8. MORE RIGID BODY DYNAMICS

- General equations:

$$\sum \underline{F} = m\ddot{\underline{r}}_C \,, \qquad \sum \underline{M}_P = \frac{d\underline{H}_P}{dt} + \underline{\rho}_C \times m\ddot{\underline{r}}_P$$

- In the last chapter, we derived the relations:

$$\underline{H}_P = H_x \underline{i} + H_y \underline{j} + H_z \underline{k}$$

where $(\underline{i}, \underline{j}, \underline{k})$ is an orthogonal triad set located at the point P. In component form, we have

$$H_x = I_{xx}\omega'_x + I_{xy}\omega'_y + I_{xz}\omega'_z$$

$$H_y = I_{yx}\omega'_x + I_{yy}\omega'_y + I_{yz}\omega'_z$$

$$H_z = I_{zx}\omega'_x + I_{zy}\omega'_y + I_{zz}\omega'_z$$

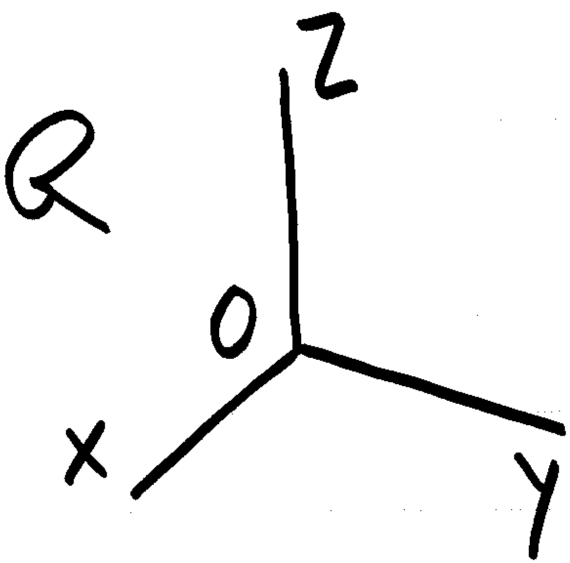
or, in vector-matrix notation:

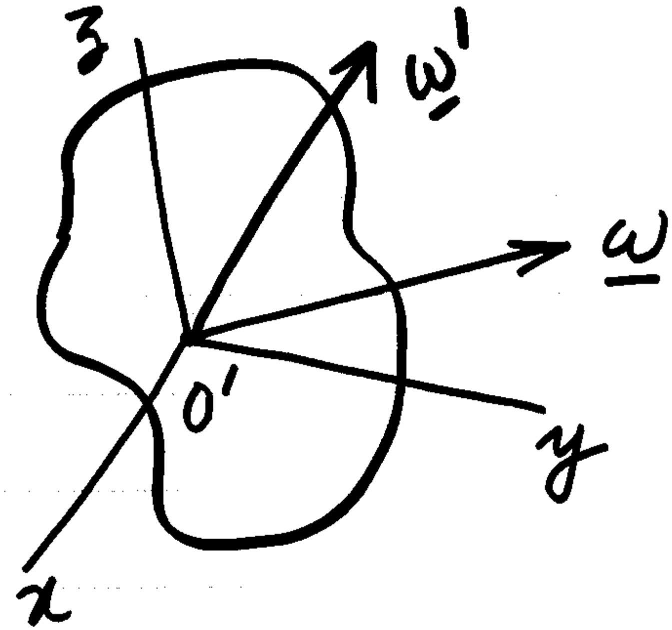
$$\{H\} = [I]\{\omega'\}$$

where $\underline{\omega'} = \omega'_x \underline{i} + \omega'_y \underline{j} + \omega'_z \underline{k}$ is the <u>angular velocity of the rigid body</u> with respect to the inertial reference frame,where it is expressed in terms of the unit vectors of the triad located at P.

- Now, we need the rate of change of angular momentum with respect the inertial frame in the equation of motion: $\dfrac{d\underline{H}_P}{dt} = \left(\dfrac{d\underline{H}_P}{dt}\right)_{rel} + \underline{\omega} \times \underline{H}_P$

(this holds for any vector when expressed in a rotating frame of reference; $\underline{\omega}$ is the angular velocity of the rotating frame defined by the triad $(\underline{i}, \underline{j}, \underline{k})$)





thus, $\underline{\omega}$- ang. vel. of the rigid body w.r.t. the XYZ,

$\underline{\omega'}$- ang. vel. of the moving frame (not attached to the body, i.e., not a body-fixed reference frame).

