ME 562 Advanced Dynamics
Summer 2010
HOMEWORK # 7

Due: August 2, 2010

Q1. (see Problem 6-24 in the text). A disk of radius R rotates about its fixed vertical axis of symmetry at a constant rate ω. A simple pendulum of length l and particle mass m is attached at a point on the edge of the disk. As generalized coordinates, let θ be the angle of the pendulum from the downward vertical, and let ϕ be the angle between the vertical plane of the pendulum and the vertical plane of the radial line from the center of the disk to the attachment point, where positive $\dot{\phi}$ is in the same sense as ω.

1. Find T_2, T_1, and T_0.
2. Obtain the differential equations of motion.
3. Assuming that $R = l$, $\omega^2 = g/2l$, and the initial conditions are $\theta(0) = 0$, $\dot{\theta}(0)=0$, show that θ cannot exceed 72.93°.

Q2. (see Problem 6-11 in the text for a figure). Consider the system shown. It consists of particles m_1 and m_2 that are connected by a massless rod of length l. These particles move on a frictionless horizontal plane, the motion of m_1 being confined to a fixed frictionless circular track of radius R. Denote the generalized coordinates for describing the dynamics of the system by the angles θ and ϕ. Then, derive the following:

1. The expression for kinetic energy of the system.
2. The equations of motion of the system for the coordinates θ and ϕ.
3. Identify the ignorable coordinate in the system, and define the generalized momentum p associated with it.
4. Define the Routhian function for the system and use it to derive the equation of motion for the reduced one degree-of-freedom system.

Q3. (see Problem 7-2 the text). Four particles, each of mass $m/4$, are located at $(x_0, 0, 0)$, $(-x_0/3, y_0, -z_0/2)$, $(-x_0/3, -y_0, -z_0/2)$, $(-x_0/3, 0, z_0)$, relative to the xyz axes. The particles are connected by rigid massless rods. Solve for the values of x_0, y_0, and z_0 such that the system has principal moments of inertia I_{xx}, I_{yy}, and I_{zz}.

Q4. (see Problem 7-6 in the text). A certain rigid body has the following inertia matrix with respect to some reference point:

$$[I] = \begin{bmatrix} 450 & -60 & 100 \\ -60 & 500 & 7 \\ 100 & 7 & 550 \end{bmatrix} kg-m^2$$

a. Solve for the principal moments of inertia at the same reference point.
b. Find the rotation matrix such that one can transform (rotate) from the xyz system to the principal coordinate system.

7-13

Q5. (see Problem ~~#9~~ in the text). A solid homogeneous sphere of mass m_0 and radius r rolls without slipping on a triangular block of mass m which can slide on a frictionless floor. Assume that the system is initially motionless.

a. Derive the equations of motion for the system.
b. Find the velocity of the block as a function of time.

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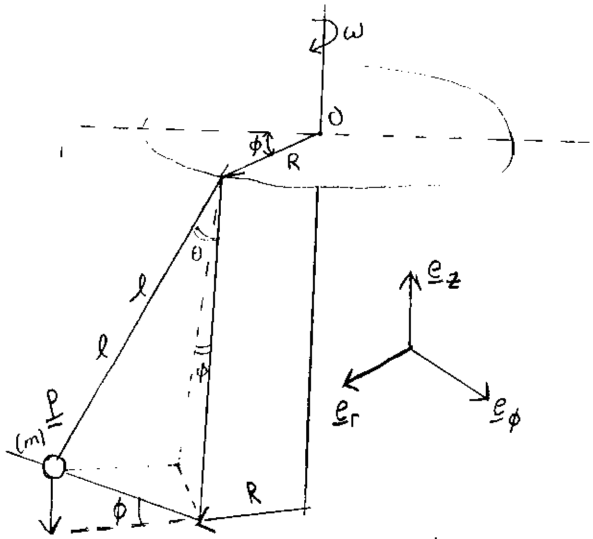
$\boxed{Q1}$



