ME 562 Advanced Dynamics
Summer 2010
HOMEWORK # 6

Due: July 23, 2010

Q1. Two wheels, each of mass *m*, are connected by a massless axle of length *l*. Each wheel is considered to have its mass concentrated as a particle at its hub. The wheels roll without slipping on a horizontal plane. The hub of wheel A is attached by a spring of stiffness *k* and unstressed length *l* to a fixed point O. Use *r*, θ, and ϕ as generalized coordinates. Then, find: (a) the constraints satisfied by these three variables; (b) the relationships between the virtual displacements in the three variables; (c) if the constraints are holonomic or non-holonomic. **(see Problem 6-25 in the text for a figure).**

Q2. (see Problem 6-7 in the text for a figure). A double pendulum consists of two massless rods of length *l* and two particles of mass *m* which can move in the vertical plane. Assume frictionless joints and define the configuration of the system using the coordinates θ and ϕ. Recall that the system is in the vertical plane.

(i) Derive the generalized forces for the generalized coordinates θ and ϕ corresponding to the weights forces of the two particles.

(ii) Then, use Lagrange's equations for holonomic systems and derive the differential equations of motion for the system.

Q3. (see Problem 6-13 in the text for a figure). A smooth tube in the form of a circle of radius *r* is pinned at O and rotates in its vertical plane with a constant angular velocity ω. The position of a particle of mass *m* that slides inside the tube is given by the relative coordinate ϕ. ϕ is the angle that the line joining the center of the ring/tube (O') to the particle makes with OO'. Use Lagrange's equations for holonomic systems to derive the differential equation for ϕ, the only generalized coordinate. Note that $\dot{\theta} = \omega$ is constant and is specified, thus it is not a generalized coordinate.

Q4. (see Problem 6-22 in the text for a figure). A dumbbell is composed of two particles, each of mass *m*, connected by a massless rod of length *l*. One particle of the dumbbell is connected by a pin to the edge of a disk of radius *r*, which is massless except for a particle of mass *m* at its center. The disc can roll without slipping on a horizontal surface. Assume frictionless joints and define the configuration of the system using the coordinates θ and ϕ which are absolute rotation angles. The system is in the vertical plane. Then, use Lagrange's equations for holonomic systems and derive the differential equations of motion for the system.

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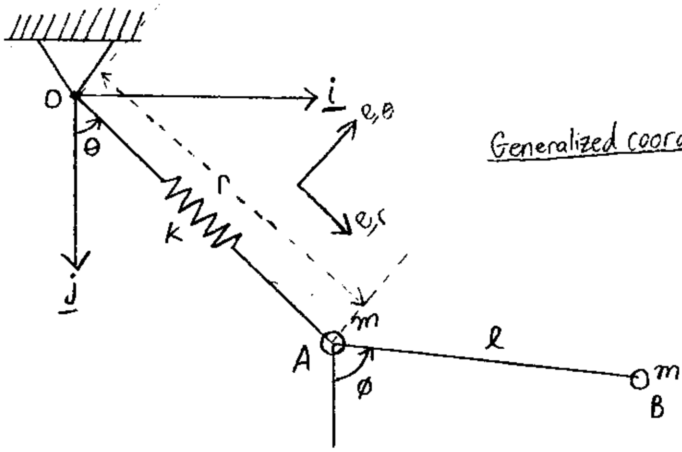
Q1



Generalized coordinates: r, θ, ϕ

a) Find constraints satisfied by $r, \theta, \& \phi$

$$\underline{r}_{AB} = \ell\cos(\phi-\theta)\,\underline{e}_r + \ell\sin(\phi-\theta)\,\underline{e}_\theta$$

$$\dot{\underline{r}}_{AB} = -\ell(\dot\phi-\dot\theta)\sin(\phi-\theta)\,\underline{e}_r + \ell\cos(\phi-\theta)\,\dot{\underline{e}}_r + \ell(\dot\phi-\dot\theta)\cos(\phi-\theta)\,\underline{e}_\theta + \ell\sin(\phi-\theta)\,\dot{\underline{e}}_\theta$$

$$\left(\dot{\underline{e}}_r = \dot\theta\,\underline{k}\times\underline{e}_r = \dot\theta\,\underline{e}_\theta \quad \& \quad \dot{\underline{e}}_\theta = \dot\theta\,\underline{k}\times\underline{e}_\theta = -\dot\theta\,\underline{e}_r\right)$$