- Note: in this situation, in the quantity $\left(\frac{d\underline{H}_P}{dt}\right)_{rel}$ both, the inertia matrix components (defined by the orientation of the body relative to the xyz system), and the components of the angular velocity vector relative to xyz system ($\omega'_x, \omega'_y, \omega'_z$) change. To avoid the added complexity of the components of $[I]$ changing, one usually sets $\underline{\omega} = \underline{\omega}'$,i.e. the moving frame $(\underline{i}, \underline{j}, \underline{k})$is attached (fixed) to the body.

then $\{H\} = [I]\{\omega\}$

and $\left(\frac{d\underline{H}_P}{dt}\right)_{rel}$ can be written in components

form as $\{\dot{H}\}_{Rel} = [I]\{\dot{\omega}\} = [I]\{\alpha\}$ where $\{\alpha\}$ consists of the components of the angular acceleration vector $\underline{\alpha}$ for the rigid body, expressed in terms of the rotating frame of reference $(\underline{i}, \underline{j}, \underline{k})$.

In explicit form, we can write

$$\left.\begin{aligned}\dot{H}_x &= I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z \\ \dot{H}_y &= I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \\ \dot{H}_z &= I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z\end{aligned}\right\} \Leftarrow$$

We also need to calculate the term

$$\underline{\omega} \times \underline{H}_P = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_x & \omega_y & \omega_z \\ H_x & H_y & H_z \end{vmatrix}$$

Combining, the <u>equations for rotational motion of a rigid body</u> (with P as a fixed point or the center of mass of the body)take the form

$$\left.\begin{aligned}
M_x &= I_{xx}\dot{\omega}_x + I_{xy}(\dot{\omega}_y - \omega_x\omega_z) + I_{xz}(\dot{\omega}_z + \omega_x\omega_y) \\
&\quad + (I_{zz} - I_{yy})\omega_y\omega_z + I_{yz}(\omega_y^2 - \omega_z^2) \\
M_y &= I_{yy}\dot{\omega}_y + I_{yz}(\dot{\omega}_z - \omega_x\omega_y) + I_{xy}(\dot{\omega}_x + \omega_y\omega_z) \\
&\quad + (I_{xx} - I_{zz})\omega_x\omega_z + I_{xz}(\omega_z^2 - \omega_x^2) \\
M_z &= I_{zz}\dot{\omega}_z + I_{xz}(\dot{\omega}_x - \omega_y\omega_z) + I_{yz}(\dot{\omega}_y + \omega_x\omega_z) \\
&\quad + (I_{yy} - I_{xx})\omega_x\omega_y + I_{xy}(\omega_x^2 - \omega_y^2)
\end{aligned}\right\} \quad (8\text{-}11)$$

Equations of motion for rotational motion

$$\Sigma \underline{M}_P = \frac{d}{dt}\underline{H}_P + 0$$

P – a fixed pt.
or G.

Now, suppose that the body fixed axes $(\underline{i},\underline{j},\underline{k})$ are chosen to be the principal axes of the rigid body

