Note on Ignorable Coordinates

Consider a holonomic system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i' \ , \quad i = 1, 2, \ldots, n$$

Suppose that, for some generalized coordinate q_s, the Lagrangian <u>does not</u> depend on q_s (though <u>it does</u> depend on $\dot{q}_s$). Also, $Q_s' = 0$, i.e., there is no generalized force corresponding to that q_s. $\Rightarrow$ $\frac{\partial L}{\partial q_s} = 0$, $Q_s' = 0$

Clearly then, the equation for q_s is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \underbrace{\cancel{\frac{\partial L}{\partial q_s}}}_{=0} = \underbrace{\cancel{Q_s'}}_{=0}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) = \frac{d}{dt}\left(p_s\right) = 0$$

generalized momentum for s^{th} coordinate.

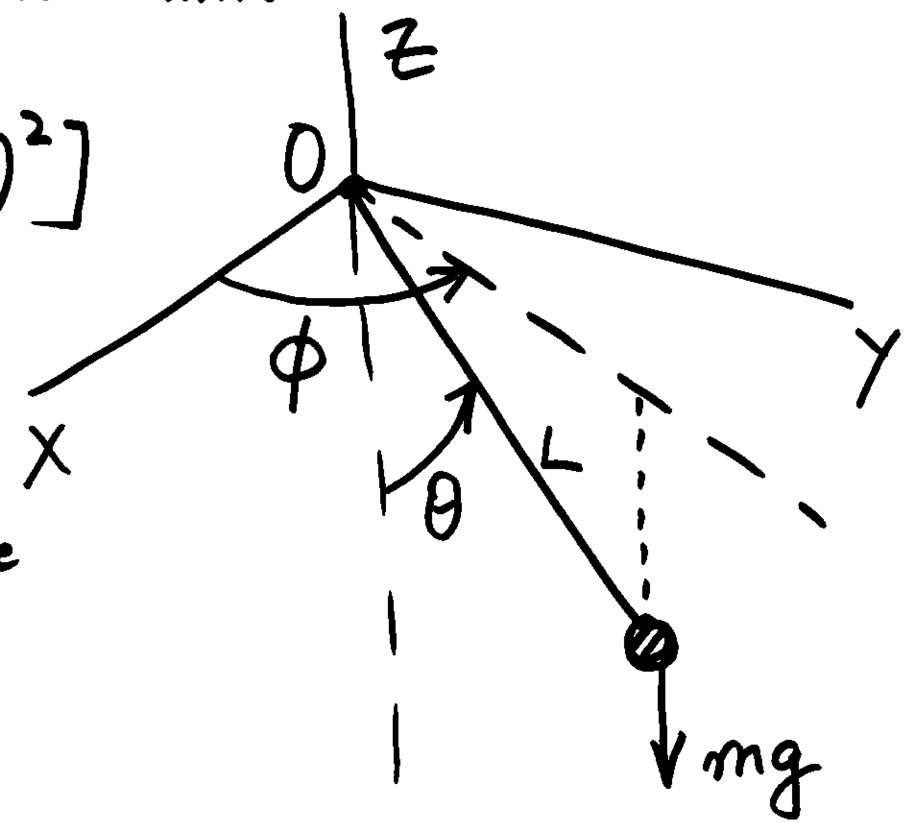
$$\Rightarrow \quad \boxed{p_s = \text{Const.}}$$ – depends on initial conditions.

Since $p_s = \dfrac{\partial L}{\partial \dot{q}_s}$ (function of $q_1, \ldots, q_{s-1}, q_{s+1}, \ldots, q_n,$ $\dot{q}_1, \ldots, \dot{q}_s, \ldots, \dot{q}_n$),

$p_s = \beta_s$ (a const.)

= a combination of other coordinates and velocities.

$\underline{\underline{ex}}$: For the spherical pendulum



$$L = \frac{1}{2} m \left[(L\dot{\theta})^2 + (\dot{\phi} L \sin\theta)^2 \right]$$

$$- mgL(1 - \cos\theta)$$

ϕ – ignorable coordinate

$$\frac{\partial L}{\partial \dot{\phi}} = p_\phi \text{ – const.}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = (mL^2 \sin^2\theta)\,\dot{\phi} \equiv \beta_\phi \quad \text{— Constant.}$$

$$\Rightarrow \quad \dot{\phi} = \beta_\phi / (mL^2 \sin\theta)$$

$\Rightarrow$ $\dot{\phi}$ changes in time along with θ.

