ME 562 Advanced Dynamics
Summer 2010
HOMEWORK # 5

Due: July 16, 2010

Q1. (see Problem 6-1 in the text for the figure). A fixed smooth rod makes an angle of 30° with the floor. A small (negligible radius) ring (say P_1) of mass m can slide on the rod and supports a string which has one end connected to a point on the floor; the other end is attached to a particle (say P_2) of mass $2m$. *Assume that the string and the rod lie in the same vertical plane.* Since there are two objects with mass, one can use their Cartesian coordinates with respect to a (x,y) system with origin at the lower left corner. Let l be the length of the string. Since the ring is constrained to move along the rod, and l is constant, the four coordinates (x_1,y_1) and (x_2,y_2) are constrained. Also, let a be the coordinate of the point on the floor where the string is attached, and b be the height of the point where the rod is attached to the vertical wall. Then,

(a) Show that the coordinates (x_1,y_1) and (x_2,y_2) satisfy the constraints:

$$y_1 = -x_1/\sqrt{3} + b$$

$$\sqrt{(x_1-a)^2 + y_1^2} + \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = l$$

(b) Derives constraints on velocities by differentiating the constraints in (a) with respect to time;

(c) Derive the relations in virtual displacements in the four coordinates (x_1,y_1) and (x_2,y_2).

Q2. Use the developments in Q1. and the principle of virtual work to find the static equilibrium position of the system. Note that this will give you four expressions for the coordinates (x_1,y_1) and (x_2,y_2), the variables that have been used to define the configuration of the system at any given instant of time. Two of the equations are the constraint equations in Q1. and two equations will come additionally from using the principle of virtual work. Finally, use the definition of angle ψ, as given in the figure, to express the equilibrium position.

Q3. (see Problem 6-3 in the text for a figure). One end of a thin uniform rod of mass m and length $3l$ rests against a smooth vertical wall. The other end of the rod is attached by a string of length l to a fixed point O which is located at a distance $2l$ from the wall. The rod and the string remain in the same vertical plane perpendicular to the wall. Let the position of the rod be defined by the coordinates (x_G,y_G) of the center of mass (denoted as G) and its angular orientation θ. The origin of the coordinate system is located at the fixed point O. Then,

(a) Show that the coordinates (x_G,y_G) and the angular orientation θ satisfy the constraints:

$$x_G + 3l\sin\theta/2 = 2l$$

$$(x_G - 3l\sin\theta/2)^2 + (y_G + 3l\cos\theta/2)^2 = l^2$$

(b) Now, use the principle of virtual work to find the static equilibrium position of the system. Note that this will give you expressions for the coordinates (x_G,y_G) and the angular orientation θ, the variables that have been used to define the configuration of the system at any given instant of time.

Q4. (see Problem 6-7 in the text for a figure). A double pendulum consists of two massless rods of length l and two particles of mass m which can move in the vertical plane. Assume frictionless joints and define the configuration of the system using the coordinates θ and ϕ. The system is in the vertical plane. Then, use the D'Alembert's principle and derive the differential equations of motion for the system. Assume now that the angles θ and ϕ, as well as their time derivatives, remain small during the motion. Then, obtain the linearized equations of motion for small θ and ϕ.

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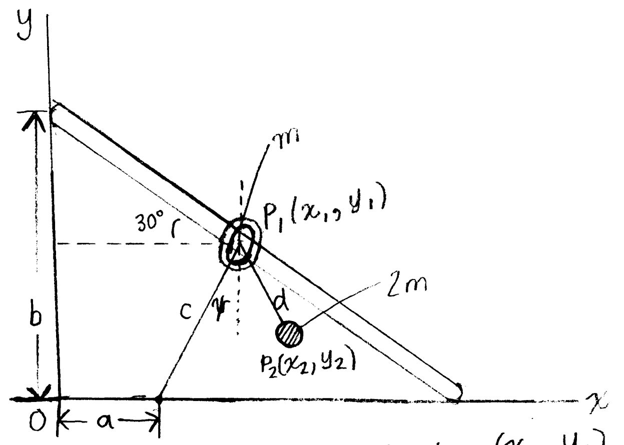
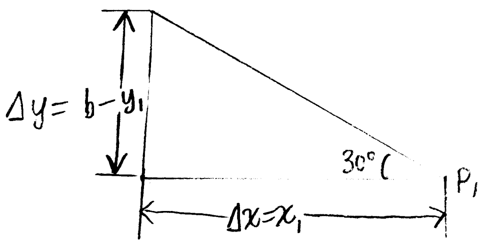
Homework #5