i.e., in the xyz system

$$I_{xy} = I_{xz} = I_{yz} = 0$$

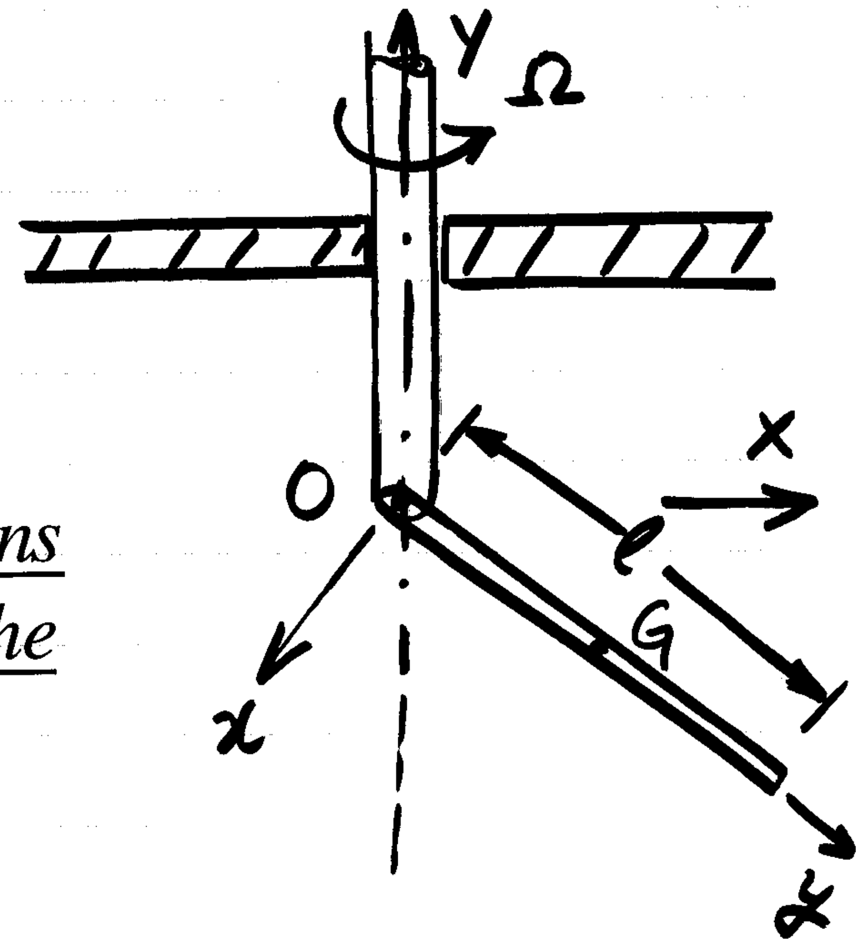
Then, the equations for rotational motion take the simplified form:

$$\left.\begin{aligned}
M_x &= I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z \\
M_y &= I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z \\
M_z &= I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y
\end{aligned}\right\} \quad (8\text{-}12)$$

These are Euler's equations for a rigid body

Ex 8.2: This example illustrates the use of the Euler's equations of motion for a rigid body, and the concepts of steady motions and their linearized stability.

Consider a rod of length *l*, pinned to the vertical spindle at O, and rotating about a space fixed vertical axis. *Determine the equations of motion for the rod, find the steady state motions, and evaluate their stability*

