_b}{ds}$$

Also $\dfrac{d}{ds}(\underline{e}_b \cdot \underline{e}_b = 1) \rightarrow \dfrac{d\underline{e}_b}{ds} \cdot \underline{e}_b = 0$

Thus $\rightarrow \underline{e}_b \perp^r \dfrac{d\underline{e}_b}{ds}$

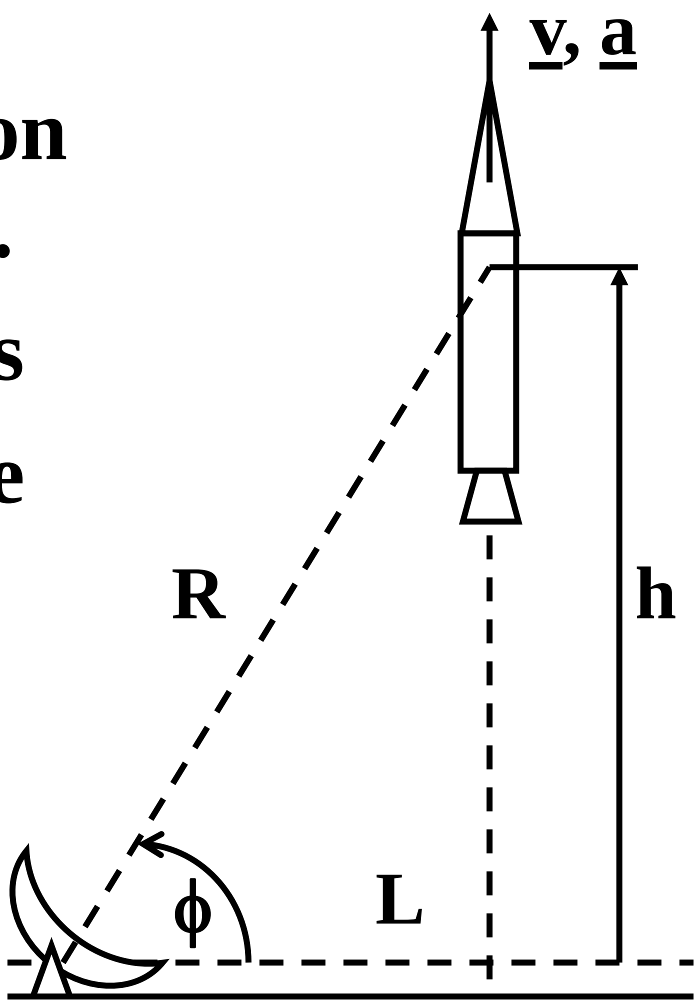
$$\rightarrow \quad \frac{d\underline{e}_b}{ds} \text{ is only along } \underline{e}_n$$

Let $$\boxed{\frac{d\underline{e}_b}{ds} = -\frac{\underline{e}_n}{\tau} \equiv -\tau_w \underline{e}_n}$$

τ - torsion , $\frac{1}{\tau} = \tau_w$ **- twist**

Torsion and twist are like radius of curvature and curvature.

Ex 3: A rocket lifts-off straight up. A radar station is located L distance away. At height H, the rocket has speed *v*, and rate of change of speed $\dot{v}$.



Find: $R, \dot{R}, \ddot{R}, \dot{\phi}, \ddot{\phi}$ **the variables measured by the tracking station.**

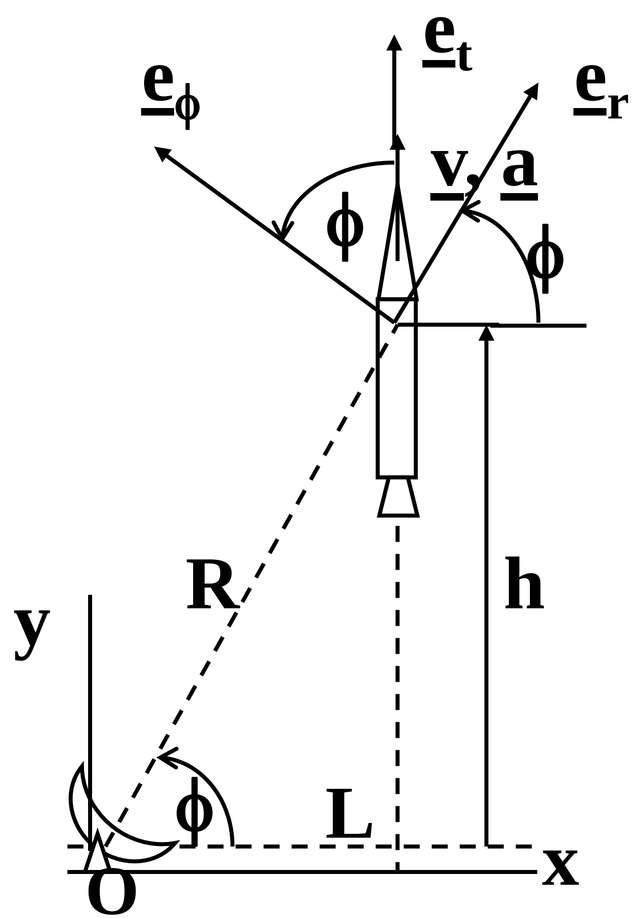
We start with velocity:

$$\underline{v}_P = v\underline{e}_t = \dot{R}\underline{e}_r + R\dot{\phi}\underline{e}_\phi$$

$$\underline{e}_t = \cos\phi\,\underline{e}_\phi + \sin\phi\,\underline{e}_r$$

$$v\underline{e}_t = v(\cos\phi\,\underline{e}_\phi + \sin\phi\,\underline{e}_r)$$

$$= \dot{R}\underline{e}_r + R\dot{\phi}\underline{e}_\phi$$



Comparing on two sides:

$$\underline{e}_\phi : \quad v\cos\phi = R\dot{\phi} \rightarrow \dot{\phi} = v\cos\phi / R$$

$$\underline{e}_r : \quad v\sin\phi = \dot{R}; \quad \text{Also, } R = L/\cos\phi$$

$$\rightarrow \quad \boxed{v\sin\phi = \dot{R}} \quad \boxed{\dot{\phi} = v\cos^2\phi / L}$$

Similarly:

$$\underline{a}_P = \dot{v}\underline{e}_t + \frac{v^2}{\rho}\underline{e}_n = (\ddot{R} - R\dot{\phi}^2)\underline{e}_r + (R\ddot{\phi} + 2\dot{R}\dot{\phi})\underline{e}_\phi$$

$$= \dot{v}(\sin\phi\,\underline{e}_r + \cos\phi\,\underline{e}_\phi)$$

comparing on the two sides:

$$\underline{e}_r: \quad \dot{v}\sin\phi = \ddot{R} - R\dot{\phi}^2$$

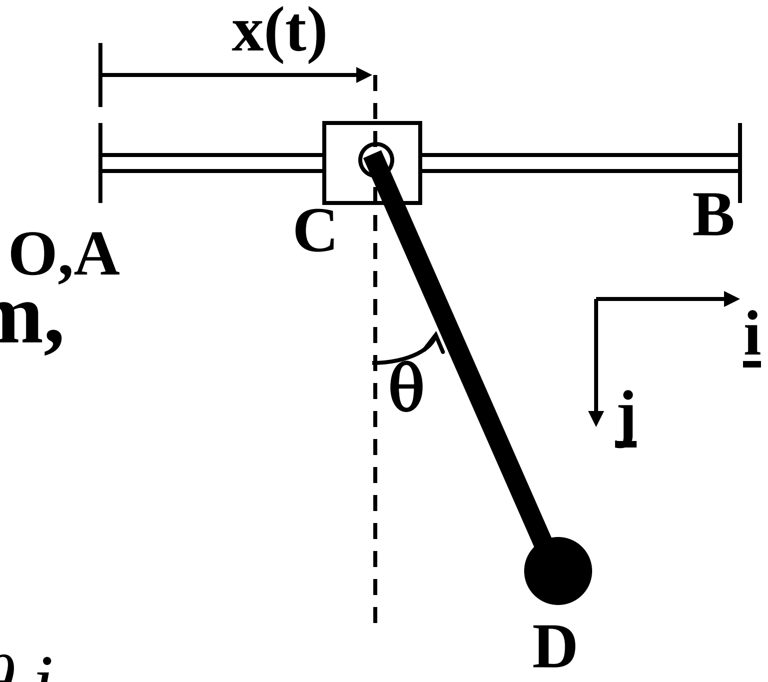
$$or \quad \boxed{\ddot{R} = \dot{v}\sin\phi + v^2\cos^3\phi / L}$$

$$\underline{e}_\phi: \quad \dot{v}\cos\phi = R\ddot{\phi} + 2\dot{R}\dot{\phi}$$

$$or \quad \boxed{\ddot{\phi} = \cos\phi[\dot{v}\cos\phi - 2v\sin\phi\cos^2\phi v / L] / L}$$

Ex 4: A block C slides along the horizontal rod, while a pendulum attached to the block can swing in the vertical plane

Find: The acceleration of the pendulum mass D.



Using (x,y,z) coord. system,

$$\underline{r}_C = x(t)\,\underline{i}$$

$$\underline{r}_D = [x(t) + l\sin\theta]\,\underline{i} + l\cos\theta\,\underline{j}$$

$$\underline{\dot{r}}_D = [\dot{x}(t) + l\dot{\theta}\cos\theta]\,\underline{i} - l\dot{\theta}\sin\theta\,\underline{j}$$

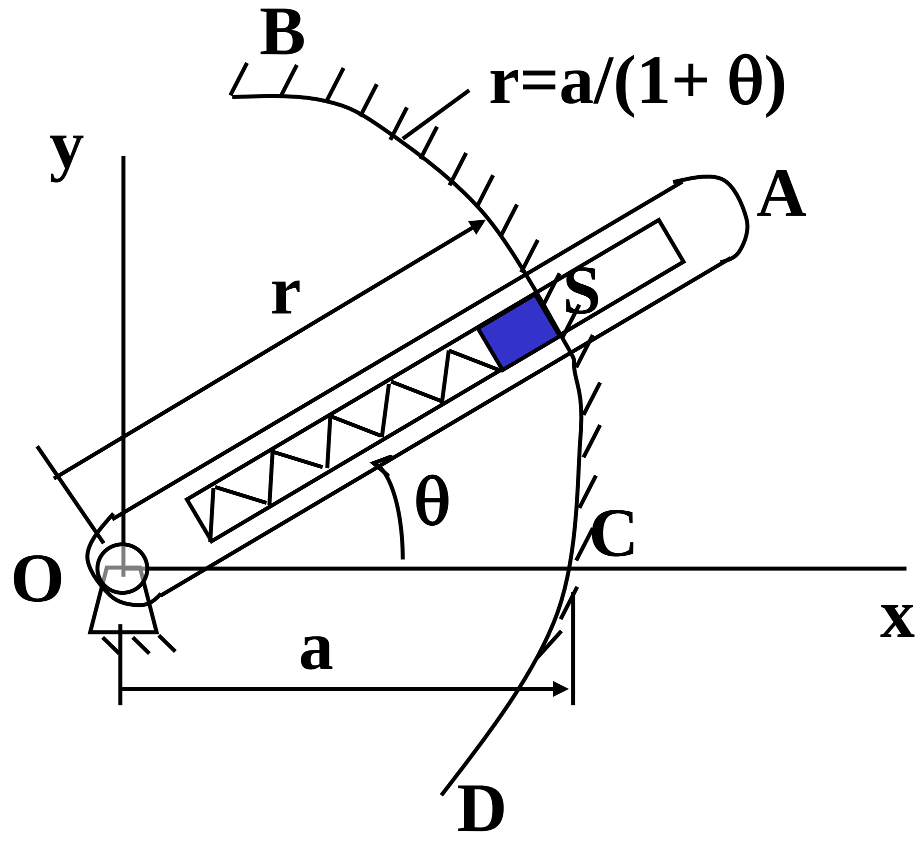
$$\underline{\ddot{r}}_D = [\ddot{x}(t) + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta]\,\underline{i}$$

$$- l[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta]\,\underline{j}$$

Ex 5: A slider S is constrained to follow the fixed surface defined by the curve BCD:

$r = a/(1+\theta)$; r is in meters, θ is in radians.