(i) In cylindrical coordinates,

$$\underline{r}_{OP} = -l\cos\theta\, \underline{e}_z + (l\sin\theta\cos\phi + R)\,\underline{e}_R + (l\sin\theta\sin\phi)\,\underline{e}_\phi$$

$$\dot{\underline{r}}_{OP} = (l\dot\theta\sin\theta)\underline{e}_z + [l\sin\theta(-\sin\phi)\dot\phi + l\cos\theta(\cos\phi)\dot\theta]\underline{e}_r + (l\sin\theta\cos\phi + R)(\omega)(\underline{e}_z\times\underline{e}_r)$$

$$+\ \underline{e}_\phi[l\cos\theta(\dot\theta)\sin\phi + l\sin\theta\cos\phi\,\dot\phi] + (\omega)(\underline{e}_z\times\underline{e}_\phi)(l\sin\theta\sin\phi)$$

$$= l\dot\theta\sin\theta\,\underline{e}_z + (-l\dot\phi\sin\theta\sin\phi + l\dot\theta\cos\theta\cos\phi)\underline{e}_r + (\omega l\sin\theta\cos\phi + \omega R)\underline{e}_\phi$$

$$+\ (l\dot\theta\cos\theta\sin\phi + l\dot\phi\sin\theta\cos\phi)\underline{e}_\phi + (\omega l\sin\theta\sin\phi)(-\underline{e}_r)$$

$$\dot{\underline{r}}_{OP} = (l\dot\theta\sin\theta)\underline{e}_z + [-(\omega+\dot\phi)l\sin\theta\sin\phi + l\dot\theta\cos\theta\cos\phi]\underline{e}_r + [(\omega+\dot\phi)l\sin\theta\cos\phi$$

$$+\ l\dot\theta\cos\theta\sin\phi + \omega R]\underline{e}_\phi$$

$$\dot{\underline{r}}_{OP}\cdot\dot{\underline{r}}_{OP} = (l\dot\theta\sin\theta)^2 + [-(\omega+\dot\phi)l\sin\theta\sin\phi + l\dot\theta\cos\theta\cos\phi]^2 + [(\omega+\dot\phi)l\sin\theta\cos\phi + l\dot\theta\cos\theta\sin\phi + \omega R]^2$$

$$= l^2\dot\theta^2 + (\omega+\dot\phi)^2 l^2\sin^2\theta + \omega^2R^2 + 2(\omega+\dot\phi)l^2\dot\theta\sin\theta\sin\phi\cos\theta\cos\phi + 2\omega Rl(\omega+\dot\phi)\sin\theta\cos\phi$$

$$+\ 2\omega Rl\dot\theta\cos\theta\sin\phi - 2(\omega+\dot\phi)l^2\dot\theta\sin\theta\cos\phi\cos\theta\sin\phi$$

$$= l^2\dot\theta^2 + \omega^2R^2 + (\omega+\dot\phi)^2l^2\sin^2\theta + 2\omega Rl(\omega+\dot\phi)\sin\theta\cos\phi + 2\omega Rl\dot\theta\cos\theta\sin\phi$$

$$\therefore\ K.E. = T = \frac{1}{2}m\,\dot{\underline{r}}_{OP}\cdot\dot{\underline{r}}_{OP} =$$

$$\boxed{T = \frac{1}{2}m[l^2\dot\theta^2 + \omega^2R^2 + (\omega+\dot\phi)^2l^2\sin^2\theta + 2\omega Rl(\omega+\dot\phi)\sin\theta\cos\phi + 2\omega Rl\dot\theta\cos\theta\sin\phi]}$$

Thus expression for T_2, T_1, and T_0 are

$$\boxed{\begin{aligned} T_2 &= \frac{1}{2}ml^2(\dot\theta^2 + \dot\phi^2\sin^2\theta) \\ T_1 &= m\omega l^2\left(\dot\phi\sin^2\theta + \frac{R\dot\theta}{l}\cos\theta\sin\phi + \frac{R\dot\phi}{l}\sin\theta\cos\phi\right) \\ T_0 &= \frac{1}{2}m\omega^2(l^2\sin^2\theta + 2lR\sin\theta\cos\phi + R^2) \end{aligned}}$$

(ii) $\frac{\partial T}{\partial \dot\theta} = \frac{1}{2} m\left[2\ell^2\dot\theta + 2\omega R\ell\cos\theta\sin\phi\right] = m(\ell^2\dot\theta + \omega R\ell\cos\theta\sin\phi)$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = m\left[\ell^2\ddot\theta + \omega R\ell(\dot\phi\cos\theta\cos\phi - \dot\theta\sin\theta\sin\phi)\right]$$

$$\frac{\partial T}{\partial \theta} = \frac{1}{2}m\left[(\omega+\dot\phi)^2\ell^2\cdot 2\sin\theta\cos\theta + 2\omega R\ell(\omega+\dot\phi)\cos\theta\cos\phi - 2\omega R\ell\dot\theta\sin\theta\sin\phi\right]$$