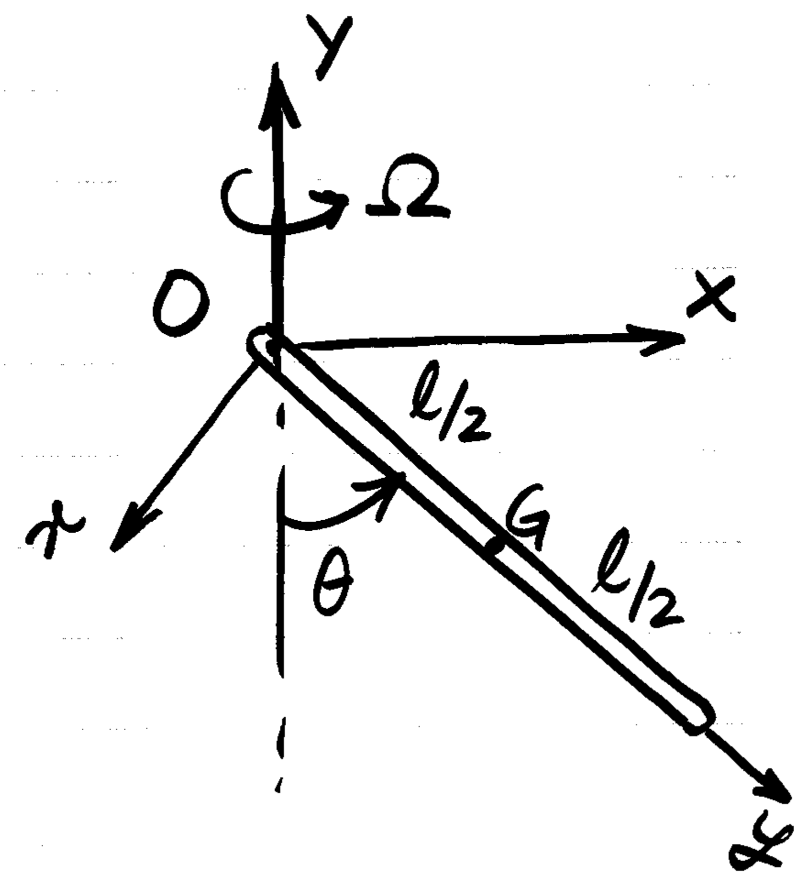


First step: define a fixed system XYZ
define a moving system xyz and its origin;

Recall: It is useful to have the moving coordinate system attached to the rigid body (i.e., the body and the moving coordinate system have the same angular velocity with respect to the XYZ system). Also, it is convenient if the xyz system is the principal system for the body-the products of inertia will be then zero.

XYZ and xyz systems are chosen, as shown:

- xyz is attached to the rod - its orientation changes in time; they are principal axes of the rod.
- The rod has only one degree-of-freedom; denote it by angle θ



$$I_{xx} = I_{zz} = ml^2/3,\ \ I_{yy} = 0$$

$$\underline{\omega} = \Omega \underline{J} + \dot{\theta}\underline{k} \quad (\underline{always})$$

$$\underline{\alpha} = \underline{\dot{\omega}} = \dot{\Omega}\underline{J} + \ddot{\theta}\underline{k} + \dot{\theta}(\underline{\omega} \times \underline{k})$$

Now

$$\underline{J} = -\cos\theta\,\underline{j} - \sin\theta\,\underline{i} \Rightarrow \underline{\omega} = -\Omega\cos\theta\,\underline{j} - \Omega\sin\theta\,\underline{i} + \dot{\theta}\,\underline{k}$$

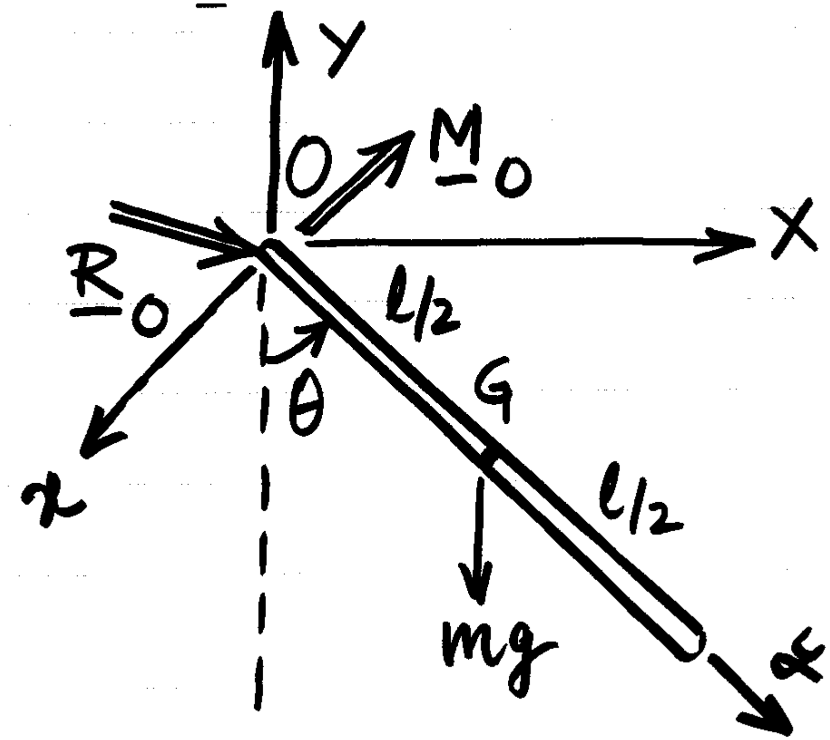
Thus

$$\underline{\alpha} = \ddot{\theta}\,\underline{k} + \dot{\theta}(-\Omega\cos\theta\,\underline{j} - \Omega\sin\theta\,\underline{i} + \dot{\theta}\,\underline{k}) \times \underline{k}$$

$$= \ddot{\theta}\,\underline{k} + \dot{\theta}(-\Omega\cos\theta\,\underline{i} + \Omega\sin\theta\,\underline{j})$$

$$= -\dot{\theta}\Omega\cos\theta\,\underline{i} + \dot{\theta}\Omega\sin\theta\,\underline{j} + \ddot{\theta}\,\underline{k}$$