Q1



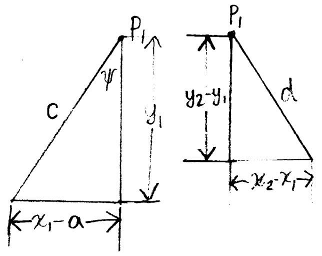
a) constraints w.r.t coordinates (x_1, y_1) & (x_2, y_2)



$$m = \frac{\Delta y}{\Delta x} = \frac{b - y_1}{x_1} = \tan 30° = \frac{1}{\sqrt{3}}$$

$$b - y_1 = \frac{1}{\sqrt{3}} x_1$$

① $\Rightarrow \boxed{y_1 = -\frac{1}{\sqrt{3}} x_1 - b}$



$$\ell = c + d$$

$$c = \sqrt{y_1^2 + (x_1 - a)^2}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

② $\boxed{\ell = \sqrt{y_1^2 + (x_1 - a)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$

b) velocity constraints

① $\Rightarrow \boxed{\dot{y}_1 = -\frac{1}{\sqrt{3}} \dot{x}_1}$ ③

② $\Rightarrow$ $$0 = \frac{1}{\cancel{2}}\left[(x_1 - a)^2 + y_1^2\right]^{-1/2}\left[(x_1 - a)\cancel{2}\dot{x}_1 + \cancel{2} y_1 \dot{y}_1\right]$$

$$+ \frac{1}{\cancel{2}}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]^{-1/2}\left[\cancel{2}(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + \cancel{2}(y_2 - y_1)(\dot{y}_2 - \dot{y}_1)\right]$$

$$\boxed{\frac{(x_1 - a)\dot{x}_1 + y_1 \dot{y}_1}{\sqrt{(x_1 - a)^2 + y_1^2}} + \frac{(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + (y_2 - y_1)(\dot{y}_2 - \dot{y}_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = 0}$$ ④

c) relations in virtual displacements in the four coordinates (x_1, y_1, x_2, y_2)

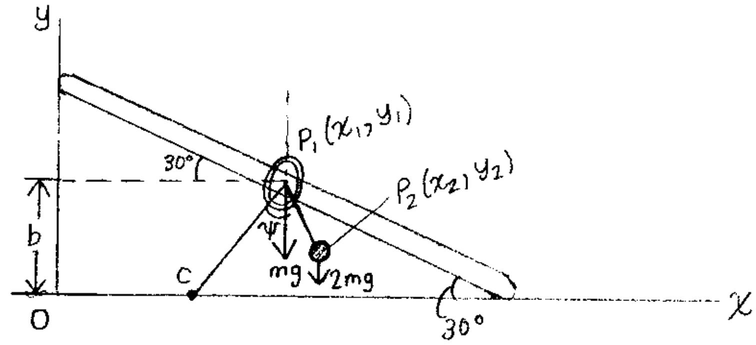
③ $\Rightarrow \boxed{\delta y_1 + \frac{1}{\sqrt{3}} \delta x_1 = 0}$

④ $\Rightarrow \left[\frac{(x_1-a)}{\sqrt{(x_1-a)^2+y_1^2}} - \frac{(x_2-x_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} \right] \delta x_1$

$$+ \left[\frac{y_1}{\sqrt{(x_1-a)^2+y_1^2}} - \frac{(y_2-y_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} \right] \delta y_1$$

$$+ \frac{(x_2-x_1)\delta x_2}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} + \frac{(y_2-y_1)\delta y_2}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} = 0$$