$$\dot{\underline{r}}_{AB} = -\ell(\dot\phi-\dot\theta)\sin(\phi-\theta)\,\underline{e}_r + \ell\dot\theta\cos(\phi-\theta)\,\underline{e}_\theta + \ell(\dot\phi-\dot\theta)\cos(\phi-\theta)\,\underline{e}_\theta - \ell\dot\theta\sin(\phi-\theta)\,\underline{e}_r$$

$$= -\ell\dot\phi\sin(\phi-\theta)\,\underline{e}_r + \ell\dot\phi\cos(\phi-\theta)\,\underline{e}_\theta$$

$$\underline{r}_{OA} = r\,\underline{e}_r$$

$$\dot{\underline{r}}_{OA} = \dot r\,\underline{e}_r + r\,\dot{\underline{e}}_r$$

$$\dot{\underline{r}}_{OA} = \dot r\,\underline{e}_r + r\dot\theta\,\underline{e}_\theta$$

$$\underline{n} = \frac{r_{AB}}{|r_{AB}|} = \frac{\ell\cos(\phi-\theta)\,\underline{e}_r + \ell\sin(\phi-\theta)\,\underline{e}_\theta}{\sqrt{\ell^2\left(\cos^2(\phi-\theta)+\sin^2(\phi-\theta)\right)}} = \cos(\phi-\theta)\,\underline{e}_r + \sin(\phi-\theta)\,\underline{e}_\theta$$

$$\dot{\underline{r}}_{OA}\cdot n = \left(\dot r\,\underline{e}_r + r\dot\theta\,\underline{e}_\theta\right)\left[\cos(\phi-\theta)\,\underline{e}_r + \sin(\phi-\theta)\,\underline{e}_\theta\right] = \boxed{\dot r\cos(\phi-\theta) + r\dot\theta\sin(\phi-\theta) = 0}$$

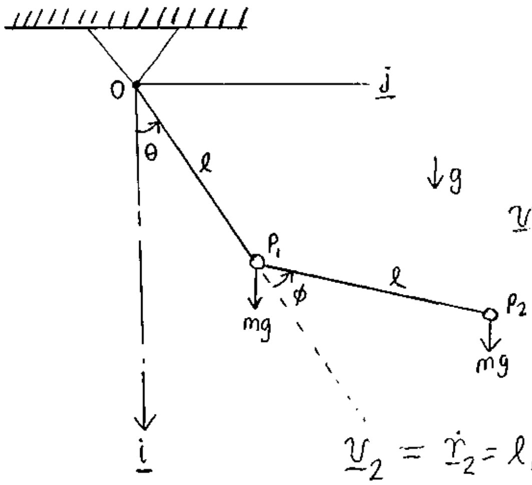
$$\left(a_{11} = \cos(\phi-\theta),\; a_{12} = r\sin(\phi-\theta),\; a_{13} = a_{1t} = 0\right)$$

b) Relationship between virtual displacements in $r, \theta, \&, \phi$

$$\boxed{\cos(\phi-\theta)\,\delta r + r\sin(\phi-\theta)\,\delta\theta = 0}$$

c) Constraint is kinematic, thus non-holonomic

Q2



$$\underline{r}_1 = \ell(\cos\theta\,\underline{i} + \sin\theta\,\underline{j})$$

$$\underline{v}_1 = \dot{\underline{r}}_1 = \ell\dot\theta(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j})$$

$$\underline{r}_2 = \ell(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}) + \ell\{\cos(\theta+\phi)\underline{i} + \sin(\theta+\phi)\underline{j}\}$$

$$= \ell[\{\cos\theta + \cos(\theta+\phi)\}\underline{i} + \{\sin\theta + \sin(\theta+\phi)\}\underline{j}]$$

$$\underline{v}_2 = \dot{\underline{r}}_2 = \ell[\{-\dot\theta\sin\theta - (\dot\theta+\dot\phi)\sin(\theta+\phi)\}\underline{i} + \{\dot\theta\cos\theta + (\dot\theta+\dot\phi)\cos(\theta+\phi)\}\underline{j}]$$

$$T = \frac{1}{2}m\underline{v}_1\cdot\underline{v}_1 + \frac{1}{2}m\underline{v}_2\cdot\underline{v}_2$$