Now, consider the FBD:



$$\underline{R}_O = R_x\,\underline{i} + R_y\,\underline{j} + R_z\,\underline{k}$$

$$\underline{M}_O = M_x\,\underline{i} + M_y\,\underline{j} + 0\,\underline{k}$$

- Euler's equations:

$$M_x = I_{xx}(-\Omega\dot{\theta}\cos\theta) + I_{zz}(-\Omega\dot{\theta}\cos\theta)$$

$$or \quad M_x = -\frac{2}{3}ml^2\,\Omega\dot{\theta}\cos\theta \qquad (1)$$

$$M_y = (0)\dot{\omega}_y + (0)\omega_x\omega_z = 0 \qquad (2)$$

$$M_z = -mg\frac{l}{2}\sin\theta = \frac{1}{3}ml^2\ddot{\theta} - \frac{1}{3}ml^2(\Omega\dot{\theta}\sin\theta)(-\Omega\dot{\theta}\cos\theta)$$

$$or \quad \boxed{\ddot{\theta} + (\frac{3g}{2l} - \Omega^2\cos\theta)\sin\theta = 0} \qquad (3)$$

z-equation
$\perp^r$ to plane

Consider small oscillations about the vertical:

$$|\theta| << 1 \Rightarrow \sin\theta = \theta - \theta^3/3! + \ldots, \cos\theta = 1 - \theta^2/2! + \ldots,$$

(3) $\Rightarrow\ \ddot{\theta}+[\frac{3g}{2l}-\Omega^2(1-\frac{\theta^2}{2}+\ldots.)](\theta-\frac{\theta^3}{6}+\ldots.)=0$

Neglecting nonlinear terms $\Rightarrow$

$$\ddot{\theta}+(\frac{3g}{2l}-\Omega^2)\theta=0$$

This equation has <u>harmonic solutions</u> with <u>oscillation frequency</u>

$$\omega_n=\sqrt{(\frac{3g}{2l}-\Omega^2)}$$

<u>Instability</u> when $\quad \omega_n=0 \quad or \quad \Omega^2=3g/2l$

What to expect when the angular rotation rate exceeds $\Omega>\sqrt{3g/2l}$?

- Equilibrium solutions

$$\ddot{\theta} = 0 \ - \ \textit{steady conical motion}$$

$$\Rightarrow \ \left(\frac{3g}{2l} - \Omega^2 \cos\theta_0\right)\sin\theta_0 = 0$$

$$(a) \quad \sin\theta_0 = 0 \quad \Rightarrow \boxed{\theta_0 = 0 \ \ or \ \ \pi}$$

$$(b) \quad \cos\theta_0 = \frac{3g}{2l\Omega^2} \quad \Rightarrow \boxed{\theta_0 = \pm\cos^{-1}\left(\frac{3g}{2l\Omega^2}\right)}$$

This steady motion exists only when $\dfrac{3g}{2l\Omega^2} \le 1$

$or \ \ \Omega^2 \ge \dfrac{3g}{2l}$ *(compare with condition for stability of* $\theta_0 = 0$*)*

- <u>Linearize around a steady motion:</u>

(small oscillations around θ_0)

$Let \quad \theta = \theta_0 + \phi$

$$\Rightarrow \sin(\theta_0 + \phi) = \sin\theta_0 \cos\phi + \cos\theta_0 \sin\phi$$

$$= \sin\theta_0 (1 - \phi^2/2! + \ldots) + \cos\theta_0 (\phi - \phi^3/3! + \ldots)$$

$$Also \ \cos(\theta_0 + \phi) = \cos\theta_0 \cos\phi - \sin\theta_0 \sin\phi$$

$$= \cos\theta_0 (1 - \phi^2/2! + \ldots) - \sin\theta_0 (\phi - \phi^3/3! + \ldots)$$