<u>In general</u>, let us say, k generalized coordinates are ignorable, i.e.,

$$L(q_{k+1}, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$$

and $Q'_1 = Q'_2 = Q'_3 = \cdots = Q'_k = 0$

(k generalized forces are zero).

Define a new function (<u>Routhian function</u>)

$$R(q_{k+1}, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \beta_1, \ldots, \beta_k, t)$$

$$\equiv L(q_{k+1}, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) - \sum_{i=1}^{k} \beta_i \dot{q}_i \qquad (1)$$

Recall, $\beta_i \equiv p_i = \dfrac{\partial L}{\partial \dot{q}_i}$, $i = 1, 2, \ldots, k$ (2)

for the k ignorable coordinates.

Now, consider a variation in R:

$$\delta R = \sum_{i=k+1}^{n} \frac{\partial R}{\partial q_i}\delta q_i + \sum_{i=k+1}^{n} \frac{\partial R}{\partial \dot{q}_i}\delta \dot{q}_i + \sum_{i=1}^{k} \frac{\partial R}{\partial \beta_i}\delta \beta_i + \frac{\partial R}{\partial t}\delta t \qquad (3)$$

Similarily, for L,

$$\delta L = \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{k} \frac{\partial L}{\partial \dot q_i}\delta \dot q_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \dot q_i}\delta \dot q_i + \frac{\partial L}{\partial t}\delta t \qquad (4)$$

Also, (2) $\Rightarrow$

$$\delta\left(\sum_{i=1}^{k} \beta_i \dot q_i\right) = \sum_{i=1}^{k} \frac{\partial L}{\partial \dot q_i}\delta \dot q_i + \sum_{i=1}^{k} \dot q_i \,\delta\beta_i \qquad (5)$$

so that

$$\delta\left(L - \sum_{i=1}^{k} \beta_i \dot q_i\right) = \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \dot q_i}\delta \dot q_i$$

$$- \sum_{i=1}^{k} \dot q_i \,\delta\beta_i + \frac{\partial L}{\partial t}\delta t \qquad (6)$$

Comparing (3) and (6) $\Rightarrow$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial R}{\partial \dot{q}_i} \quad , \quad \frac{\partial L}{\partial q_i} = \frac{\partial R}{\partial q_i} \tag{7}$$

$$i = k+1, \ldots, n$$

and
$$\dot{q}_i = -\frac{\partial R}{\partial \beta_i} \quad , \quad i = 1, 2, \ldots, k \tag{8}$$

$$\frac{\partial L}{\partial t} = \frac{\partial R}{\partial t}$$

Then, Lagrange's equations for q_i, $i = k+1, \ldots, n$ become

$$\boxed{\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = Q_i' \quad , \quad i = k+1, \ldots, n.} \tag{9}$$

for the <u>non-ignorable</u> coordinates

Then, for the ignorable coordinates:

$$\left.\begin{aligned} \dot{q}_i &= -\frac{\partial R}{\partial \beta_i}, \quad i = 1, \ldots, k \\ \text{and} \quad q_i(t) &= -\int \frac{\partial R}{\partial \beta_i}\, dt \end{aligned}\right\} \quad (10)$$

ex Spherical pendulum:

$$R = L - \beta_\phi \dot{\phi}$$

$$= \frac{1}{2} m \left[(L\dot{\theta})^2 + \dot{\phi}^2 L^2 \sin^2\theta \right] - mgL(1 - \cos\theta)$$

$$- \beta_\phi \left[\beta_\phi / (mL^2 \sin^2\theta) \right]$$

$$= \frac{1}{2} m \left[L^2 \dot{\theta}^2 + \left\{ \beta_\phi^2 / (mL^2 \sin^2\theta)^2 \right\} L^2 \sin^2\theta \right]$$

$$- mgL(1 - \cos\theta) - \beta_\phi^2 / (mL^2 \sin^2\theta)$$

$$R = \frac{1}{2} m L^2 \dot{\theta}^2 + \frac{1}{2} \cancel{m}\, \beta_\phi^2 \frac{\cancel{L^2 \sin^2\theta}}{m^{\cancel{2}} (L^2 \sin^2\theta)^{\cancel{2}}}$$

$$- mgL(1-\cos\theta) - \beta_\phi^2 / (m L^2 \sin^2\theta)$$

or

$$\boxed{R = \frac{1}{2} m L^2 \dot{\theta}^2 - \frac{1}{2} \beta_\phi^2 / (m L^2 \sin^2\theta) - mgL(1-\cos\theta)}$$

The equation of motion is then

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) - \frac{\partial R}{\partial \theta} = 0$$

only one equation for θ, with β_ϕ – a const.