$$\frac{\partial T}{\partial \dot\phi} = \frac{1}{2}m\left[2(\omega+\dot\phi)\ell^2\sin^2\theta + 2\omega R\ell\sin\theta\cos\phi\right]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) = m\left(\ell^2\ddot\phi\sin^2\theta + \ell^2\dot\theta(\omega+\dot\phi)\sin\theta\cos\theta + R\ell\omega(\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi)\right)$$

$$\frac{\partial T}{\partial \phi} = m\omega R\ell(\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi - \omega\sin\theta\sin\phi)$$

$\underline{V = -mg\ell\cos\theta}$

$$\frac{\partial V}{\partial \theta} = mg\ell\sin\theta \quad \& \quad \frac{\partial V}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0$$

$\therefore$ For θ:

$$\boxed{m\ell^2\ddot\theta - \frac{1}{2}m\ell(\omega+\dot\phi)^2\sin 2\theta - m\omega^2 R\ell\cos\theta\cos\phi + mg\ell\sin\theta = 0}$$

For ϕ:

$$\boxed{m\ell^2\ddot\phi\sin^2\theta + m\ell^2\dot\theta(\omega+\dot\phi)\sin 2\theta + mR\ell\omega^2\sin\theta\sin\phi = 0}$$

(iii) $R = \ell$

$\omega^2 = g/2\ell$

$\theta(0) = 0$

$\dot\theta(0) = 0$

Lagrangian is independent of time,

$T_2 - T_0 + V = \text{constant}$

$$\therefore \frac{1}{2}m\ell^2(\dot\theta^2 + \dot\phi^2\sin^2\theta) - \frac{1}{2}m\omega^2(\ell^2\sin^2\theta + 2\ell R\sin\theta\cos\phi + R^2) + mg\ell\cos\theta = \text{constant}$$

$$\frac{1}{2}m\ell^2(\dot\theta^2 + \dot\phi^2\sin^2\theta) - \frac{1}{2}m\omega^2(\ell^2\sin^2\theta + 2\ell R\sin\theta\cos\phi) + mg\ell\cos\theta = -mg\ell$$

Putting $R=l$ and $\omega^2 = \frac{g}{2l}$

$$\therefore \frac{1}{2}ml^2(\dot\theta^2+\dot\phi^2\sin^2\theta) - mgl\left(\frac{\sin^2\theta}{4}+\frac{\sin\theta\cos\phi}{2}\right) - mgl\cos\theta = -mgl$$

Now when $\dot\theta = 0$ & $\theta = \theta_{max}$ as potential energy decreases with θ

$\theta_{max} \Rightarrow V_{max}$

Now $(T_2 - T_0) + V_{max} = \text{constant}$

$\therefore$ we have to look for $(T_2-T_0)_{min}$ i.e. $(T_2)_{min} + (-T_0)_{min}$

$\Rightarrow (T_2)_{min}$ occurs only when $\dot\theta = 0$ and $\dot\phi = 0$

$\therefore \dot\phi = 0 \Rightarrow$ assumption

$(-T_0)_{min}$ occurs for any value at a provided $\cos\phi = 1$ i.e. $\phi = 0°$

$$\therefore -mgl\left(\frac{\sin^2\theta}{4}+\frac{\sin\theta}{2}\right) - mgl\cos\theta = -mgl$$

$$\frac{\sin^2\theta}{4} + \frac{\sin\theta}{2} + \cos\theta = 1 \Rightarrow \frac{\sin^2\theta}{4} + \frac{\sin\theta}{2} - 1 = \cos\theta$$

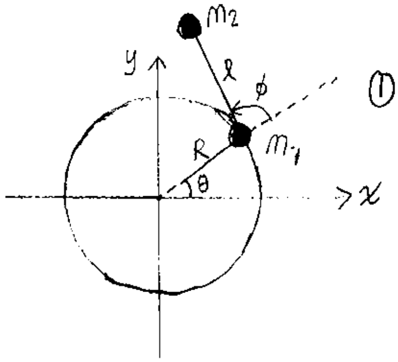
$$\frac{\sin^4\theta}{16} + \frac{\sin^2\theta}{4} + \cancel{1} + 2\frac{\sin^3\theta}{8} - \frac{2\sin^2\theta}{4} - \frac{2\sin\theta}{2} = \cos^2\theta = \cancel{1} - \sin^2\theta$$