Q2



Constraints: $\boxed{y_1 = -\frac{1}{\sqrt{3}} x_1 + b}$ ①

$$\boxed{l = \left[\sqrt{(x_1-a)^2+y_1^2} + \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\right]} \quad ②$$

Virtual displacements:

$$\boxed{\delta y_1 = -\delta x_1/\sqrt{3}} \quad ③$$

$$\boxed{\frac{(x_1-a)\delta x_1 + y_1 \delta y_1}{\sqrt{(x_1-a)^2+y_1^2}} + \frac{(x_2-x_1)(\delta x_2 - \delta x_1) + (y_2-y_1)(\delta y_2 - \delta y_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} = 0} \quad ④$$

The principle of virtual work gives

$$-mg\,\underline{j}\cdot\delta \underline{r}_{P_1} - 2mg\,\underline{j}\cdot\delta\underline{r}_{P_2} = 0$$

$$-mg\,\delta y_1 - 2mg\,\delta y_2 = 0$$

$$\boxed{\delta y_1 + 2\delta y_2 = 0} \quad ⑤$$

$$④ \Rightarrow \left[\frac{(x_1-a) - y_1/\sqrt{3}}{\sqrt{(x_1-a)^2+y_1^2}}\right]\delta x_1 + \frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\Big[(x_2-x_1)\delta x_2 - (x_2-x_1)\delta x_1$$

$$+ (y_2-y_1)\delta y_2 + (y_2-y_1)\delta x_1/\sqrt{3}\Big] = 0$$

$$\left[\frac{1}{\sqrt{(x_1-a)^2+y_1^2}}\left\{(x_1-a) - \frac{y_1}{\sqrt{3}}\right\} - \frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\left\{(x_2-x_1) - (y_2-y_1)\frac{1}{\sqrt{3}}\right\}\right]\delta x_1$$

$$+ \frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(x_2-x_1)\delta x_2 + \frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(y_2-y_1)\delta y_2 = 0$$

(6)

③ & ⑤ $\Longrightarrow$ ⑥ : $\left(\delta y_2 = -\dfrac{\delta y_1}{2} = \dfrac{\delta x_1}{2\sqrt{3}}\right)$

$$\left[\frac{1}{\sqrt{(x_1-a)^2+y_1^2}}\left\{(x_1-a)-\frac{y_1}{\sqrt{3}}\right\}-\frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\left\{(x_2-x_1)-(y_2-y_1)\frac{1}{\sqrt{3}}\right\}\right]\delta x_1$$

$$+\frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(x_2-x_1)\,\delta x_2+\frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(y_2-y_1)\frac{\delta x_1}{2\sqrt{3}}=0$$

$$\boxed{\begin{aligned}&\left[\frac{1}{\sqrt{(x_1-a)^2+y_1^2}}\left\{(x_1-a)-\frac{y_1}{\sqrt{3}}\right\}-\frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(x_2-x_1)+\frac{\sqrt{3}}{2\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(y_2-y_1)\right]\delta x_1\\&+\left[\frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(x_2-x_1)\right]\delta x_2=0\end{aligned}} \quad ⑦$$

<u>δx_1 component</u>:

$$\frac{1}{\sqrt{(x_1-a)^2+y_1^2}}\left\{(x_1-a)-\frac{y_1}{\sqrt{3}}\right\}-\frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(x_2-x_1)+\frac{\sqrt{3}}{2\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(y_2-y_1)=0 \quad ⑧$$

<u>δx_2 component</u>: $x_2-x_1=0$ ⑨

⑨ $\Longrightarrow$ $\boxed{x_1=x_2}$ at static equilibrium

⑧ $\Rightarrow$ $$\boxed{\frac{1}{\sqrt{(x_1-a)^2+y_1^2}}\left\{(x_1-a)-\frac{y_1}{\sqrt{3}}\right\}+\frac{\sqrt{3}}{2\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(y_2-y_1)=0} \quad ⑩$$ at static equilibrium