$$= \frac{1}{2}m\ell^2\dot\theta^2 + \frac{1}{2}m\ell^2[\dot\theta^2\sin^2\theta + (\dot\theta+\dot\phi)^2\sin^2(\theta+\phi) + 2\dot\theta(\dot\theta+\dot\phi)\sin\theta\sin(\theta+\phi)$$

$$+ \dot\theta^2\cos^2\theta + (\dot\theta+\dot\phi)^2\cos^2(\theta+\phi) + 2\dot\theta(\dot\theta+\dot\phi)\cos\theta\cos(\theta+\phi)]$$

$$= \frac{1}{2}m\ell^2\dot\theta^2 + \frac{1}{2}m\ell^2[\dot\theta^2 + (\dot\theta+\dot\phi)^2 + 2\dot\theta(\dot\theta+\dot\phi)\cos\phi] \quad ①$$

$$\delta W = Q_\theta\,\delta\theta + Q_\phi\,\delta\phi = mg\underline{i}\cdot\delta\underline{r}_1 + mg\underline{i}\cdot\delta\underline{r}_2$$

$$= -mg\ell\sin\theta\,\delta\theta - mg\ell[\sin\theta\,\delta\theta + \sin(\theta+\phi)(\delta\theta+\delta\phi)]$$

$$\boxed{\begin{aligned} Q_\theta &= -2mg\ell\sin\theta - mg\ell\sin(\theta+\phi) \\ Q_\phi &= -mg\ell\sin(\theta+\phi) \end{aligned}}$$

Then, Lagrange's equations are

$\underline{\underline{\theta}}$: $$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} = Q_\theta$$

$$\frac{\partial T}{\partial\dot\theta} = m\ell^2\dot\theta + m\ell^2[\dot\theta + (\dot\theta+\dot\phi) + (\dot\theta+\dot\phi)\cos\phi + \dot\theta\cos\phi]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = 3m\ell^2\ddot\theta + m\ell^2\ddot\phi + m\ell^2[(\ddot\theta+\ddot\phi)\cos\phi - (\dot\theta+\dot\phi)\dot\phi\sin\phi + \ddot\theta\cos\phi - \dot\theta\dot\phi\sin\phi]$$

$$= 3m\ell^2\ddot\theta + m\ell^2\ddot\phi + 2m\ell^2\ddot\theta\cos\phi + m\ell^2\ddot\phi\cos\phi - m\ell^2[2\dot\theta\dot\phi\sin\phi + \dot\phi^2\sin\phi]$$

$$\frac{\partial T}{\partial\theta} = 0$$

$$\boxed{m\ell^2[(3+2\cos\phi)\ddot\theta + (1+\cos\phi)\ddot\phi - (\dot\phi^2+2\dot\theta\dot\phi)\sin\phi] + mg\ell[2\sin\theta + \sin(\theta+\phi)] = 0}$$

$\underline{\underline{\phi}}$: $$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) - \frac{\partial T}{\partial\phi} = Q_\phi$$

$$\frac{\partial T}{\partial\dot\phi} = m\ell^2(\dot\theta+\dot\phi) + m\ell^2\dot\theta\cos\phi$$

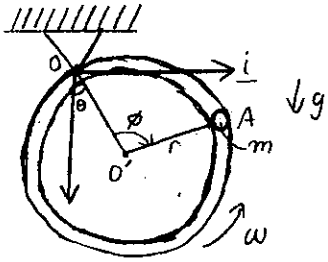
$$\frac{\partial T}{\partial \dot{\phi}} = m\ell^2(\dot{\theta}+\dot{\phi}) + m\ell^2\dot{\theta}\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = m\ell^2(\ddot{\theta}+\ddot{\phi}) + m\ell^2\ddot{\theta}\cos\phi - m\ell^2\dot{\theta}\dot{\phi}\sin\phi$$

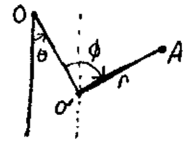
$$\frac{\partial T}{\partial \phi} = -m\ell^2\dot{\theta}\sin\phi(\dot{\theta}+\dot{\phi})$$

$$\boxed{m\ell^2\ddot{\theta}(1+\cos\phi) + m\ell^2\ddot{\phi} + m\ell^2\dot{\theta}^2\sin\phi + mg\ell\sin(\theta+\phi) = 0}$$