Find: $\underline{\mathbf{v}}_S$, $\underline{\mathbf{a}}_S$



Consider the solution using the cylindrical coordinate system: the unit vectors are $\underline{e}_r$ and $\underline{e}_\theta$

The position is: $\underline{r}_S = r\underline{e}_r$

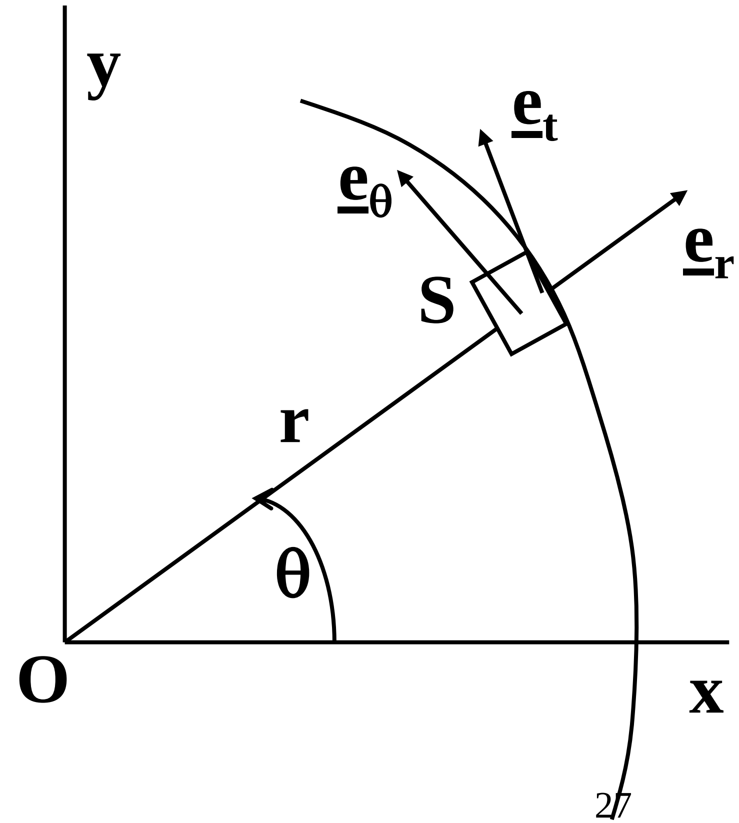
The velocity is $\underline{v}_S = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta;$

Now $r = a/(1+\theta),$

$$\theta = c\sin(\omega t),\ \dot{\theta} = -c\omega\cos(\omega t)$$

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta};\ \ \frac{dr}{d\theta} = -\frac{a}{(1+\theta)^2}$$

$$\underline{v}_S = -\frac{a\dot{\theta}}{(1+\theta)^2}\underline{e}_r + \frac{a\dot{\theta}}{(1+\theta)}\underline{e}_\theta$$



Now we consider the acceleration of the block:

The expression is:

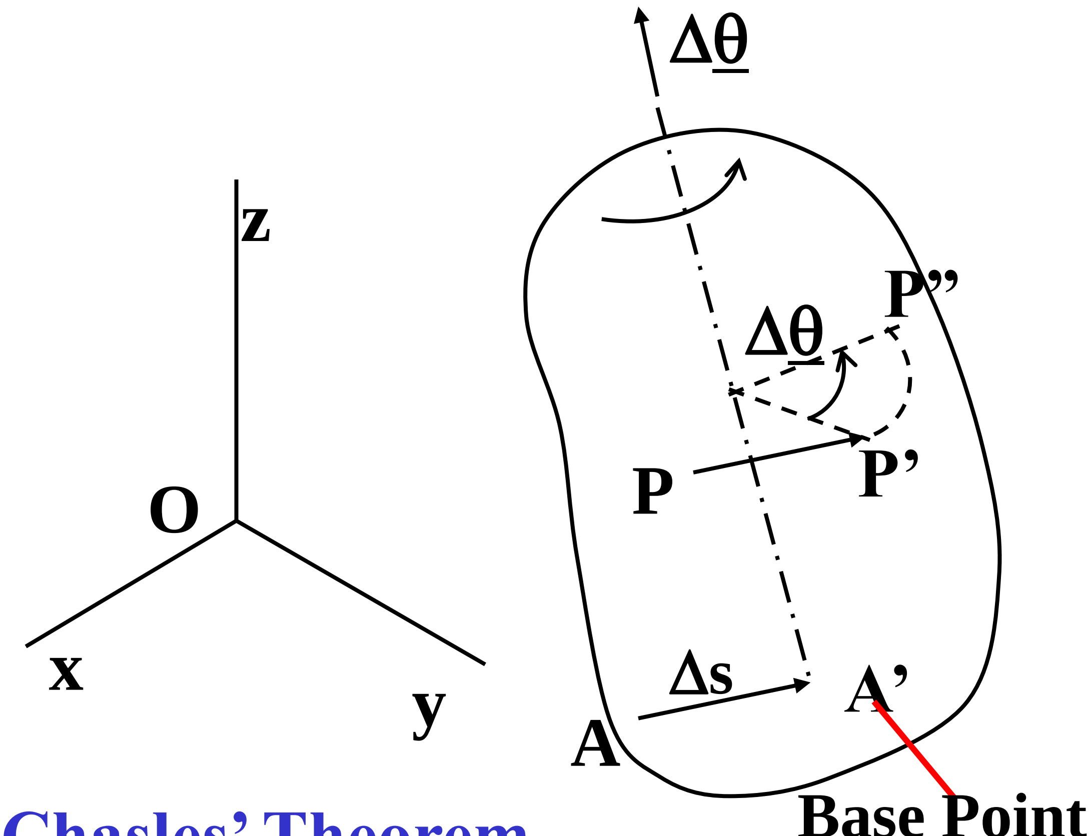
$$\underline{a}_S = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$$

The various terms in this expression are:

$$\dot{\theta} = -c\omega\cos(\omega t);\ \ddot{\theta} = -c\omega^2\sin(\omega t)$$

$$\ddot{r} = \frac{d}{dt}\dot{r} = \frac{d}{dt}\left(-\frac{a\dot{\theta}}{(1+\theta)^2}\right) = -\frac{a\ddot{\theta}}{(1+\theta)^2} + \frac{2a\dot{\theta}\,\dot{\theta}}{(1+\theta)^3}$$

2.2 ANGULAR VELOCITY: It defines the rate of change of orientation of a rigid body - or, a coordinate frame with respect to another.



Chasles' Theorem

Consider displacement in time Δt.

(displ. + rot.)

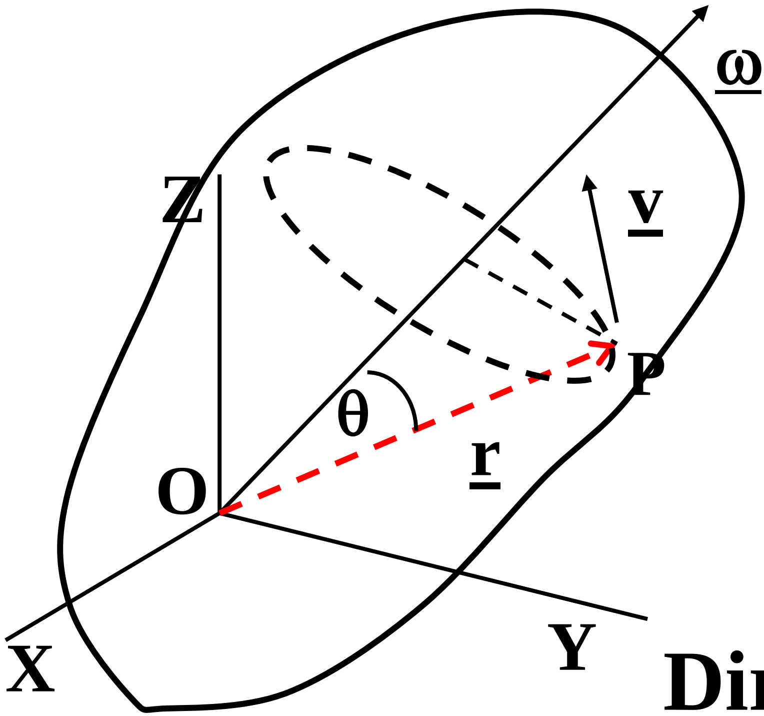
Shown is an infinitesimal displacement of a rigid body

$$\underline{\omega} \equiv \lim_{\Delta t \to 0} \frac{\Delta \underline{\theta}}{\Delta t}$$ **defines the angular velocity** $\underline{\omega}$

- **The angular velocity does not depend on the base point** A'**. Rather, it is a property for the whole body.**

- **The angular velocity vector will usually change both its magnitude** $|\underline{\omega}|$ **and direction** $\underline{e}_{\omega} = \underline{\omega} / |\underline{\omega}|$ **continuously with time.**

2.3 Rigid Body Motion about a Fixed Point:



The rigid body rotates about point O (fixed base point). P - a point fixed in the body.

$\underline{\omega}$ - angular velocity of the body relative to XYZ axes.

Direction of $\underline{\omega}$ - instant. axis of rotation. Speed of P $\dot{s} = \omega r \sin\theta$.

→ **velocity** $\boxed{\underline{v} = \underline{\omega} \times \underline{r}}$ **(along tangent to circle)**

The acceleration is now calculated, using the definition that it is the time-derivative of velocity: $\underline{a} = d(\underline{v})/dt = d(\underline{\omega} \times \underline{r})/dt$

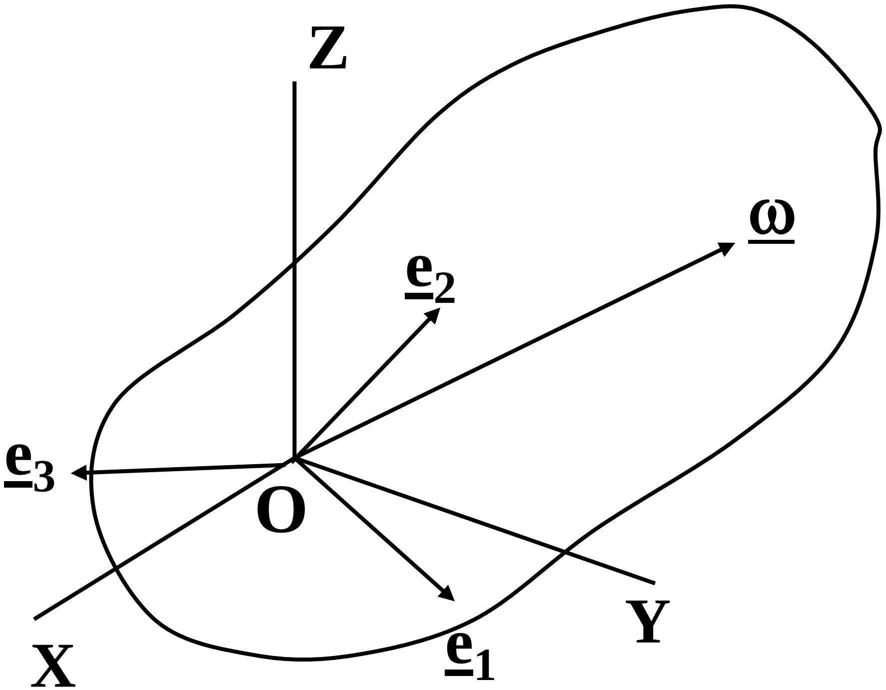
$$\underline{a} = \underline{\dot{\omega}} \times \underline{r} + \underline{\omega} \times \underline{\dot{r}}$$

or $$\boxed{\underline{a} = \underline{\dot{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})}$$

(Recall: these rates of change are ω.r.t.**XYZ).**

- $(\underline{\omega} \times (\underline{\omega} \times \underline{r}))$ **is directed towards the instantan axis from P - centripetal acceleration.**
- $\underline{\dot{\omega}} \times \underline{r}$ **- tangential acceleration (not really tangent to the path of P).**

2.4 The Derivative of a Unit Vector:



Let $\underline{e}_1, \underline{e}_2, \underline{e}_3$ be an independent set of unit vectors attached to a rigid body rotating with angular velocity $\underline{\omega}$. The body rotates relative to the reference frame XYZ. Thus, for the unit vectors:

$$\dot{\underline{e}}_1 = \underline{\omega} \times \underline{e}_1, \quad \dot{\underline{e}}_2 = \underline{\omega} \times \underline{e}_2, \quad \dot{\underline{e}}_3 = \underline{\omega} \times \underline{e}_3$$

Assume that the set ($\underline{e}_1, \underline{e}_2, \underline{e}_3$) is - orthonormal

Thus, $\underline{e}_1 \perp^r \underline{e}_2$; $\underline{e}_1 \perp^r \underline{e}_3$ and $\underline{e}_2 \perp^r \underline{e}_3$

This can also be stated as: $\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$, etc.

Let $\underline{\omega} = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3$

(expressed in moving basis)

$$\rightarrow \dot{\underline{e}}_1 = (\omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3) \times \underline{e}_1 = \omega_3 \underline{e}_2 - \omega_2 \underline{e}_3$$

$$\dot{\underline{e}}_2 = (\omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3) \times \underline{e}_2 = \omega_1 \underline{e}_3 - \omega_3 \underline{e}_1$$

$$\dot{\underline{e}}_3 = (\omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3) \times \underline{e}_3 = \omega_2 \underline{e}_1 - \omega_1 \underline{e}_2$$

Ex: $\underline{e}_1 = \underline{i}$, $\underline{e}_2 = \underline{j}$, $\underline{e}_3 = \underline{k}$

$$\rightarrow \underline{\omega} = \omega_x \underline{i} + \omega_y \underline{j} + \omega_z \underline{k}$$

Then, $$d\underline{i}/dt = \underline{\omega} \times \underline{i} = \omega_z \underline{j} - \omega_y \underline{k}$$

$$d\underline{j}/dt = \underline{\omega} \times \underline{j} = \omega_x \underline{k} - \omega_z \underline{i}$$

$$d\underline{k}/dt = \underline{\omega} \times \underline{k} = \omega_y \underline{i} - \omega_x \underline{j}$$

IMPORTANT:

- **The rates of change of unit vectors have been calculated with respect to the (X,Z,Y) system - also called "relative to XYZ".**
- **These rates (vectors) have been expressed in terms of the unit vectors moving with the body.**