$$\frac{1}{16}\sin^4\theta + \frac{1}{4}\sin^3\theta + \frac{3}{4}\sin^2\theta - \sin\theta = 0$$

$$\therefore \sin^3\theta + 4\sin^2\theta + 12\sin\theta - 16 = 0 \Rightarrow \sin\theta = 0.955939$$

$$\Rightarrow \boxed{\theta_{max} = 72.93°}$$

Q2



① $\underline{r}_1 = R\cos\theta\,\underline{i} + R\sin\theta\,\underline{j}$

$\dot{\underline{r}}_1 = -R\dot{\theta}\sin\theta\,\underline{i} + R\dot{\theta}\cos\theta\,\underline{j}$

$\dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1 = R^2\dot{\theta}^2$

$\underline{r}_2 = [R\cos\theta + \ell\cos(\theta+\phi)]\underline{i} + [R\sin\theta + \ell\sin(\theta+\phi)]\underline{j}$

$\dot{\underline{r}}_2 = [-R\dot{\theta}\sin\theta + \ell(-\sin(\theta+\phi))(\dot{\theta}+\dot{\phi})]\underline{i} + [R\dot{\theta}\cos\theta + \ell\cos(\theta+\phi)(\dot{\theta}+\dot{\phi})]\underline{j}$

$\dot{\underline{r}}_2 \cdot \dot{\underline{r}}_2 = R^2\dot{\theta}^2 + \ell^2(\dot{\theta}+\dot{\phi})^2 + 2R\ell(\dot{\theta}+\dot{\phi})\dot{\theta}\cos\phi$

$K.E. = T = T_1 + T_2$

$$\boxed{T = \frac{1}{2}m_1R^2\dot{\theta}^2 + \frac{1}{2}m_2[R^2\dot{\theta}^2 + \ell^2(\dot{\theta}+\dot{\phi})^2 + 2R\ell(\dot{\theta}+\dot{\phi})\dot{\theta}\cos\phi]}$$

② $\frac{\partial T}{\partial \theta} = 0$

$\frac{\partial T}{\partial \dot{\theta}} = m_1R^2\dot{\theta} + \frac{1}{2}m_2[2R^2\dot{\theta} + 2\ell^2(\dot{\theta}+\dot{\phi}) + 2R\ell\cos\phi(2\dot{\theta}+\dot{\phi})]$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = m_1R^2\ddot{\theta} + \frac{1}{2}m_2[2R^2\ddot{\theta} + 2\ell^2(\ddot{\theta}+\ddot{\phi}) + 2R\ell\cos\phi(2\ddot{\theta}+\ddot{\phi})$
$-2R\ell\dot{\phi}\sin\phi(2\dot{\theta}+\dot{\phi})]$

$= m_1R^2\ddot{\theta} + m_2[R^2\ddot{\theta} + \ell^2(\ddot{\theta}+\ddot{\phi}) + R\ell\cos\phi(2\ddot{\theta}+\ddot{\phi}) - R\ell\dot{\phi}\sin\phi(2\dot{\theta}+\dot{\phi})]$

$\frac{\partial T}{\partial \phi} = \frac{1}{2}m_2[-2R\ell\dot{\theta}(\dot{\theta}+\dot{\phi})\sin\phi] = -m_2R\ell\dot{\theta}(\dot{\theta}+\dot{\phi})\sin\phi$

$\frac{\partial T}{\partial \dot{\phi}} = \frac{1}{2}m_2[2\ell^2(\dot{\theta}+\dot{\phi}) + 2R\ell\dot{\theta}\cos\phi] = m_2[\ell^2(\dot{\theta}+\dot{\phi}) + R\ell\dot{\theta}\cos\phi]$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = m_2[\ell^2(\ddot{\theta}+\ddot{\phi}) + R\ell\ddot{\theta}\cos\phi - R\ell\dot{\theta}\dot{\phi}\sin\phi]$

– Equation of motion –

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i$

As motion is in a horizontal plane (frictionless) with an external force applied: $\delta W = \sum Q_i\,\delta q = 0$ and $Q_i = 0$.