$\boxed{Q3}$



$\dot{\theta} = \omega = \text{constant}$



$$\underline{r}_{OA} = \left\{ r\sin\theta + r\sin(\phi-\theta) \right\} \underline{i} + \left\{ r\cos\theta - r\cos(\phi-\theta) \right\} \underline{j}$$

$$\delta \underline{r}_A = r\left\{ \cos\theta\,\delta\theta + \cos(\phi-\theta)(\delta\phi - \delta\theta) \right\} \underline{i} - r\left\{ \sin\theta\,\delta\theta - \sin(\phi-\theta)(\delta\phi-\delta\theta) \right\} \underline{j}$$

$$= \left\{ r\cos(\phi-\theta)\,\underline{i} + r\sin(\phi-\theta)\,\underline{j} \right\} \delta\phi + \left\{ \left(r\cos\theta - \cos(\phi-\theta)\right) \underline{i} - \left(r\sin\theta + r\sin(\phi-\theta)\right) \underline{j} \right\} \delta\theta$$

$$\underline{F}_A = mg\,\underline{j}$$

$$\delta \underline{W}_A = \underline{F}_A \cdot \delta \underline{r}_A = \underbrace{\left[mgr\sin(\phi-\theta) \right]}_{Q_\phi} \delta\phi - mgr\left[\sin\theta + \sin(\phi-\theta) \right] \delta\theta$$

$$Q_\phi = mgr\sin(\phi-\theta)$$

$$\dot{\underline{r}}_{OA} = \left[r\dot{\theta}\cos\theta + r(\dot{\phi}-\dot{\theta})\cos(\phi-\theta) \right] \underline{i} + \left[-r\dot{\theta}\sin\theta + r(\dot{\phi}-\dot{\theta})\sin(\phi-\theta) \right] \underline{j}$$

$$= \left[r\omega\cos\theta + r(\dot{\phi}-\omega)\cos(\phi-\theta) \right] \underline{i} - \left[r\omega\sin\theta - r(\dot{\phi}-\omega)\sin(\phi-\theta) \right] \underline{j}$$

$$\dot{\underline{r}}_{OA} \cdot \dot{\underline{r}}_{OA} = r^2\omega^2\cos^2\theta + 2r^2\omega(\dot{\phi}-\omega)\cos\theta\cos(\phi-\theta) + r^2(\dot{\phi}-\omega)^2\cos^2(\phi-\theta) + r^2\omega^2\sin^2\theta - 2r^2\omega(\dot{\phi}-\omega)\sin\theta\sin(\phi-\theta) + r^2(\dot{\phi}-\omega)^2\sin^2(\phi-\theta)$$

$$= r^2\omega^2 + 2r^2\omega(\dot{\phi}-\omega)\left[\underbrace{\cos\theta(\cos\phi\cos\theta + \sin\phi\sin\theta) - \sin\theta(\sin\phi\cos\theta - \cos\phi\sin\theta)}_{\cos\phi} \right] + r^2(\dot{\phi}-\omega)^2$$

$$= r^2\omega^2 + 2r^2\omega(\dot{\phi}-\omega)\cos\phi + r^2(\dot{\phi}-\omega)^2$$

$$T = \frac{1}{2}m\left(\dot{\underline{r}}_{OA} \cdot \dot{\underline{r}}_{OA}\right) = \frac{1}{2}mr^2\left[\omega^2 + 2\omega(\dot{\phi}-\omega)\cos\phi + (\dot{\phi}-\omega)^2 \right]$$

$$= \frac{1}{2}mr^2\left[\omega^2 + 2\omega(\dot{\phi}-\omega)\cos\phi + \dot{\phi}^2 - 2\dot{\phi}\omega + \omega^2 \right]$$

$$= \frac{1}{2}mr^2\left[2\omega^2 + 2\omega(\dot{\phi}-\omega)\cos\phi + \dot{\phi}^2 - 2\dot{\phi}\omega \right]$$

$$\frac{\partial T}{\partial \dot{\phi}} = \frac{1}{2}mr^2\left[2\omega\cos\phi + 2\dot{\phi} - 2\omega \right] = mr^2(\omega\cos\phi + \dot{\phi} - \omega) = mr^2\left[\omega(\cos\phi - 1) + \dot{\phi} \right]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}} \right) = mr^2\left[-\omega\dot{\phi}\sin\phi + \ddot{\phi} \right]$$