$\Rightarrow$

$$\ddot{\phi} + [\frac{3g}{2l} - \Omega^2 \{\cos\theta_0 - \sin\theta_0 (\phi - \ldots)\}] \bullet$$

$$[\sin\theta_0 + \phi\cos\theta_0 + \ldots] = 0$$

$$or \quad \ddot{\phi} + [\frac{3g}{2l} - \Omega^2 \cos\theta_0]\cos\theta_0\phi + \Omega^2 \sin\theta_0\phi = 0$$

$$(retain\ only\ linear\ terms)$$

$$or \quad \ddot{\phi} + [\frac{3g}{2l}\cos\theta_0 - \Omega^2(\cos^2\theta_0 - \sin^2\theta_0)]\phi = 0$$

$$For\ \theta_0\ given\ by \qquad \cos\theta_0 = 3g/2l\Omega^2$$

$$\cos^2\theta_0 = (3g/2l\Omega^2)^2, \quad \sin^2\theta_0 = 1 - (3g/2l\Omega^2)^2$$

$$\Rightarrow \quad \cos^2\theta_0 - \sin^2\theta_0 = 2(3g/2l\Omega^2)^2 - 1$$

$$(3g/2l)\cos\theta_0 = (3g/2l)^2/\Omega^2$$

$$(3g/2l)\cos\theta_0 - \Omega^2(\cos^2\theta_0 - \sin^2\theta_0)$$

$$= (\frac{3g}{2l})^2 \frac{1}{\Omega^2} - 2(\frac{3g}{2l})^2 \frac{1}{\Omega^2} + \Omega^2$$

$$= \Omega^2 - (\frac{3g}{2l})^2 \frac{1}{\Omega^2}$$

$\Rightarrow$ *the equation for small motions around* θ_0 *is*

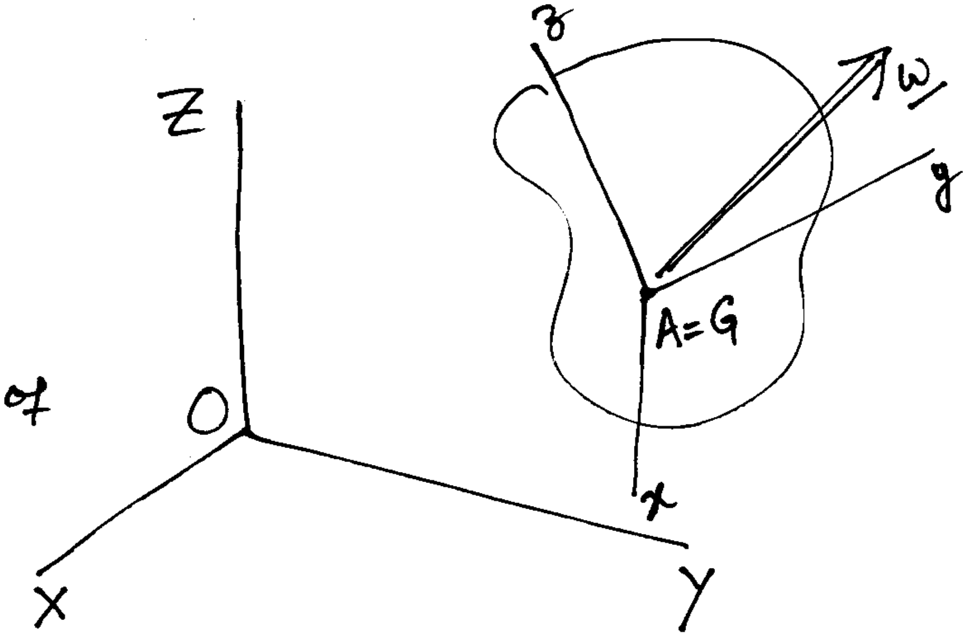
$$\ddot{\phi} + [\Omega^2 - (\frac{3g}{2l})^2 \frac{1}{\Omega^2}]\phi = 0$$

$\Rightarrow$ *the frequency of small oscillations around* θ_0 *is*

$$\omega_n = \sqrt{\Omega^2 - (\frac{3g}{2l})^2 \frac{1}{\Omega^2}}$$

Recall :

- Consider a rigid body rotating under the action of forces and moments.