For θ: $\boxed{(m_1+m_2)R^2\ddot{\theta} + m_2[\ell^2(\ddot{\theta}+\ddot{\phi}) + R\ell\cos\phi(2\ddot{\theta}+\ddot{\phi}) - R\ell\dot{\phi}\sin\phi(2\dot{\theta}+\dot{\phi})] = 0}$

For ϕ: $m_2 \ell^2(\ddot\theta+\ddot\phi) + m_2 R\ell\ddot\theta\cos\phi - m_2 R\ell\dot\theta\dot\phi\sin\phi + m_2 R\ell\dot\theta(\dot\theta+\dot\phi)\sin\phi = 0$

$$\boxed{m_2\ell^2(\ddot\theta+\ddot\phi) + m_2 R\ell\ddot\theta\cos\phi + m_2 R\ell\dot\theta^2\sin\phi = 0}$$

③ Lagrangian $L = T - V$

$$L = \frac{1}{2}\left[R^2\dot\theta^2(m_1+m_2) + m_2\ell^2(\dot\theta+\dot\phi)^2 + 2m_2 R\ell\dot\theta(\dot\theta+\dot\phi)\cos\phi\right] - 0$$

Lagrangian is independent of θ so θ is the ignorable coordinate

Thus $p = \frac{\partial L}{\partial\dot\theta} = \frac{\partial T}{\partial\dot\theta} = (m_1+m_2)R^2\dot\theta + m_2\ell^2(\dot\theta+\dot\phi) + m_2R\ell(2\dot\theta+\dot\phi)\cos\phi$

$$\boxed{P_\theta = (m_1+m_2)R^2\dot\theta + m_2\ell^2(\dot\theta+\dot\phi) + m_2R\ell(2\dot\theta+\dot\phi)\cos\phi}$$

④ Routhian = $R = L - \beta_\theta\dot q_\theta$

$$\boxed{R = \frac{1}{2}\left\{R^2\dot\theta^2(m_1+m_2) + m_2\ell^2(\dot\theta+\dot\phi)^2 + 2m_2R\ell\dot\theta(\dot\theta+\dot\phi)\cos\phi\right\} - \beta_\theta\dot\theta}$$

Routhian function defined as $R = L - \beta_s\dot q_s$

β_s = constant value of generalized moment $= \beta_\theta$

$\dot q_s = \dot\theta$

$$P_\theta = \beta_\theta = (m_1+m_2)R^2\dot\theta + m_2\ell^2(\dot\theta+\dot\phi) + m_2R\ell(2\dot\theta+\dot\phi)\cos\phi$$

$$= \left[(m_1+m_2)R^2 + m_2\ell^2 + 2m_2R\ell\cos\phi\right]\dot\theta + m_2\ell^2\dot\phi + m_2R\ell\dot\phi\cos\phi$$

$$\dot\theta = \frac{\beta_\theta - m_2(\ell^2 + 2R\ell\cos\phi)\dot\phi}{(m_1+m_2)R^2 + m_2\ell^2 + 2m_2R\ell\cos\phi}$$

restart;

$eqt1 := \frac{1}{2}\cdot[R^2\cdot thetadot^2\cdot(m1+m2) + m2\cdot l^2\cdot(thetadot+phidot)^2 + 2\cdot m2\cdot R\cdot l\cdot thetadot\cdot(thetadot + phidot)\cdot\cos(phi2) - \mathrm{beta}\cdot thetadot];$ $\mathcal{R}$

$$\left[\frac{1}{2}R^2\,thetadot^2\,(m1+m2) + \frac{1}{2}m2\,l^2\,(thetadot+phidot)^2 + m2\,R\,l\,thetadot\,(thetadot + phidot)\cos(\phi 2) - \frac{1}{2}\beta\,thetadot\right] \quad (1)$$

$thetadot := \left(\frac{\mathrm{beta} - m2\cdot(l^2 + 2\cdot R\cdot l\cdot\cos(phi2))\cdot phidot}{(m1+m2)\cdot R^2 + m2\cdot l^2 + 2\cdot m2\cdot R\cdot l\cdot\cos(phi2)}\right)$

$$\frac{\beta - m2\,(l^2 + 2\,R\,l\cos(\phi 2))\,phidot}{(m1+m2)\,R^2 + m2\,l^2 + 2\,m2\,R\,l\cos(\phi 2)} \quad (2)$$

$\dot{\theta} = \text{fn of}(\dot{\phi})$

$eqt2 := simplify(eqt1)$

$$\left[\frac{1}{2}\,\frac{m2\,phidot\,l^2\,(\beta + phidot\,R^2\,m1 + phidot\,R^2\,m2)}{R^2\,m1 + R^2\,m2 + m2\,l^2 + 2\,m2\,R\,l\cos(\phi 2)}\right] \quad (3)$$

simplified $\mathcal{R}$ from (1)