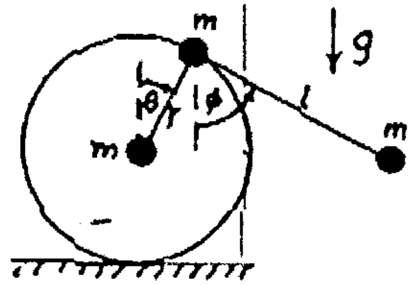
$$\frac{\partial T}{\partial \phi} = \frac{1}{2}mr^2\left[-2\omega(\dot{\phi}-\omega)\sin\phi \right] = -mr^2\omega(\dot{\phi}-\omega)\sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = Q_\phi$$

$$mr^2\left[-\omega \phi \sin\phi + \ddot{\phi} + \omega(\dot{\phi} - \omega)\sin\phi\right] = mgr\sin(\phi - \theta)$$

$$\boxed{mr^2\ddot{\phi} - mr^2\omega^2\sin\phi - mgr\sin(\phi - \theta) = 0}$$

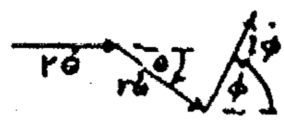
6-22. Adding individual kinetic energies,



$$T = \tfrac{1}{2} m r^2 \dot\theta^2 + \tfrac{1}{2} m\left[(r\dot\theta)^2 + (r\dot\theta)^2 + 2(r\dot\theta)^2 \cos\theta\right]$$
$$+ \tfrac{1}{2} m\left[(r\dot\theta + r\dot\theta\cos\theta + l\dot\phi\cos\phi)^2 + (-r\dot\theta\sin\theta + l\dot\phi\sin\phi)^2\right]$$

or

$$T = \tfrac{1}{2} m\left\{r^2\dot\theta^2(5+4\cos\theta) + l^2\dot\phi^2 + 2rl\dot\theta\dot\phi\left[\cos(\theta+\phi)+\cos\phi\right]\right\}$$



$$V = mg(2r\cos\theta - l\cos\phi), \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

$$\frac{\partial T}{\partial \dot\theta} = mr^2\dot\theta(5+4\cos\theta) + mrl\dot\phi\left[\cos(\theta+\phi)+\cos\phi\right], \quad \frac{\partial V}{\partial \theta} = -2mgr\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = mr^2\ddot\theta(5+4\cos\theta) + mrl\ddot\phi\left[\cos(\theta+\phi)+\cos\phi\right] - 4mr^2\dot\theta^2\sin\theta$$
$$- mrl\dot\phi\left[(\dot\theta+\dot\phi)\sin(\theta+\phi) + \dot\phi\sin\phi\right]$$

$$\frac{\partial T}{\partial \theta} = -2mr^2\dot\theta^2\sin\theta - mrl\dot\theta\dot\phi\sin(\theta+\phi)$$

θ equation: $$mr^2\ddot\theta(5+4\cos\theta) + mrl\ddot\phi\left[\cos(\theta+\phi)+\cos\phi\right] - 2mr^2\dot\theta^2\sin\theta$$
$$- mrl\dot\phi^2\left[\sin(\theta+\phi)+\sin\phi\right] - 2mgr\sin\theta = 0$$

$$\frac{\partial T}{\partial \dot\phi} = ml^2\dot\phi + mrl\dot\theta\left[\cos(\theta+\phi)+\cos\phi\right], \quad \frac{\partial V}{\partial \phi} = mgl\sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) = ml^2\ddot\phi + mrl\ddot\theta\left[\cos(\theta+\phi)+\cos\phi\right] - mrl\dot\theta\left[(\dot\theta+\dot\phi)\sin(\theta+\phi) + \dot\phi\sin\phi\right]$$

$$\frac{\partial T}{\partial \phi} = -mrl\dot\theta\dot\phi\left[\sin(\theta+\phi)+\sin\phi\right]$$

ϕ equation: $$ml^2\ddot\phi + mrl\ddot\theta\left[\cos(\theta+\phi)+\cos\phi\right] - mrl\dot\theta^2\sin(\theta+\phi) + mgl\sin\phi = 0$$