2.6 EXAMPLES:

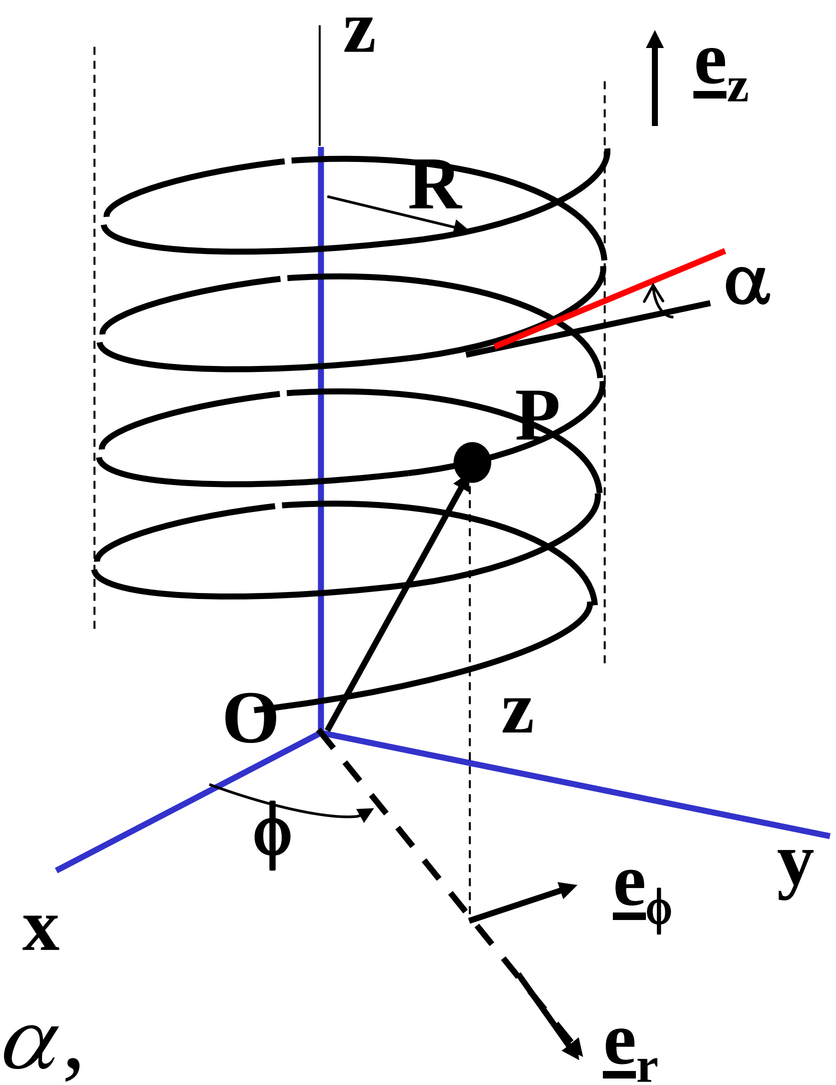
ii) **Helical Motion:**

A particle moves along a helical path. The helix is defined in terms of the **Cylindrical Coordinates:**

$$r = R(\text{constant})$$

$$z = kR\phi, \text{ where } k = \tan\alpha,$$

$$\alpha - \text{helix angle}$$



Clearly, as ϕ changes with time, so does z.

So, $\dot{r} = 0,\ \ddot{r} = 0,\ \dot{\phi} = \omega,\ \ddot{\phi} = \dot{\omega}$

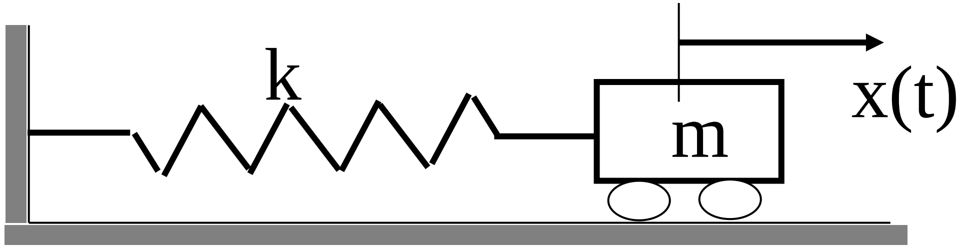
$$\dot{z} = kR\dot{\phi},\ \ddot{z} = kR\ddot{\phi} = kR\dot{\omega}$$

- **$\underline{\mathbf{v}}_\mathbf{P}$=** $\dot{r}\underline{e}_r + r\dot{\phi}\underline{e}_\phi + \dot{z}\underline{e}_z = R\omega\underline{e}_\phi + kR\omega\underline{e}_z$
- **$\underline{\mathbf{a}}_\mathbf{P}$=** $-R\omega^2\underline{e}_r + R\dot{\omega}\underline{e}_\phi + kR\dot{\omega}\underline{e}_z$
- **speed** $\dot{s} = \left|\underline{v}_P\right| = \sqrt{(R\omega)^2 + (kR\omega)^2} = R\omega\sqrt{1+k^2}$
- **constant or uniform speed** $\rightarrow \ddot{s} = 0,\ \dot{\omega} = 0$

$$\rightarrow \underline{a} = -R\omega^2\underline{e}_R = (\dot{s}^2/\rho)\underline{e}_n$$

- $\boxed{\rho = R(1+k^2)}$ **- radius of curvature of the path of the particle**

iii) Harmonic motion: (Reading assignment)



m - mass, k - stiffness of the spring

key point: the force is directly proportional to the distance of the particle from some point →

$$\ddot{x} = -\omega^2 x;$$ **x- displacement**

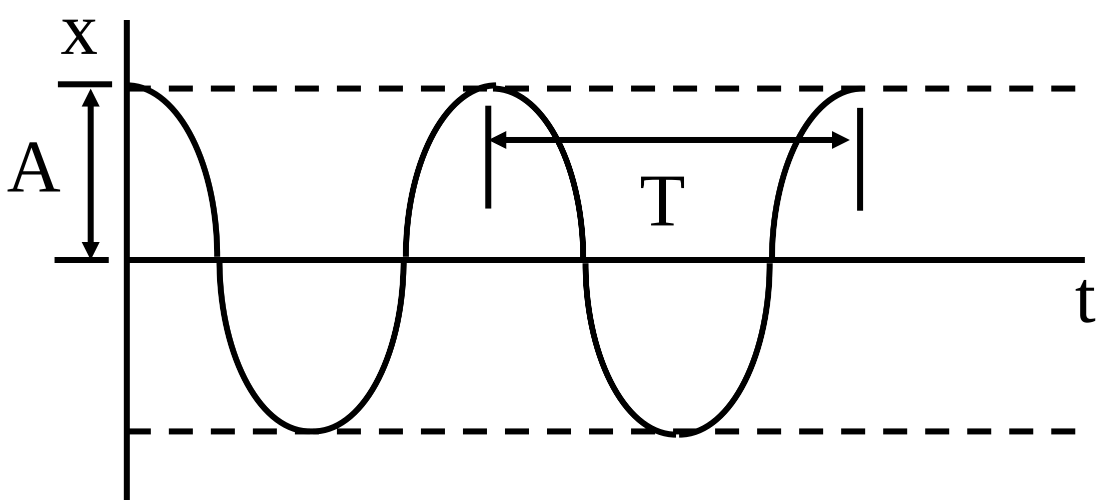
$\omega^2 > 0$ **- a constant (square of natural freq.)**

Solution: Let x(t) = A cos(ωt + α)

Here, A - amplitude, α - phase angle

(these are determined by initial conditions x, $\dot{x}$ at t = 0).

If α = 0, i.e., $\dot{x}(t) = 0$ at t = 0, $\boxed{x(t) = A \cos \omega t}$



simple harmonic motion

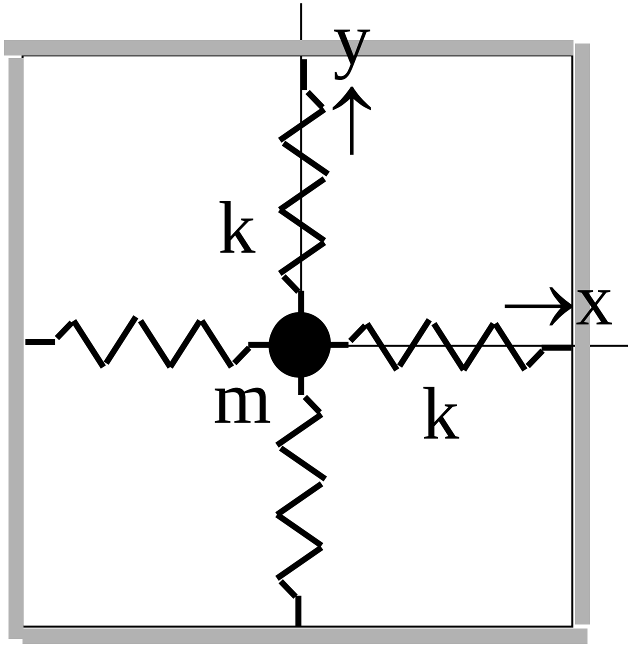
T = 2π/ω - time period of harmonic motion

ω - circular frequency

- **x(t) = A cos ωt**
- $\dot{x}(t) = -A\omega \sin \omega t = A\omega \cos(\omega t + \frac{\pi}{2})$
- $\ddot{x}(t) = -A\omega^2 \cos \omega t = A\omega^2 \cos(\omega t + \pi)$

→ **In simple harmonic motion, extreme values of position and acceleration occur when the velocity vanishes. Also, the velocity is out of phase with position by π/2, and the acceleration is out of phase by π.**

Two-dimensional harmonic motion:



- **Consider the spring-mass system shown:**

$$\ddot{x} = -\omega^2 x; \quad \ddot{y} = -\omega^2 y$$

$$x(t) = A\cos(\omega t + \alpha);$$

$$y(t) = B\cos(\omega t + \beta)$$

- **Choose reference time such that α = 0** $\rightarrow$ $\underline{r}(t) = x(t)\,\underline{i} + y(t)\,\underline{j}$

$$= A\cos\omega t\,\underline{i} + B\cos(\omega t + \beta)\,\underline{j}$$

$$\underline{v}(t) = -\omega A\sin\omega t\,\underline{i} - \omega B\sin(\omega t + \beta)\,\underline{j}$$

$$\underline{a}(t) = -\omega^2 A\cos\omega t\,\underline{i} - \omega^2 B\cos(\omega t + \beta)\,\underline{j}$$

Now: $\cos \omega t = x/A$, and $\cos(\omega t + \beta) = y/B$

or, $\cos \omega t \cos\beta - \sin\omega t \sin\beta = y/B$

Using expression for $x/A \rightarrow$

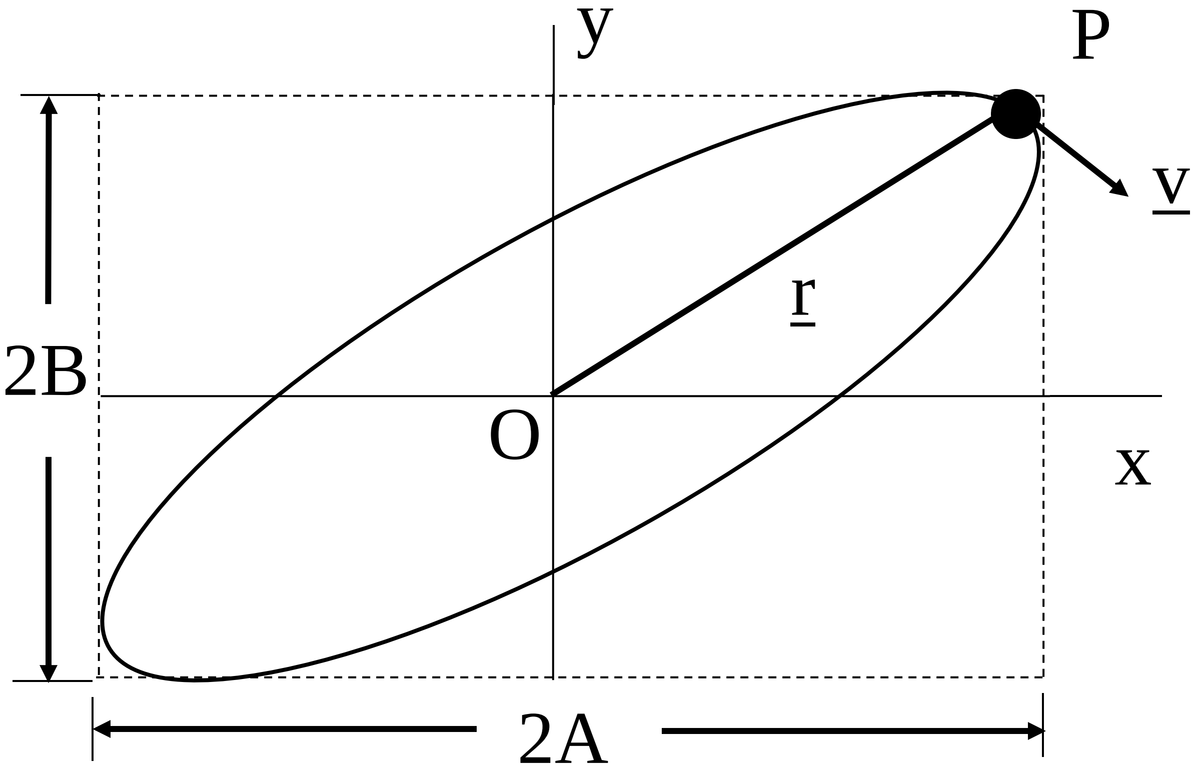
$y/B = (x/A)\cos\beta - \sin\omega t \sin\beta$