$eqt3 := \frac{\partial}{\partial phi2}\,eqt2$

$$\left[\frac{m2^2\,phidot\,l^3\,(\beta + phidot\,R^2\,m1 + phidot\,R^2\,m2)\,R\sin(\phi 2)}{(R^2\,m1 + R^2\,m2 + m2\,l^2 + 2\,m2\,R\,l\cos(\phi 2))^2}\right] \quad (4)$$

$\frac{\partial \mathcal{R}}{\partial \phi}$

$eqt4 := \frac{\partial}{\partial phidot}\,eqt2$

$$\left[\frac{1}{2}\,\frac{m2\,l^2\,(\beta + phidot\,R^2\,m1 + phidot\,R^2\,m2)}{R^2\,m1 + R^2\,m2 + m2\,l^2 + 2\,m2\,R\,l\cos(\phi 2)} + \frac{1}{2}\,\frac{m2\,phidot\,l^2\,(R^2\,m1 + R^2\,m2)}{R^2\,m1 + R^2\,m2 + m2\,l^2 + 2\,m2\,R\,l\cos(\phi 2)}\right] \quad (5)$$

$\frac{\partial \mathcal{R}}{\partial \dot{\phi}}$

$phi2 := \mathrm{phi}(t)$

$$\phi(t) \quad (6)$$

$phidot := \frac{d}{dt}\,\mathrm{phi}(t)$

$$\frac{d}{dt}\,\phi(t) \quad (7)$$

$\frac{d}{dt}\left(\frac{\partial \mathcal{R}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{R}}{\partial \phi} = 0$

$eqt5 := simplify\left(\left(\frac{d}{dt}\,eqt4\right) - eqt3 = 0.0\right)$ $\leftarrow$

$$\left[\left(m2\,l^2\,R^2\,(m1+m2)\left(\left(\frac{d^2}{dt^2}\phi(t)\right)R^2\,m1 + \left(\frac{d^2}{dt^2}\phi(t)\right)R^2\,m2 + \left(\frac{d^2}{dt^2}\phi(t)\right)m2\,l^2 + 2\left(\frac{d^2}{dt^2}\phi(t)\right)m2\,R\,l\cos(\phi(t)) + m2\left(\frac{d}{dt}\phi(t)\right)^2 l\,R\sin(\phi(t))\right)\right)\Big/\left(R^4\,m1^2 + 2\,m2\,R^4\,m1 + 2\,m2\,l^2\,R^2\,m1 + 4\,R^3\,m1\,m2\,l\cos(\phi(t)) + m2^2\,R^4 + 2\,m2^2\,l^2\,R^2 + 4\,R^3\,m2^2\,l\cos(\phi(t)) + m2^2\,l^4 + 4\,m2^2\,l^3\,R\cos(\phi(t)) + 4\,m2^2\,R^2\,l^2\cos(\phi(t))^2\right)\right] = 0. \quad (8)$$

$$eqt6 := numer\left(simplify\left(\left(\frac{d}{dt}\, eqt4\right) - eqt3\right)\right) = 0$$

$$\left[m2\, l^2 R^2 (m1 + m2) \left(\left(\frac{d^2}{dt^2}\phi(t)\right) R^2 m1 + \left(\frac{d^2}{dt^2}\phi(t)\right) R^2 m2 + \left(\frac{d^2}{dt^2}\phi(t)\right) m2\, l^2 + 2\left(\frac{d^2}{dt^2}\phi(t)\right) m2\, R\, l \cos(\phi(t)) + m2 \left(\frac{d}{dt}\phi(t)\right)^2 l\, R \sin(\phi(t))\right)\right] = 0 \tag{9}$$

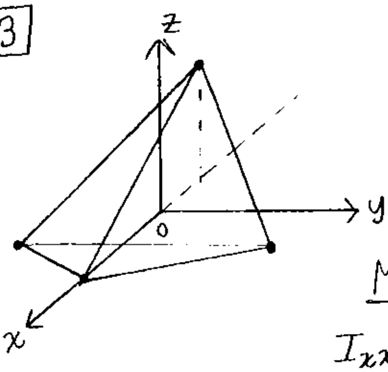
$simplify(eqt6)$

$$\boxed{\left[m2\, l^2 R^2 (m1 + m2) \left(\left(\frac{d^2}{dt^2}\phi(t)\right) R^2 m1 + \left(\frac{d^2}{dt^2}\phi(t)\right) R^2 m2 + \left(\frac{d^2}{dt^2}\phi(t)\right) m2\, l^2 + 2\left(\frac{d^2}{dt^2}\phi(t)\right) m2\, R\, l \cos(\phi(t)) + m2 \left(\frac{d}{dt}\phi(t)\right)^2 l\, R \sin(\phi(t))\right)\right] = 0} \tag{10}$$