- xyz - body-fixed principal axes - products of inertia absent.

- Euler's equations :

$$M_x = I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\,\omega_y\,\omega_z$$

$$M_y = I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\,\omega_z\,\omega_x$$

$$M_z = I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\,\omega_x\,\omega_y$$

Body-axis Translational Motion :

$$\underline{F} = m\dot{\underline{v}} \quad , \quad \underline{v} \text{ - absolute velocity of } G.$$

Let $\underline{F} = F_x \underline{i} + F_y \underline{j} + F_z \underline{k}$

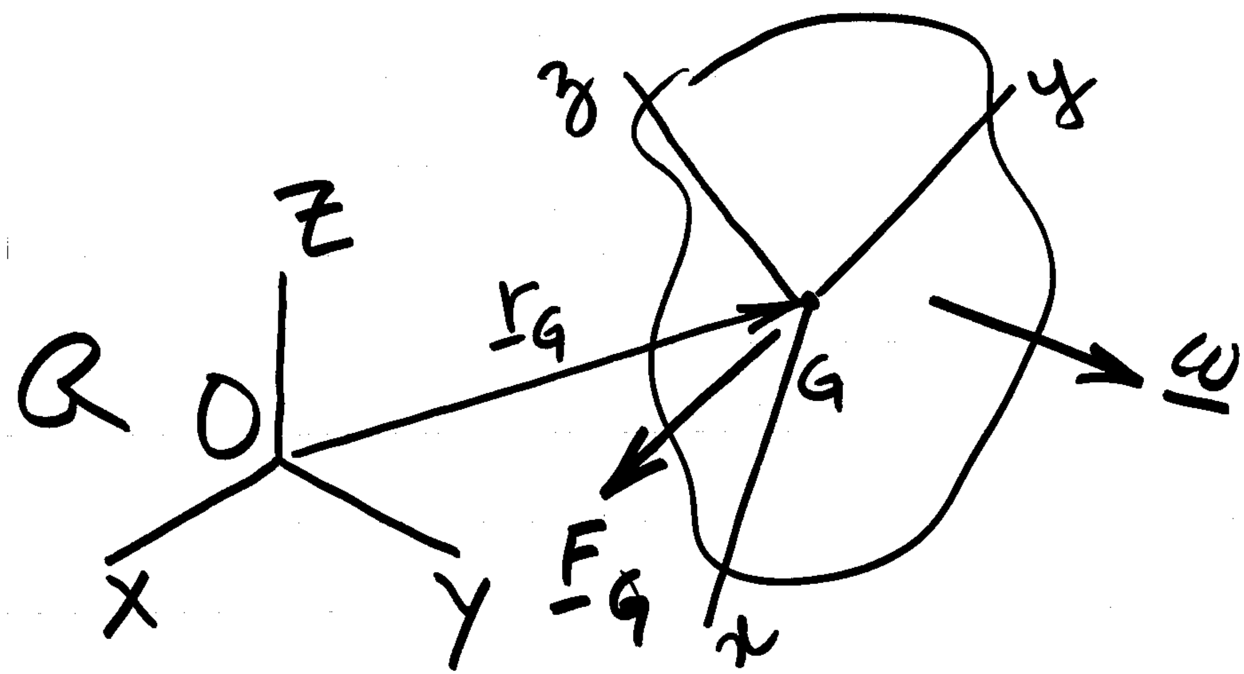
$$\underline{v} = v_x \underline{i} + v_y \underline{j} + v_z \underline{k}$$

$$\dot{\underline{v}} = (\dot{\underline{v}})_r + \underline{\omega} \times \underline{v} \quad ; \quad (\dot{\underline{v}})_r = \dot{v}_x \underline{i} + \dot{v}_y \underline{j} + \dot{v}_z \underline{k}$$

$$\Rightarrow \boxed{\begin{aligned} F_x &= m(\dot{v}_x + v_z \omega_y - v_y \omega_z) \\ F_y &= m(\dot{v}_y + v_x \omega_z - v_z \omega_x) \\ F_z &= m(\dot{v}_z + v_y \omega_x - v_x \omega_y) \end{aligned}}$$

$\Rightarrow$ Translational and rotational equations are coupled.