$\rightarrow \sin\omega t = [(x/A)\cos\beta - y/B] \sin\beta$

Since $\cos^2 \omega t + \sin^2 \omega t = 1 \rightarrow$

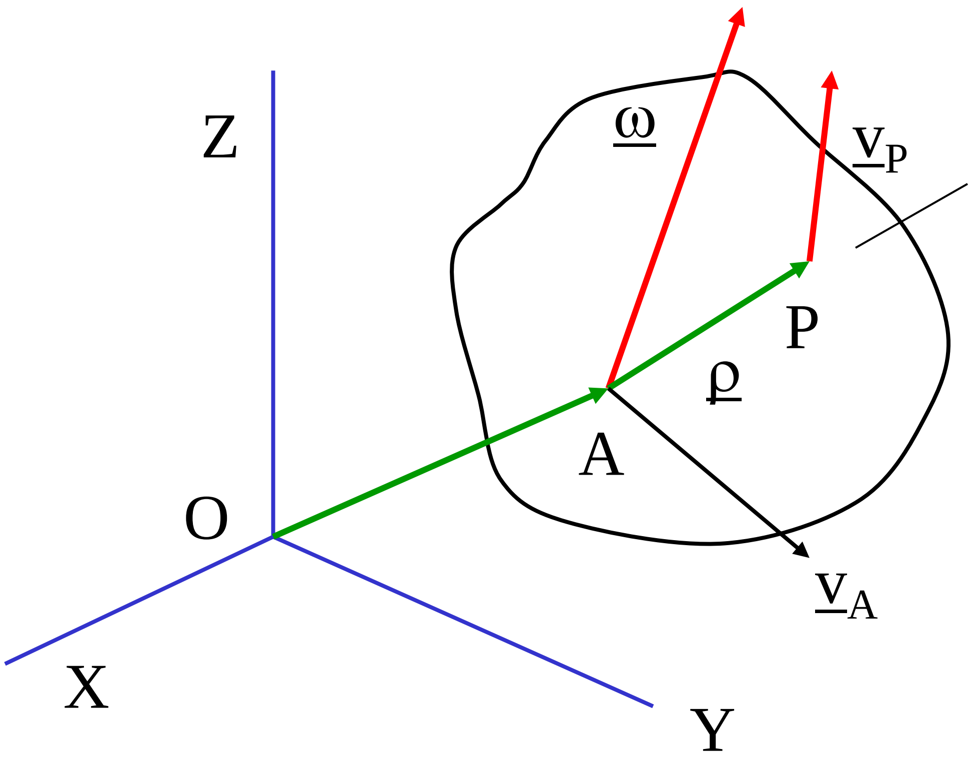
$$(\sin\beta)^{-2}[(x/A)^2 + (y/B)^2 - 2(x/A)(y/B)\cos\beta] = 1$$

(equation of an ellipse)



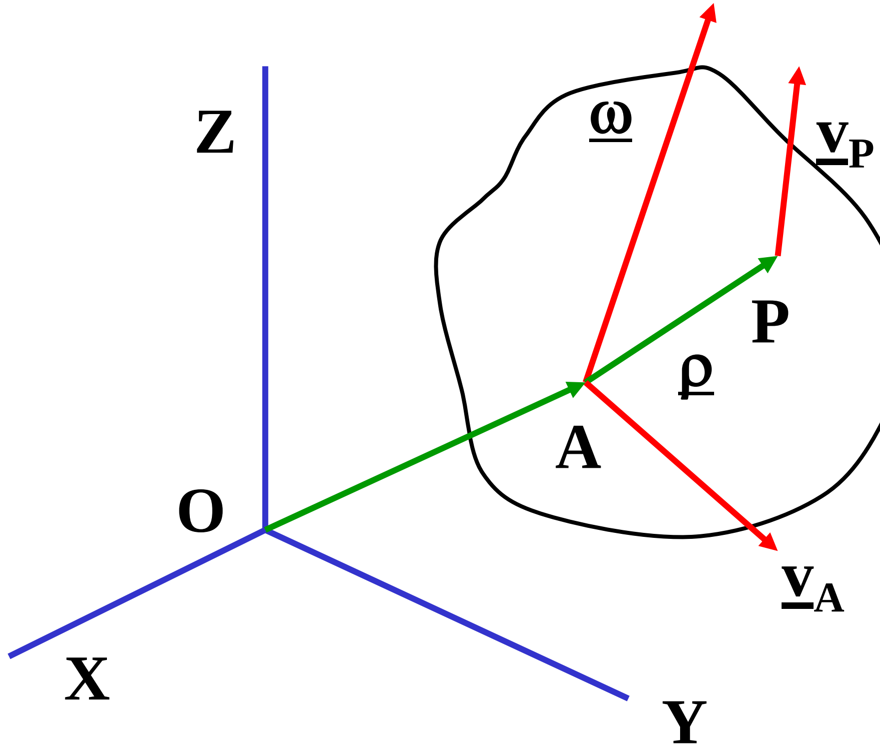
Harmonic motion in two dimensions (a plane)

2.7 Velocity and Acceleration of a Point in a Rigid Body



rigid body rotating at the rate of $\underline{\omega}$ relative to XYZ.

A, P are two points on the same rigid body



ρ - position of P as measured from A.

Now, $\underline{r}_{OP} = \underline{r}_{OA} + \underline{\rho}$

So, $\underline{v}_P = \underline{v}_A + d\underline{\rho}/dt$

$\rightarrow \boxed{\underline{v}_P = \underline{v}_A + \underline{\omega} \times \underline{\rho}}$

(since | ρ |= const., ρ changes only in orientation)

- $\underline{v}_P - \underline{v}_A = \underline{\omega} \times \underline{\rho} = \underline{v}_{P/A}$ **velocity of P ω.r.t. Point A, as viewed in the reference frame XYZ.**

Now, consider the acceleration:

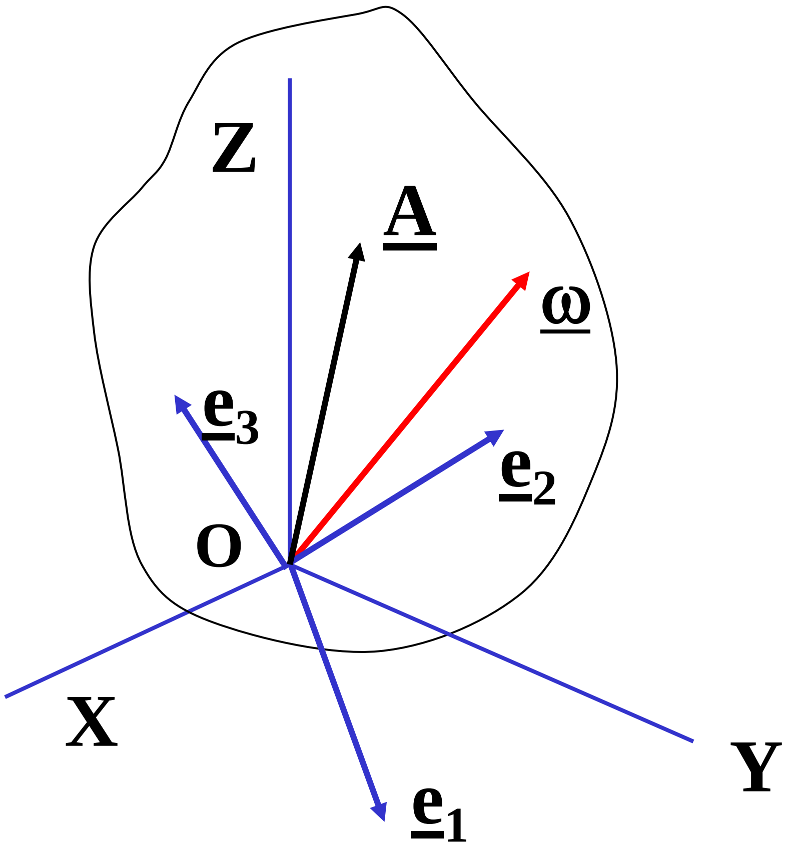
$$\underline{a}_P = d\underline{v}_P / dt = d(\underline{v}_A + \underline{\omega} \times \underline{\rho}) / dt$$

$$= \underline{a}_A + \dot{\underline{\omega}} \times \underline{\rho} + \underline{\omega} \times \dot{\underline{\rho}}$$

or $$\boxed{\underline{a}_P = \underline{a}_A + \dot{\underline{\omega}} \times \underline{\rho} + \underline{\omega} \times (\underline{\omega} \times \underline{\rho})}$$

This is the acceleration of point P in a rigid body as viewed ω.r.t the frame XYZ; the point P is in the rigid body which is rotating at angular velocity $\underline{\omega}$ relative to XYZ, and this rotation rate is changing at the rate $\dot{\underline{\omega}}$.

2.8 Vector Derivative in Rotating Systems



- **O - a fixed point in the body**
- $\underline{e}_1, \underline{e}_2, \underline{e}_3$ **- triad of unit vectors in the body**
- $\underline{\omega}$ **- angular velocity of the body**
- **Consider now an arbitrary vector** $\underline{A}$

It can be represented as $\underline{A} = A_1 \underline{e}_1 + A_2 \underline{e}_2 + A_3 \underline{e}_3$

There are

two observers - $\begin{cases} \text{stationary with XYZ} \\ \text{moving with the body } (\underline{\omega}). \end{cases}$

$$\text{Then, } \frac{d\underline{A}}{dt} = \begin{cases} \text{can be with respect to XYZ } ({}^{XYZ}d\underline{A}/dt) \\ \text{can be with respect to the moving body } ({}^{\Re}d\underline{A}/dt) \end{cases}$$

(depends on the observer)

Let $\dot{\underline{A}} = {}^{XYZ}d\underline{A}/dt \equiv$ **rate of change ω.r.t. XYZ**

Then

$$\dot{\underline{A}} = \underbrace{\dot{A}_1\underline{e}_1 + \dot{A}_2\underline{e}_2 + \dot{A}_3\underline{e}_3}_{} + A_1\dot{\underline{e}}_1 + A_2\dot{\underline{e}}_2 + A_3\dot{\underline{e}}_3$$

$(\dot{\underline{A}})_r$ **- the rate of change w.r.t. the body in which** $\underline{e}_i$ **are fixed**

Now $\dot{\underline{e}}_1 = \underline{\omega} \times \underline{e}_1; \quad \dot{\underline{e}}_2 = \underline{\omega} \times \underline{e}_2; \quad \dot{\underline{e}}_3 = \underline{\omega} \times \underline{e}_3$

$$\rightarrow A_1\dot{\underline{e}}_1 + A_2\dot{\underline{e}}_2 + A_3\dot{\underline{e}}_3 = \underline{\omega} \times \underline{A}$$

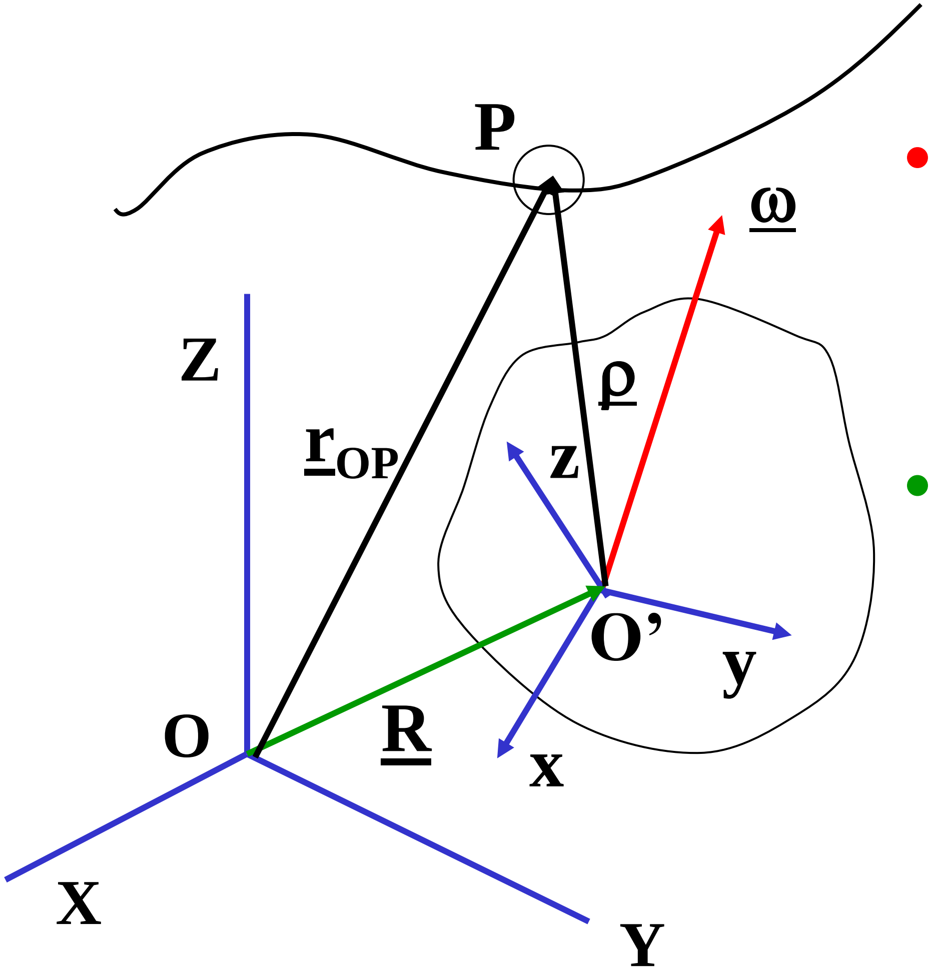
$$\rightarrow \boxed{\dot{\underline{A}} = (\dot{\underline{A}})_r + \underline{\omega} \times \underline{A}}$$

In a more general sense, let A and B be two bodies; $\underline{\omega}_{A/B}$ - angular velocity of A as viewed (by an observer) from B;

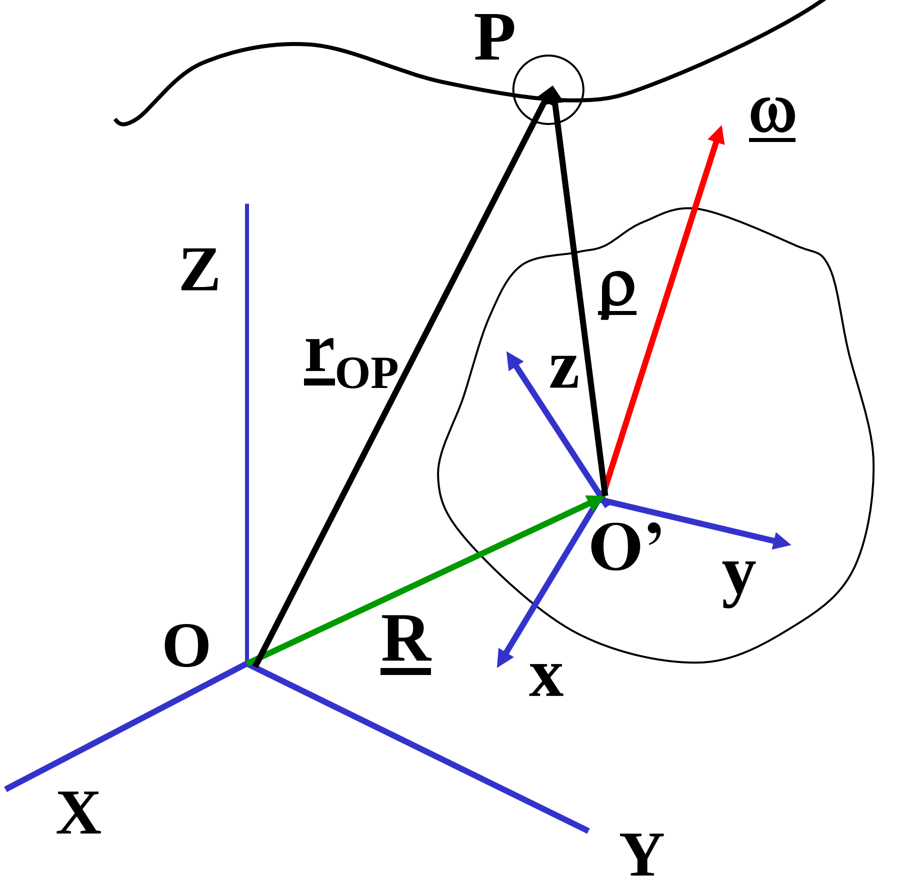
(Note $\underline{\omega}_{B/A} = -\underline{\omega}_{A/B}$ - angular velocity of B as viewed from A). Then

$$\boxed{(\dot{\underline{A}})_A = (\dot{\underline{A}})_B + \underline{\omega}_{B/A} \times \underline{A}}$$

2.9 Motion of a Particle in a Moving Coordinate System



- XYZ - fixed reference frame (really, a given frame)
- xyz - a frame denoting a moving body with angular velocity $\underline{\omega}$ relative to XYZ.