E.O.M.

by $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\phi}}\right) - \frac{\partial R}{\partial \phi} = 0$

a3



Four particles, each of mass $m/4$

$(x_0, 0, 0)$, $(-x_0/3, y_0, -z_0/2)$, $(-x_0/3, -y_0, -z_0/2)$,

$(-x_0/3, 0, z_0)$

Moments of Inertia

$$I_{xx} = \frac{1}{4} m \left[y_0^2 + \left(-\frac{z_0}{2}\right)^2 + y_0^2 + \left(-\frac{z_0}{2}\right)^2 + z_0^2 \right] = \frac{1}{4} m \left(2y_0^2 + \frac{3}{2} z_0^2 \right)$$

$$I_{yy} = \frac{1}{4} m \left[x_0^2 + \frac{x_0^2}{9} + \frac{z_0^2}{4} + \frac{x_0^2}{9} + \frac{z_0^2}{4} + \frac{x_0^2}{9} + z_0^2 \right]$$

$$= \frac{m}{4} \left(\frac{4}{3} x_0^2 + \frac{3}{2} z_0^2 \right)$$

$$I_{zz} = \frac{1}{4} m \left[x_0^2 + \frac{x_0^2}{9} + y_0^2 + \frac{x_0^2}{9} + y_0^2 + \frac{x_0^2}{9} \right] = \frac{m}{4} \left(\frac{4}{3} x_0^2 + 2 y_0^2 \right.$$

Product of inertia

$$I_{xy} = \frac{m}{4} \left[-\frac{1}{3} x_0 y_0 + \frac{1}{3} x_0 y_0 \right] = 0$$

$$I_{yz} = \frac{m}{4} \left[-\frac{1}{2} y_0 z_0 + \frac{1}{2} y_0 z_0 \right] = 0$$

$$I_{xz} = \frac{m}{4} \left[\frac{1}{6} x_0 z_0 + \frac{1}{6} x_0 z_0 - \frac{1}{3} x_0 z_0 \right] = 0$$

$$\boxed{\begin{aligned} x_0 &= \sqrt{\frac{3}{2m} (I_{yy} + I_{zz} - I_{xx})} \\ y_0 &= \sqrt{\frac{1}{m} (I_{zz} + I_{xx} - I_{yy})} \\ z_0 &= \sqrt{\frac{4}{3m} (I_{xx} + I_{yy} - I_{zz})} \end{aligned}}$$

Q4

a) $[I]=\begin{bmatrix} 450 & -60 & 100 \\ -60 & 500 & 7 \\ 100 & 7 & 550 \end{bmatrix} kg\text{-}m^2$

so the characteristic equation for the system

$$\begin{vmatrix} 450-I & -60 & 100 \\ -60 & 500-I & 7 \\ 100 & 7 & 550-I \end{vmatrix} = 0 \Rightarrow \text{obtain the roots are}$$

$$\boxed{I_1 = 365.49,\ I_2 = 516.47,\ I_3 = 618.04\ (kg\text{-}m^2)}$$

b) $$\begin{cases} -60\frac{y}{x} + 100\frac{z}{x} = -(450-I) \\ (500-I)\frac{y}{x} + 7\frac{z}{x} = 60 \\ 7\frac{y}{x} + (550-I)\frac{z}{x} = -100 \end{cases}$$

when $I_1 = 365.49$, $\frac{y}{x} = 0.4752$, $\frac{z}{x} = -0.5600$

$l_{x'x} = 1/\left[1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right]^{1/2} = 0.8060$

$l_{x'y} = 0.4752/\left[1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right]^{1/2} = 0.3830$