Stability of Rotational motion



Only force acting - through the centroid G.

- No external torques $\Rightarrow M_x = M_y = M_z = 0$

xyz - principal axes

I_{xx}, I_{yy}, I_{zz} - Principal inertias.

Euler equations for rotational motion

$$
\left.
\begin{aligned}
I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\,\omega_y\,\omega_z &= 0 \quad (1)\\
I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\,\omega_x\,\omega_z &= 0 \quad (2)\\
I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\,\omega_x\,\omega_y &= 0 \quad (3)
\end{aligned}
\right\}
$$

Steady-state :

$$\omega_x = \Omega, \quad \omega_y = \omega_z = 0$$

Stability :

Let

$$\omega_x = \Omega + \eta_x$$

$$\omega_y = 0 + \eta_y$$

$$\omega_z = 0 + \eta_z$$

perturbations about the steady state $(\Omega, 0, 0)$

$$(1) \Rightarrow I_{xx}(\dot{\Omega} + \dot{\eta}_x) + (I_{zz} - I_{yy})(0 + \eta_y)(0 + \eta_z)$$

$$= 0$$

or $$I_{xx}\dot{\eta}_x + (I_{zz} - I_{yy})\eta_y \eta_z = 0 \qquad (4)$$

$$(2) \Rightarrow I_{yy}\dot{\eta}_y + (I_{xx} - I_{zz})(\Omega + \eta_x)(0 + \eta_z) = 0$$

or $$I_{yy}\dot{\eta}_y + (I_{xx} - I_{zz})(\Omega + \eta_x)\eta_z = 0 \qquad (5)$$

(3) $\Rightarrow$

$$I_{zz}\dot{\eta}_z + (I_{yy} - I_{xx})(\Omega + \eta_x)\eta_y = 0 \qquad (6)$$

Small perturbations: retain smallest perturbation terms $\Rightarrow$ drop η_x^2, η_y^2, η_z^2, $\eta_x\eta_y$, $\eta_x\eta_z$, $\eta_y\eta_z$ etc

(4) $\Rightarrow$ $\quad I_{xx}\dot{\eta}_x = 0 \quad$ (7) — linear <u>approx.</u>

$I_{xx} \neq 0 \Rightarrow \quad \eta_x(t) = \text{constant}$

$$(5) \Rightarrow \quad I_{yy}\dot{\eta}_y + (I_{xx} - I_{zz})\,\Omega\,\eta_z = 0$$

$$(6) \Rightarrow \quad I_{zz}\dot{\eta}_z + (I_{yy} - I_{xx})\,\Omega\,\eta_y = 0$$

Combining $\Rightarrow$

$$\ddot{\eta}_y + \left[\frac{(I_{zz} - I_{xx})(I_{yy} - I_{xx})\Omega^2}{I_{yy}\, I_{zz}}\right]\eta_y = 0$$

$$\ddot{\eta}_z + \left[\frac{(I_{zz} - I_{xx})(I_{yy} - I_{xx})\Omega^2}{I_{yy}\, I_{zz}}\right]\eta_z = 0$$

or $\ddot{\eta} + \omega_n^2 \eta = 0$

where
$$\omega_n^2 = \frac{(I_{zz} - I_{xx})(I_{yy} - I_{xx})\Omega^2}{I_{yy} I_{zz}}.$$

<u>if $\omega_n^2 > 0$</u>

$$\eta(t) = \eta(0)\cos\omega_n t + \frac{\dot{\eta}(0)}{\omega_n}\sin\omega_n t$$

(Oscillatory motion)

$\omega_n^2 > 0 \Rightarrow$

$$(I_{zz} - I_{xx})(I_{yy} - I_{xx}) > 0$$

$\Rightarrow$ $I_{xx} > I_{yy}$ <u>and</u> $I_{xx} > I_{zz}$

(biggest inertia)

or

$I_{xx} < I_{yy}$ <u>and</u> $I_{xx} < I_{zz}$

(Smallest inertia)