O' - origin of coordinate system in xyz

$\underline{R}$ - position of O'

$\underline{r}_{OP}$ - position of point P (moving object)

$\underline{\rho}$ - position of P ω.r.t. O'

Then, the position of the particle is

$$\underline{r}_{OP} = \underline{R} + \underline{\rho}$$

Then, the velocity with respect to the XYZ is

$\dot{\underline{r}}_{OP} = \underline{v}_P = \dot{\underline{R}} + \dot{\underline{\rho}}$ **but** $\dot{\underline{\rho}} = (\dot{\underline{\rho}})_r + \underline{\omega} \times \underline{\rho}$

(rate of change of $\underline{\rho}$ $\omega.r.t.$ the rotating frame (the rotating body))

$\rightarrow$ $\boxed{\underline{v}_P = \dot{\underline{R}} + (\dot{\underline{\rho}})_r + \underline{\omega} \times \underline{\rho}}$ **- velocity of P** ω.r.t. **XYZ.**

$(\dot{\underline{\rho}})_r$ **- velocity of P** ω.r.t. P′ **in xyz.**

$\dot{\underline{R}}$ **- velocity of** O′ ω.r.t. **XYZ.**

$\dot{\underline{R}} + \underline{\omega} \times \underline{\rho}$ **- velocity of a point** P′ **in the rotating body which is coincident with P at this instant.**

$$\ddot{\underline{r}}_{OP} = d\underline{v}_P / dt = \underline{a}_P$$ - **acceleration in XYZ frame**

$$\text{or } \underline{a}_P = d[\dot{\underline{R}} + (\dot{\underline{\rho}})_r + \underline{\omega} \times \underline{\rho}] / dt$$

$$= \ddot{\underline{R}} + d(\dot{\underline{\rho}})_r / dt + \dot{\underline{\omega}} \times \underline{\rho} + \underline{\omega} \times d\underline{\rho} / dt$$

$$\text{Now } d(\dot{\underline{\rho}})_r / dt = (\ddot{\underline{\rho}})_r + \underline{\omega} \times (\dot{\underline{\rho}})_r$$

$$\text{and } d\underline{\rho} / dt = (\dot{\underline{\rho}})_r + \underline{\omega} \times \underline{\rho}$$

$$\rightarrow \boxed{\underline{a}_P = \ddot{\underline{R}} + \dot{\underline{\omega}} \times \underline{\rho} + (\ddot{\underline{\rho}})_r + \underbrace{2\underline{\omega} \times (\dot{\underline{\rho}})_r}_{\text{coriolis}} + \underbrace{\underline{\omega} \times (\underline{\omega} \times \underline{\rho})}_{\text{centripetal}}}$$

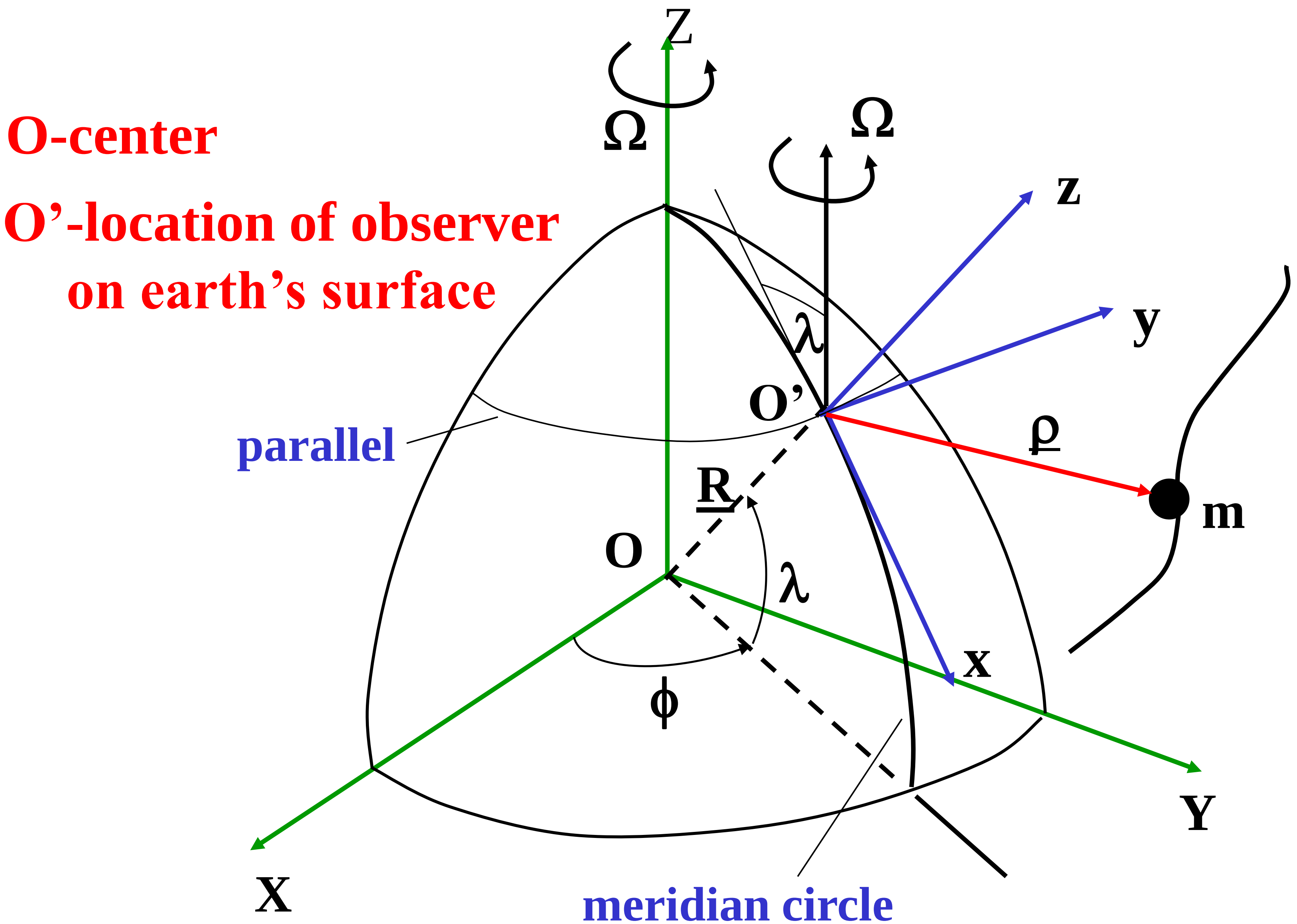
coriolis **centripetal**

This is the most general expression for accel

Ex: Motion of a Particle Relative to Rotating Earth

We now consider one of the application of the general formulation for acceleration when a rotating reference frame is quite natural. Here, an object is moving and its motion is observed by someone moving with earth. Assumptions:

- **Earth rotates about the sun.**
- **The acceleration of center of earth is, however, relatively small compared to gravity and acceleration on the earth's surface due to earth's spin, especially away from poles.**



O-center
O'-location of observer
on earth's surface
Z
Ω
Ω
z
y
λ
O'
ρ
parallel
R
m
O
λ
x
ϕ
Y
X
meridian circle

Basic definitions:

x - local south (tangent to the meridian circle)

y - local east (tangent to a parallel)

ϕ - longitude (defines location of a meridian plane relative to plane through Greenwich)

λ - latitude (defines location of a parallel relative to the Equator)

z - local vertical

XYZ - Fixed Frame located at O - center of the earth

More definitions:

OXY - Equatorial plane

OZ - axis of earth's rotation

xyz - attached to the surface of earth at O'

(at latitude - λ ; longitude - ϕ)

$\underline{\omega}$ **- angular velocity of the moving frame**

$$= -\Omega \cos\lambda \, \underline{i} + \Omega \sin\lambda \, \underline{k}$$

Note: $\underline{\omega}$ **is constant** $\rightarrow \dot{\underline{\omega}} = 0$

(for a vector to be constant, both its magnitude and the direction must remain constant w.r.t. the reference frame)

Now, we use the notation already established to define the kinematics:

$\underline{R}$ **- position vector of** O' **to O.**

$\underline{\rho}$ **- position of the mass particle relative to** O' **(the point on earth's surface)**

$$\underline{\rho} = x\underline{i} + y\underline{j} + z\underline{k}$$

$$(\dot{\underline{\rho}})_r \equiv (d\underline{\rho}/dt)_{rel\,to\,xyz} = \dot{x}\,\underline{i} + \dot{y}\,\underline{j} + \dot{z}\,\underline{k}$$

$$(\ddot{\underline{\rho}}) = \ddot{x}\,\underline{i} + \ddot{y}\,\underline{j} + \ddot{z}\,\underline{k}$$

$$\underline{\omega} \times \underline{\rho} = (-\Omega\cos\lambda\,\underline{i} + \Omega\sin\lambda\underline{k}) \times (x\underline{i} + y\underline{j} + z\underline{k})$$

$$= \Omega[-y\sin\lambda\,\underline{i} + (z\cos\lambda + x\sin\lambda)\,\underline{j} - y\cos\lambda\underline{k}]$$

$$\underline{\omega}\times(\underline{\omega}\times\underline{\rho})\equiv\Omega^2[-\sin\lambda(z\cos\lambda+x\sin\lambda)\underline{i}$$

$$-y\,j-\cos\lambda(z\cos\lambda+x\sin\lambda)\underline{k}]$$

(centripetal acceleration)

$$2\underline{\omega}\times(\dot{\underline{\rho}})_r=2(-\Omega\cos\lambda\,\underline{i}+\Omega\sin\lambda\underline{k})\times(\dot{x}\underline{i}+\dot{y}\,\underline{j}+\dot{z}\underline{k})$$

$$=2\Omega[-\dot{y}\sin\lambda\,\underline{i}+(\dot{z}\cos\lambda+\dot{x}\sin\lambda)\,\underline{j}-\dot{y}\cos\lambda\underline{k}]$$

(coriolis acceleration)

Finally, $\underline{\dot{\omega}} \times \underline{\rho} = 0$

Now:

$$\underline{a}_P = \underline{\ddot{R}} + \underline{\dot{\omega}} \times \underline{\rho} + (\underline{\ddot{\rho}})_r + 2\underline{\omega} \times (\underline{\dot{\rho}})_r + \underline{\omega} \times (\underline{\omega} \times \underline{\rho})$$

$$\underline{R} = R\underline{k} \qquad \text{(constant w.r.t. xyz)}$$

$$\underline{\dot{R}} = (d\underline{R}/dt)_{rel\ to\ XYZ} = \underline{\omega} \times \underline{R}$$

$$\underline{\ddot{R}} = \underline{\omega} \times (\underline{\omega} \times \underline{R})$$

$$= \Omega^2 R(-\cos\lambda \sin\lambda\, \underline{i} - \cos^2\lambda\, \underline{k})$$

Note: $\Omega \equiv 7.29\times10^{-5}$ rad/sec

$$\underline{a}_P = \Omega^2 R(-\cos\lambda \sin\lambda\, \underline{i} - \cos^2\lambda\, \underline{k}) + 0$$

$$+ 2\Omega[-\dot{y}\sin\lambda\, \underline{i} + (\dot{z}\cos\lambda + \dot{x}\sin\lambda)\, \underline{j}$$

$$- \dot{y}\cos\lambda \underline{k}]$$

$$+ \Omega^2[-\sin\lambda(z\cos\lambda + x\sin\lambda)\underline{i} - y\, j$$

$$- \cos\lambda(z\cos\lambda + x\sin\lambda)\underline{k}]$$

$$+ \ddot{x}\, \underline{i} + \ddot{y}\, \underline{j} + \ddot{z}\, \underline{k}$$