$l_{x'z} = -0.56/\left[1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right]^{1/2} = -0.4513$

when $I_2 = 516.47$, $\frac{y}{x} = -4.5097$, $\frac{z}{x} = -2.0411$

similar as above, we can calculate that

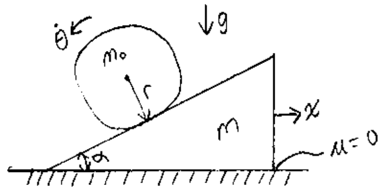
$l_{y'x} = -0.1980$, $l_{y'y} = 0.8930$, $l_{y'z} = 0.4042$

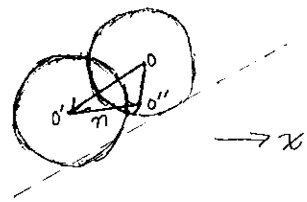
when $I_3 = 618.04$, $\frac{y}{x} = -0.4237$, $\frac{z}{x} = 1.9263$

$l_{z'x} = 0.5578$, $l_{z'y} = -0.2364$, $l_{z'z} = 0.7956$

$$[l] = \begin{bmatrix} 0.8060 & 0.3830 & -0.4513 \\ -0.1980 & 0.8930 & 0.4042 \\ 0.5578 & -0.2364 & 0.7956 \end{bmatrix}$$

Q5 (Problem 7-13, not 8-3)





a) As shown above, the motion of the sphere is $\overrightarrow{OO''}$,

$\underline{r}_{OO''} = \underline{r}_{OO'} + \underline{r}_{O'O''}$

$\underline{r}_{OO'} = -r\theta \sin\alpha \, \underline{j} - r\theta\cos\alpha \, \underline{i}, \quad \underline{r}_{O'O''} = x\,\underline{i}$

$\Rightarrow \underline{r}_{OO''} = (x - r\theta\cos\alpha)\,\underline{i} - r\theta\sin\alpha\,\underline{j}$

$\underline{v}_{O''} = \dot{\underline{r}}_{OO''} = (\dot{x} - r\dot{\theta}\cos\alpha)\,\underline{i} - (r\dot{\theta}\sin\alpha)\,\underline{j}$

$\Rightarrow \underline{v}^2_{OO''} = \dot{x}^2 + r^2\dot{\theta}^2 - 2\dot{x}\dot{\theta} r\cos\alpha$

Kinetic energy

$T = T_c + T_{rot} \; ; \; T_{rot} = \frac{1}{2} I \omega^2 = \frac{1}{2}\left(\frac{2}{5} m_0 r^2\right)\dot{\theta}^2$

$T_{rot} = \frac{1}{5} m_0 r^2 \dot{\theta}^2$

$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m_0 (\dot{x}^2 + r^2\dot{\theta}^2 - 2\dot{x}\dot{\theta} r\cos\alpha) + \frac{1}{5} m_0 r^2 \dot{\theta}^2$

and $V = -m_0 g r \theta \sin\alpha$

$\frac{\partial T}{\partial \dot{x}} = m\dot{x} + m_0\dot{x} - m_0 r\dot{\theta}\cos\alpha$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = (m_0 + m)\ddot{x} - m_0 r\ddot{\theta}\cos\alpha$

$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial V}{\partial x} = 0$

$\boxed{\text{EOM} \Rightarrow (m_0 + m)\ddot{x} - m_0 r\ddot{\theta}\cos\alpha = 0}$ ①

$\frac{\partial T}{\partial \dot{\theta}} = \frac{7}{5} m_0 r^2 \dot{\theta} - m_0 r\dot{x}\cos\alpha$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{7}{5} m_0 r^2 \ddot{\theta} - m_0 r\ddot{x}\cos\alpha$

$\frac{\partial T}{\partial \theta} = 0 \; ; \; \frac{\partial V}{\partial \theta} = -m_0 g r\sin\alpha$

$\boxed{\text{EOM} \Rightarrow \frac{7}{5} m_0 r^2\ddot{\theta} - m_0 r\ddot{x}\cos\alpha - m_0 g r\sin\alpha = 0}$ ②

b) Solve equations of ① & ②

$$① \Rightarrow \ddot{\theta} = \frac{(m+m_0)\ddot{x}}{m_0 r \cos\alpha}\ ; \text{ put into ②}$$

$$\Rightarrow \ddot{x} = \frac{5 m_0 g \sin\alpha \cos\alpha}{7(m+m_0) - 5 m_0 \cos\alpha}$$

Initial conditions: $\dot{x}(0) = 0$

$$\Rightarrow \dot{x} = \frac{5 m_0 g \sin\alpha \cos\alpha}{7(m+m_0) - 5 m_0 \cos\alpha}\, t$$