- **close to earth's surface →** $\Omega^2 x \ll \Omega^2 R,\ etc.$

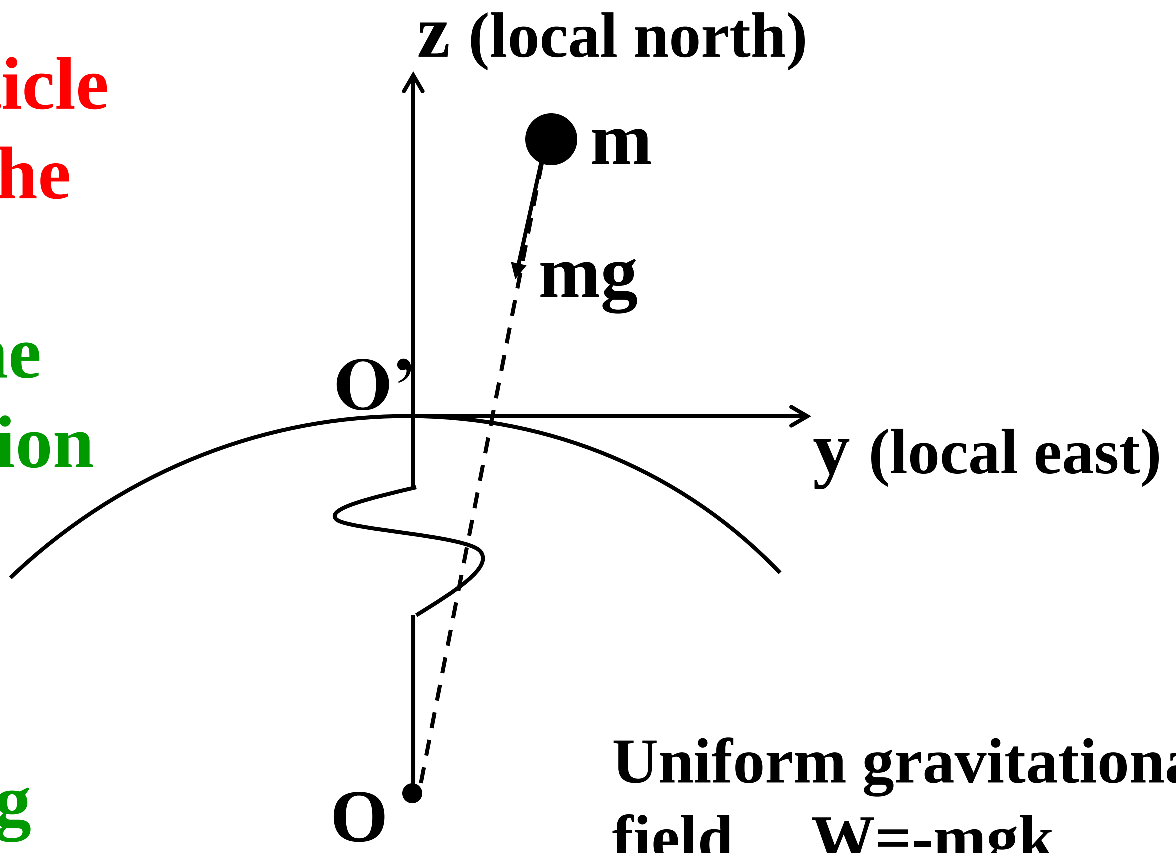
$$\underline{a}_P \approx \Omega^2 R(-\cos\lambda \sin\lambda \, \underline{i} - \cos^2\lambda \, \underline{k})$$
$$+ 2\Omega[-\dot{y}\sin\lambda \, \underline{i} + (\dot{z}\cos\lambda + \dot{x}\sin\lambda)\, \underline{j}$$
$$- \dot{y}\cos\lambda \underline{k}]$$
$$+ \ddot{x}\, \underline{i} + \ddot{y}\, \underline{j} + \ddot{z}\, \underline{k}$$

(for motions near earth's surface).

- **Since $\Omega = 7.29 \times 10^{-5}$, Ω^2 terms are also neglected in study of most motions close to earth's surface. Note that this depends also on the latitude λ of the point O'.**

Ex: Motion of a Particle in Free Fall Near Earth's Surface

- Consider a particle moving close to the earth's surfare;
- We will write the equations of motion using the co-ordinate system attached to the surface of moving earth;



Uniform gravitational field $\underline{W}=-mg\underline{k}$

Newton's 2nd Law: $\Sigma \underline{\mathrm{F}} = \mathrm{m}\underline{\mathrm{a}}_{\mathrm{P}}$
(in an inertial frame)

$$-mg\underline{k} = m\underline{a}_P \quad \rightarrow \quad -g\underline{k} = \underline{a}_P$$

Neglecting Ω^2 terms and air drag →

$$\underline{\underline{i}}: \quad \ddot{x} - 2\Omega\dot{y}\sin\lambda = 0 \qquad (1)$$

$$\underline{\underline{j}}: \quad \ddot{y} + 2\Omega(\dot{x}\sin\lambda + \dot{z}\cos\lambda) = 0 \qquad (2)$$

$$\underline{\underline{k}}: \quad \ddot{z} - 2\Omega\dot{y}\cos\lambda + g = 0 \qquad (3)$$

Note: Ω^2 terms also need to be always neglected in calculations to follow.

Initial conditions are:

x(0) =0, y(0) =0, z(0) =h; $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$

$$(1) \rightarrow \ \dot{x} - 2\Omega y \sin\lambda = cons\tan t = 0 \qquad (4)$$

$$(3) \rightarrow \ \dot{z} - 2\Omega y \cos\lambda + gt = cons\tan t = 0 \quad (5)$$

$$(4),(5)\ in\ (2) \rightarrow$$

$$\ddot{y} + 2\Omega(2\Omega y \sin^2\lambda + 2\Omega y \cos^2\lambda - gt\cos\lambda) = 0$$

$$or \quad \ddot{y} - 2\Omega gt\cos\lambda = 0$$

$$\rightarrow \boxed{y = \Omega g t^3 \cos\lambda / 3} \quad (6)$$

(4), (6) $\rightarrow \ \dot{x} = (2/3)\Omega^2 g t^3 \sin\lambda \cos\lambda \simeq 0$

$$\rightarrow \boxed{x(t) \simeq 0} \qquad (7)$$

(5), (6) $\rightarrow \ \dot{z} = (2/3)\Omega^2 g t^3 \cos^2\lambda - gt \simeq -gt$

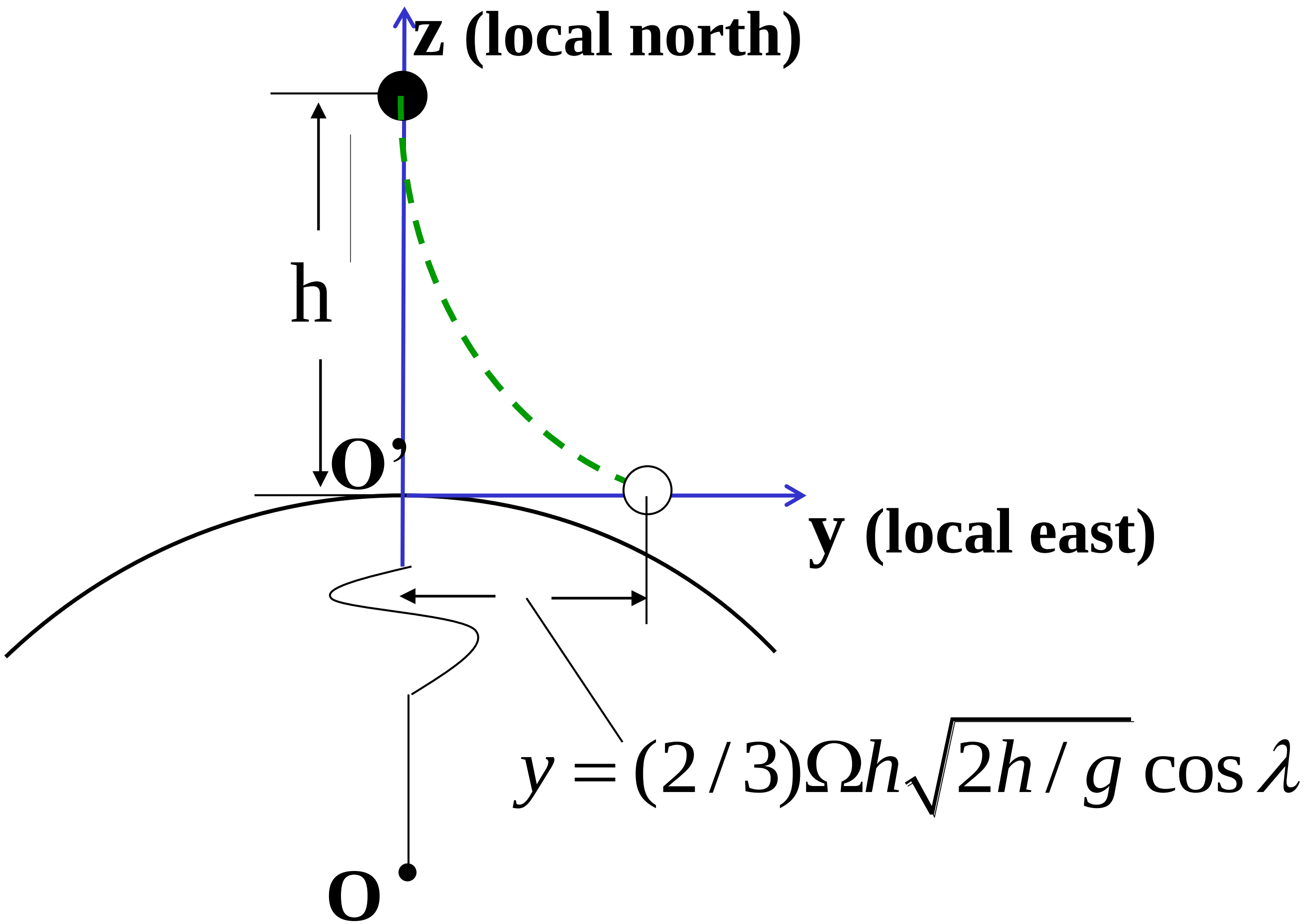
$$\rightarrow \boxed{z(t) = h - gt^2/2} \qquad (8)$$

Time to reach earth's surface: $z = 0 \rightarrow t = \sqrt{2h/g}$

Coordinate of the landing point:

$$\boxed{x = 0, \ \ y = (2/3)\Omega h \sqrt{2h/g}\cos\lambda, \ \ z = 0}$$

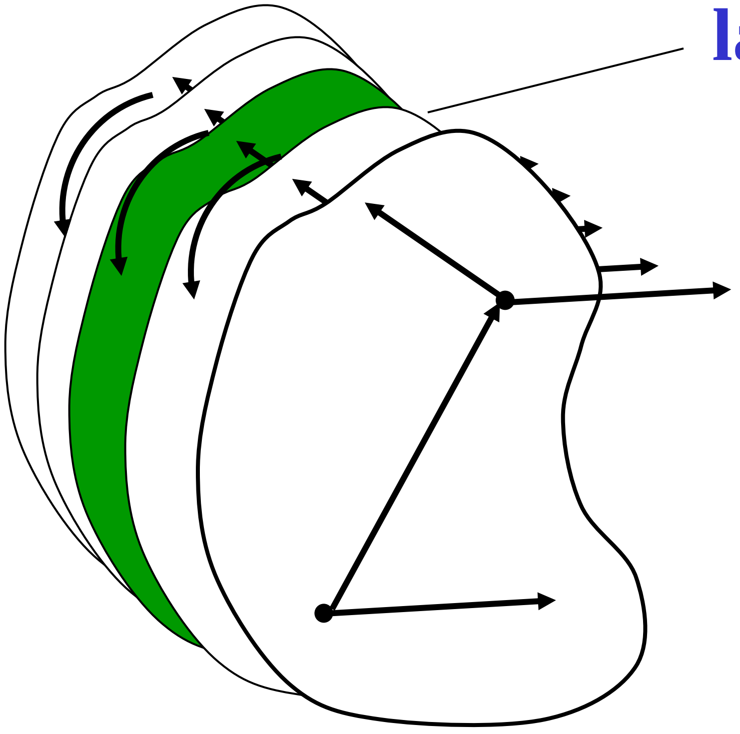
Schematics



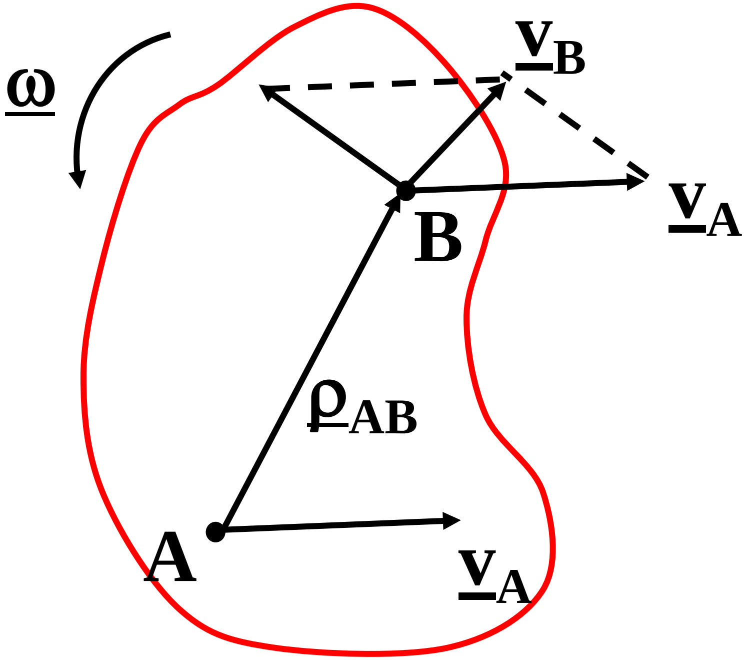
2.10 Plane Motion (motion of a 3-D rigid body in a plane)

- There exists a plane such that every point has velocity and acceleration parallel to this one fixed plane.

lamina of motion



All planes are moving parallel to the the plane colored green. This lamina contains the centroid of the rigid body.



Consider motion in the plane called the ‘lamina of motion’.

Let $\underline{\omega}$ - angular velocity of the body (same for every plane in the body and $\perp^R$ to the lamina of motion).

Let $\underline{v}_A,\ \underline{v}_B$ – velocities of A and B, two points in the lamina. Then $\underline{v}_A = \underline{v}_A + \underline{\omega} \times \underline{\rho}_{AB}$

This relates velocities of two points on the same rigid body.

Question: Does there exist a point such that its velocity is zero, even if only instantaneously ?

- **Suppose that C be such a point: $\underline{v}_C$=0.**

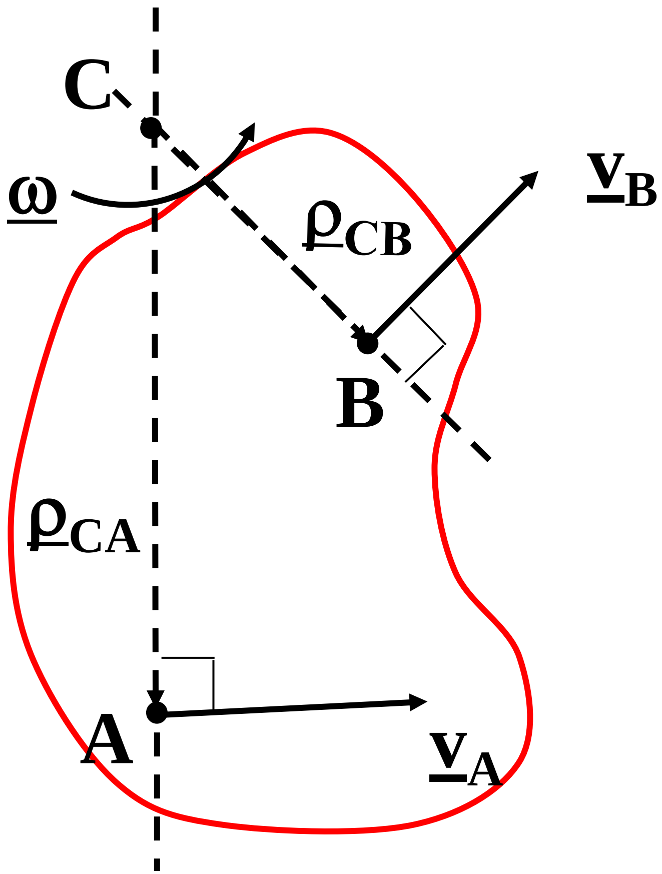
Then, considering points A and B, velocities are:

$$\underline{v}_A = \underline{v}_C + \underline{\omega} \times \underline{\rho}_{CA} \rightarrow \underline{v}_A \perp^r \underline{\rho}_{CA} \quad \text{and}$$

$$\underline{v}_B = \underline{v}_C + \underline{\omega} \times \underline{\rho}_{CB} \rightarrow \underline{v}_B \perp^r \underline{\rho}_{CB}$$

(Assuming that $\underline{v}_C = 0$)

Using these, we can construct and find the point C, as well as the angular velocity $\underline{\omega}$, given the velocities $\underline{v}_A$ and $\underline{v}_B$ for points A and B.



Consider the construction on the left. Points A and B are given with their velocities. Then, one can follow the construction and note that

$\underline{v}_A = \underline{\omega} \times \underline{\rho}_{CA}$ and

$\underline{v}_B = \underline{\omega} \times \underline{\rho}_{CB}$

(Assuming that $\underline{v}_C = 0$)

C - **instantaneous center (of zero velocity)**

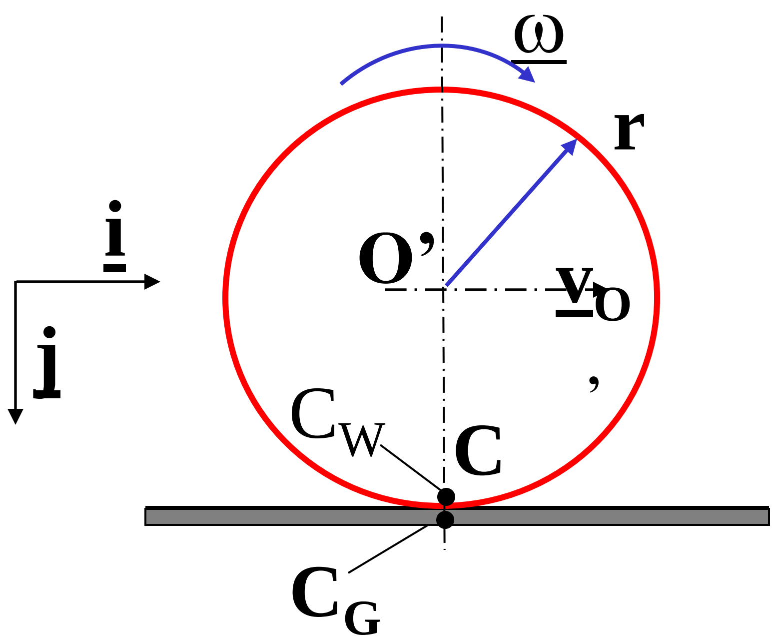
$$\omega = \left|\underline{v}_A\right| / \left|\underline{\rho}_{CA}\right| = \left|\underline{v}_B\right| / \left|\underline{\rho}_{CB}\right|$$

Ex: Rolling Motion: Consider a wheel moving on the fixed ground.

O' - center of wheel

Let: $\underline{v}_{O'}$ - velocity of O'

$\underline{\omega}$ - angular velocity of the wheel



They are not independent in rolling.

- **Two physical points** C_W , C_G **;one belongs to the wheel, the other to the ground on which it is rolling** $\rightarrow$ $\underline{v}_{C_W} = \underline{v}_{C_G}$
- **Ground fixed** $\rightarrow$ $\underline{v}_{C_G} = 0 \rightarrow \underline{v}_{C_W} = 0.$

- **wheel is one rigid body** $\underline{v}_{C_G} = 0 \rightarrow C_W$ **is the instant center (of zero velocity).**
- **Then** $\underline{v}_{O'} = \underline{v}_C + \underline{\omega} \times \underline{r}_{CO'} = \underline{\omega} \times (-r\underline{j})$

$$= \omega r \underline{i} = v_{O'} \underline{i} \rightarrow \boxed{v_{O'} = \omega r}$$

(this is always valid)

differentiating with respect to time $\rightarrow$

$$\dot{v}_{O'} = a_{O'} = \dot{\omega} r$$

- **Let** $\dot{\omega}$ **= angular acceleration of the wheel.**

$$\rightarrow a_{O'} = \dot{\omega} r \text{ or } \boxed{\underline{a}_{O'} = \dot{\omega} r \underline{i} = \alpha r \underline{i}}$$

Now, consider acceleration of the wheel center.

The relation is:

$$\underline{a}_{O'} = \underline{a}_C + \dot{\underline{\omega}} \times \underline{\rho}_{CO'} + \underline{\omega} \times (\underline{\omega} \times \underline{\rho}_{CO'})$$

$$\underline{\rho}_{CO'} = -r\underline{j};$$

$$\dot{\underline{\omega}} \times \underline{\rho}_{CO'} = \alpha \underline{k} \times (-r\underline{j}) = \alpha r \underline{i}$$

$$\rightarrow \alpha r \underline{i} = \underline{a}_C + \alpha r \underline{i} + \underline{\omega} \times (\underline{\omega} \times -r\underline{j})$$

$$\rightarrow \boxed{\underline{a}_C = -\omega \underline{k} \times \omega r \underline{i} = -\omega^2 r\underline{j}}$$

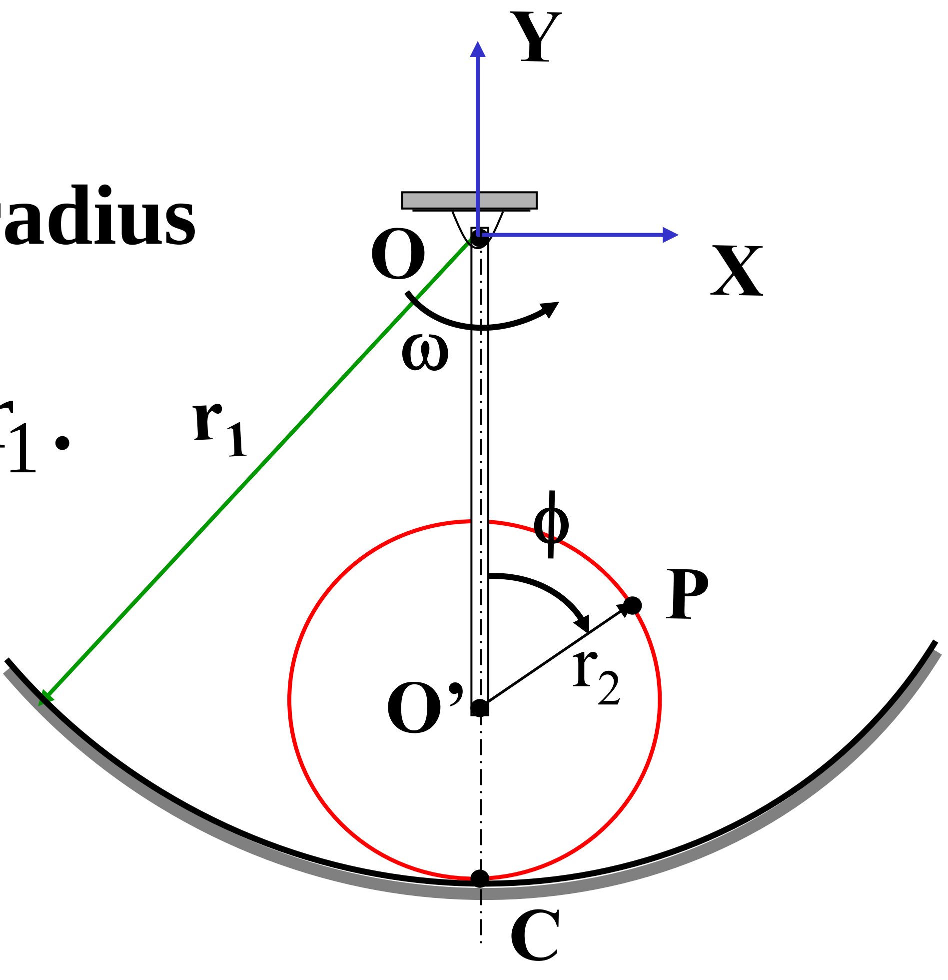
(it is directed towards the wheel center).

Reading assignment:

Examples 2.3, 2.4-2.5, 2.7

Example 2.6

Consider a wheel of radius r_2, rolling inside the fixed track of radius r_1. The arm OO' rotates at a constant angular velocity ω about the fixed point *O'*.

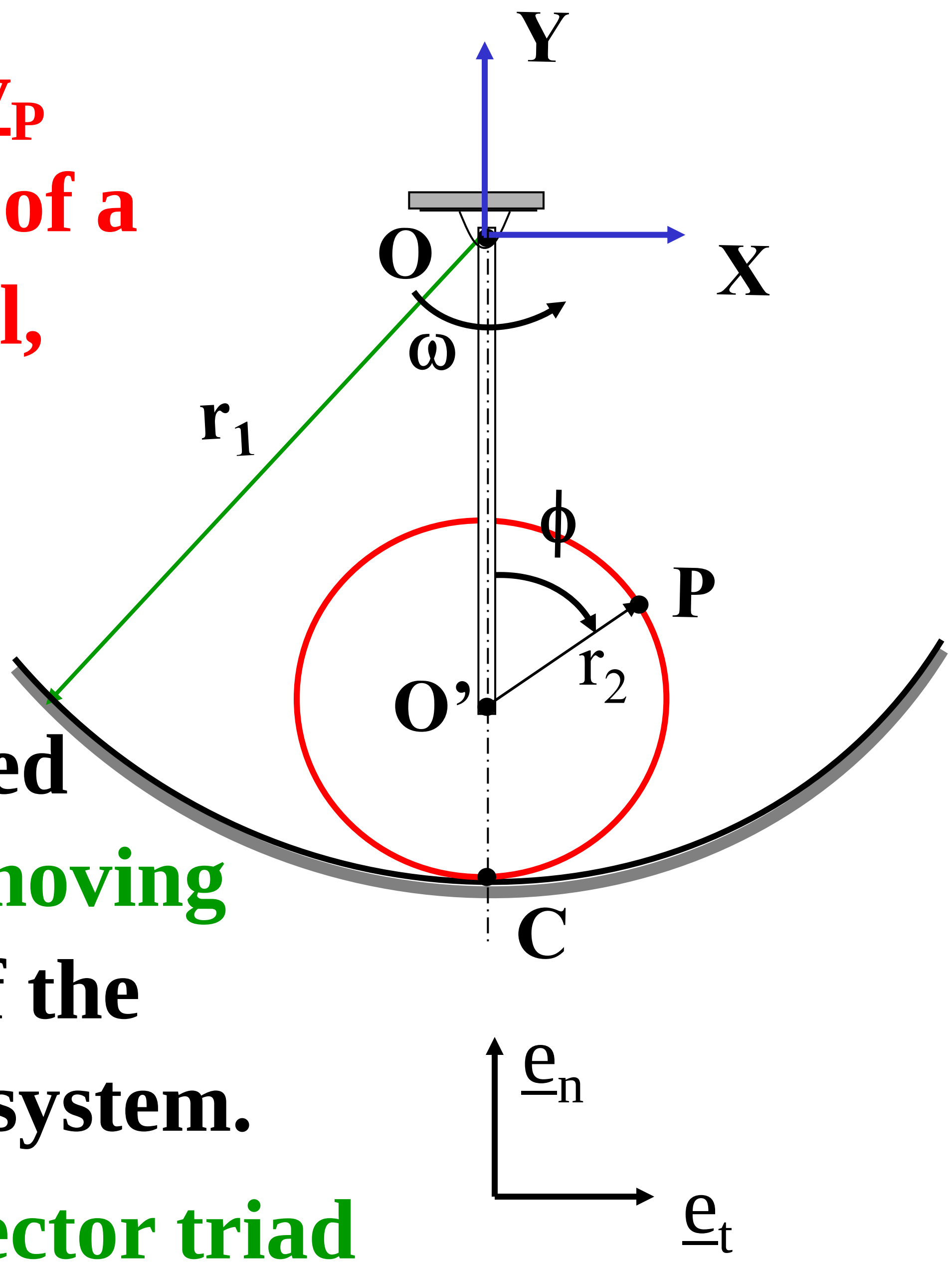


Find: **The velocity $\underline{v}_P$ and acceleration $\underline{a}_P$ of a point P on the wheel, specified by angle ϕ w.r.t. line *OO'* .**

First Method:

O - origin of the fixed frame; Arm is the moving frame; O'- origin of the moving coordinate system.

($\underline{e}_t$, $\underline{e}_n$, $\underline{e}_b$) - a unit vector triad



$$\underline{v}_P = \dot{\underline{R}} + (\dot{\underline{\rho}})_r + \underline{\omega} \times \underline{\rho} = \underline{v}_{O'} + (\dot{\underline{\rho}})_r + \underline{\omega} \times \underline{\rho}$$

where $\dot{\underline{R}} = \underline{v}_{O'}; \quad \underline{\rho} = \underline{r}_{O'P}$

Now $\boxed{\underline{R} = \underline{r}_{OO'} = -\mathrm{R}\underline{\mathrm{e}}_n = -(r_1 - r_2)\,\underline{\mathrm{e}}_n}$

So $\dot{\underline{R}} = -(r_1 - r_2)\dot{\underline{e}}_n = -(r_1 - r_2)\underline{\omega} \times \underline{\mathrm{e}}_n$

$$= -(r_1 - r_2)\omega \underline{e}_b \times \underline{\mathrm{e}}_n = (r_1 - r_2)\omega\,\underline{\mathrm{e}}_t = \underline{v}_{O'}$$

Also $\underline{\rho} = r_2(\cos\phi\,\underline{\mathrm{e}}_n + \sin\phi\,\underline{\mathrm{e}}_t)$

$\rightarrow \quad (\dot{\underline{\rho}})_r = r_2\dot{\phi}(-\sin\phi\,\underline{\mathrm{e}}_n + \cos\phi\,\underline{\mathrm{e}}_t)$

$$\underline{\omega} \times \underline{\rho} = \omega \underline{e}_b \times r_2(\cos\phi\, \underline{e}_n + \sin\phi\, \underline{e}_t)$$

$$= -\omega r_2 \cos\phi\, \underline{e}_t + \omega r_2 \sin\phi\, \underline{e}_n$$

$$\rightarrow \underline{v}_P = (r_1 - r_2)\omega \underline{e}_t + r_2\dot{\phi}(-\sin\phi\, \underline{e}_n + \cos\phi\, \underline{e}_t)$$

$$+\omega r_2(-\cos\phi\, \underline{e}_t + \sin\phi\, \underline{e}_n)$$

Rolling Constraint: $\underline{v}_C \overset{r_2}{} = 0,$

and $P = C$ when $\phi = \pi$

$$\rightarrow \underline{v}_C = (r_1 - r_2)\omega \underline{e}_t + r_2\dot{\phi}\, \underline{e}_t + \omega r_2\, \underline{e}_t = 0$$

or $\omega r_1 = \dot{\phi} r_2 \quad \rightarrow \quad \boxed{\dot{\phi} = \omega r_1 / r_2}$ (always valid)

So, $\underline{v}_P = [(r_1 - r_2) + r_1 \cos\phi - r_2 \cos\phi]\omega \underline{e}_t$

$+[-r_1 \sin\phi + r_2 \sin\phi]\omega \underline{e}_n$

or $\boxed{\underline{v}_P = (r_1 - r_2)\omega\{[1 + \cos\phi]\underline{e}_t - \sin\phi\, \underline{e}_n\}}$

Also, $\dot{\phi} = \omega r_1 / r_2 \rightarrow \boxed{\ddot{\phi} = 0}$

Acceleration:

$$\underline{a}_P = \ddot{\underline{R}} + \dot{\underline{\omega}} \times \underline{\rho} + \underline{\omega} \times (\underline{\omega} \times \underline{\rho}) + (\ddot{\underline{\rho}})_r + 2\underline{\omega} \times (\dot{\underline{\rho}})_r$$

$$\ddot{\underline{R}} = (r_1 - r_2)\omega(\omega \underline{e}_b \times \underline{e}_t) = (r_1 - r_2)\omega^2 \underline{e}_n$$

$$\dot{\underline{\omega}} \times \underline{\rho} = 0$$

Consider the other terms :

$$\underline{\omega}\times(\underline{\omega}\times\underline{\rho}) = \omega\underline{e}_b\times[-\omega r_2\cos\phi\,\underline{e}_t + \omega r_2\sin\phi\,\underline{e}_n]$$

$$= \omega^2 r_2(-\cos\phi\,\underline{e}_n - \sin\phi\,\underline{e}_t)$$

$$(\ddot{\underline{\rho}})_r = r_2\ddot{\phi}(-\sin\phi\,\underline{e}_n + \cos\phi\,\underline{e}_t)$$

$$+ r_2\dot{\phi}(-\dot{\phi}\cos\phi\,\underline{e}_n - \dot{\phi}\sin\phi\,\underline{e}_t)$$

$$= \omega^2 r_1^2(-\cos\phi\,\underline{e}_n - \sin\phi\,\underline{e}_t)/r_2$$

$$2\underline{\omega}\times(\dot{\underline{\rho}})_r = 2\omega\underline{e}_b\times\omega r_1[-\sin\phi\,\underline{e}_n + \cos\phi\,\underline{e}_t]$$

$$= 2\omega^2 r_1(\cos\phi\,\underline{e}_n + \sin\phi\,\underline{e}_t)$$

Thus, the acceleration is

$$\underline{a}_P = (r_1 - r_2)\omega^2 \underline{e}_n + \omega^2 r_2(-\cos\phi\,\underline{e}_n - \sin\phi\,\underline{e}_t)$$

$$+\,\omega^2 r_1^2(-\cos\phi\,\underline{e}_n - \sin\phi\,\underline{e}_t)/r_2$$

$$+\,2\omega^2 r_1(\cos\phi\,\underline{e}_n + \sin\phi\,\underline{e}_t)$$

$$\rightarrow \boxed{\begin{aligned}\underline{a}_P = [(r_1 - r_2)\omega^2 - \omega^2(r_1 - r_2)^2\cos\phi / r_2]\\ -[\omega^2(r_1 - r_2)^2\sin\phi / r_2]\underline{e}_t\end{aligned}}$$

Second Method:

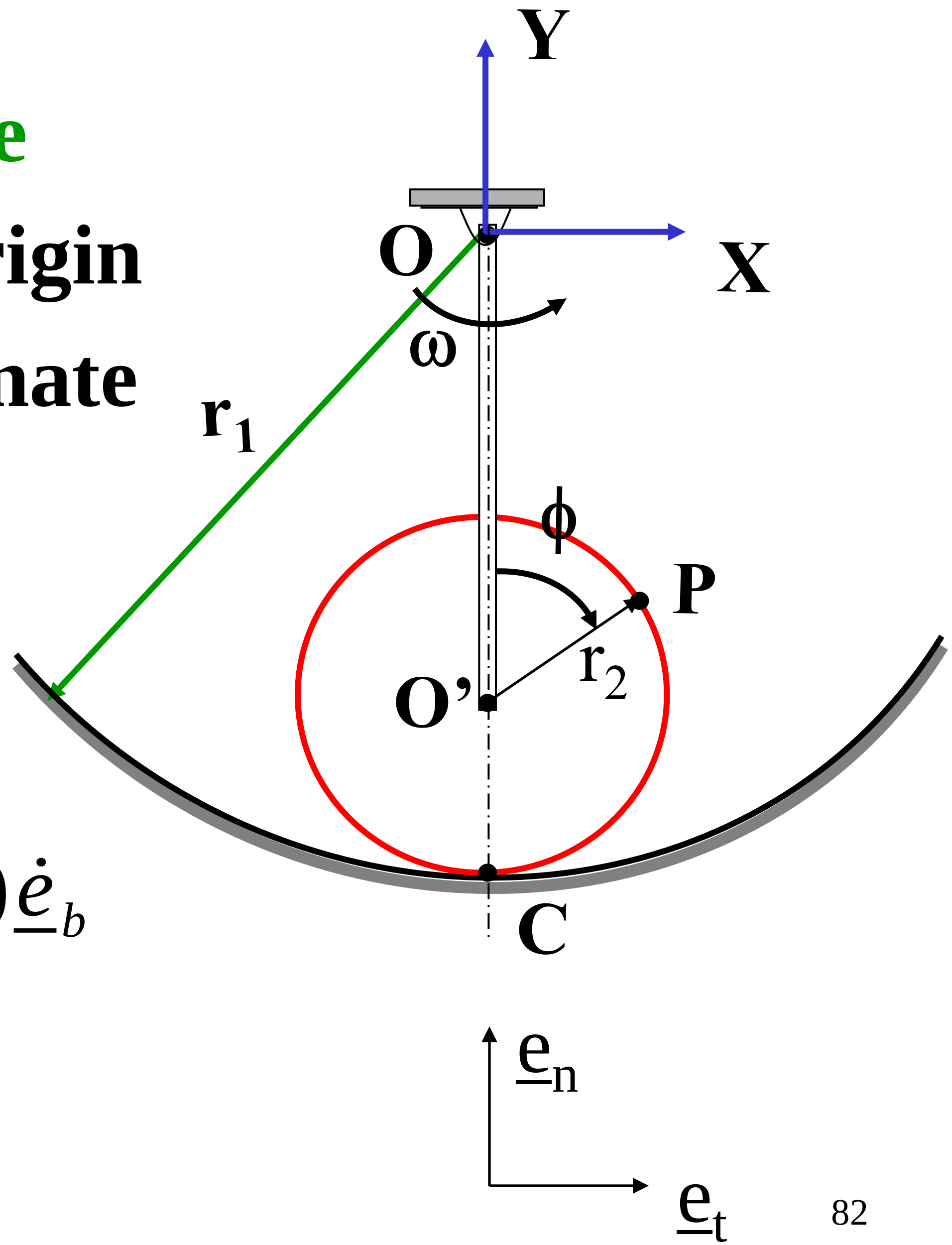
Let wheel serve as the moving frame; O' **- origin of the moving coordinate system, attached to the wheel.**

Then, $\underline{\omega} = (\omega - \dot{\phi})\underline{e}_b$

$$\underline{\alpha} = (\dot{\omega} - \ddot{\phi})\underline{e}_b + (\omega - \dot{\phi})\dot{\underline{e}}_b$$

$$\dot{\underline{e}}_b = \underline{\omega} \times \underline{e}_b = 0$$

$$\rightarrow \underline{\alpha} = \dot{\underline{\omega}} = (\dot{\omega} - \ddot{\phi})\underline{e}_b$$



$$\underline{R} = \underline{r}_{OO'} = -R\underline{e}_n \qquad (Note: \text{O' is on the arm})$$

$$\dot{\underline{R}} = \omega \underline{e}_b \times -R\underline{e}_n = \omega R \underline{e}_t \qquad (R = r_1 - r_2)$$

$$\underline{\rho} = r_2(\cos\phi\, \underline{e}_n + \sin\phi\, \underline{e}_t)$$

$$(\dot{\underline{\rho}})_r = 0 \qquad \text{(moving frame attached to the wheel)}$$

$$\underline{\omega} \times \underline{\rho} = (\omega - \dot{\phi})\underline{e}_b \times r_2(\cos\phi\, \underline{e}_n + \sin\phi\, \underline{e}_t)$$

$$= (\omega - \dot{\phi}) r_2 (\sin\phi\, \underline{e}_n - \cos\phi\, \underline{e}_t)$$

$$\rightarrow \underline{v}_P = \omega(r_1 - r_2)\underline{e}_t + (\omega - \dot{\phi}) r_2 (\sin\phi\, \underline{e}_n - \cos\phi\, \underline{e}_t)$$

Now, the constraint is Rolling

$$\rightarrow \underline{v}_C = 0 \;\; and \;\; P = C \;\; when \;\; \phi = \pi$$

Imposing constraint $\underline{v}_C$=0 →

$$\boxed{\dot{\phi} = \omega r_1 / r_2} \rightarrow \ddot{\phi} = 0 \ (\text{since } \dot{\omega} = 0)$$

$$\rightarrow \boxed{\begin{aligned} \underline{v}_P &= (r_1 - r_2)\{1 + \cos\phi\}\omega \underline{e}_t \\ &+ (r_1 - r_2)\omega \sin\phi \, \underline{e}_n \end{aligned}}$$

Acceleration:

$$(\ddot{\underline{\rho}})_r = 0; \quad \ddot{\underline{R}} = \omega^2 (r_1 - r_2) \underline{e}_n$$

.

.*etc*.

Rate of Change of a vector in a Rotating Reference Frame

- Let $\mathcal{A}$ and $\mathcal{B}$ be two reference frames, $\mathcal{B}$ moves relative to $\mathcal{A}$
- $\{\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3\}$ – orthonormal basis in $\mathcal{A}$
- $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ – orthonormal basis in $\mathcal{B}$

We can express:

$\mathbf{b}=b_1\mathbf{e}_1+b_2\mathbf{e}_2+b_3\mathbf{e}_3$

in the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$