<u>The constraint equations ① & ② & ⑨ & ⑩ all determine the static equilibrium position.</u>

Definition of angle ψ:

$$\tan\psi = \frac{x_1 - a}{y_1} \Rightarrow y_1 \tan\psi = x_1 - a \quad (11)$$

From (10):

$$\frac{(x_1-a) - \frac{y_1}{\sqrt{3}}}{\sqrt{(x_1-a)^2 + y_1^2}} + \frac{\sqrt{3}}{2} \frac{y_2 - y_1}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}} = 0$$

Apply (11) $\rightarrow$ (10):
(first term)

$$\frac{y_1 \tan\psi - \frac{y_1}{\sqrt{3}}}{\sqrt{y_1^2(\tan^2\psi + 1)}} = \frac{\tan\psi - \frac{1}{\sqrt{3}}}{\sec\psi}$$

(second term):

$$\frac{y_2 - y_1}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}} = \begin{cases} 1 \,;\; m_2 > m_1 \\ -1 \,;\; m_1 < m_2 \end{cases} = \pm 1$$

$$\frac{\tan\psi - \frac{1}{\sqrt{3}}}{\sec\psi} \pm \frac{\sqrt{3}}{2} = \sin\psi - \frac{1}{\sqrt{3}}\cos\psi \pm \frac{\sqrt{3}}{2} = 0$$

$$\sqrt{3}\sin\psi - \cos\psi = \mp\frac{3}{2}$$

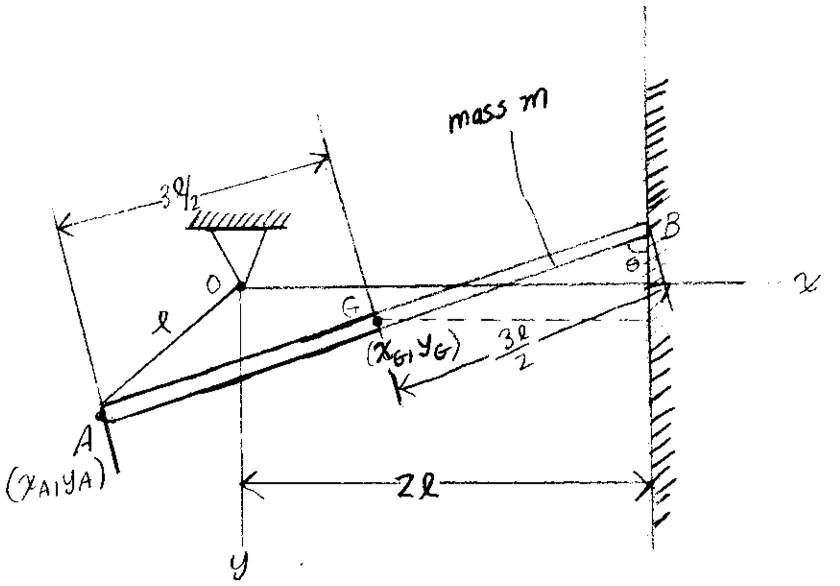
$$\frac{\sqrt{3}}{2}\sin\psi - \frac{1}{2}\cos\psi = \mp\frac{3}{4}$$

$$\sin\psi\cos 30^\circ - \sin 30^\circ \cos\psi = \mp\frac{3}{4}$$

$$\sin(\psi - 30^\circ) = \mp\frac{3}{4}$$

$$\Rightarrow \psi - 30^\circ = 48.59^\circ \Rightarrow \boxed{\psi = 78.59^\circ}$$

Q3 Problem 6-3



a) Constraints: $\boxed{x_G + \frac{3l}{2}\sin\theta = 2l \quad ①}$

Position of A $\Rightarrow$ $\left.\begin{aligned} x_A &= x_G - \frac{3l}{2}\sin\theta \\ y_A &= y_G + \frac{3l}{2}\cos\theta \end{aligned}\right\} x_A^2 + y_A^2 = l^2$

$$\boxed{l^2 = \left(x_G - \frac{3l}{2}\sin\theta\right)^2 + \left(y_G + \frac{3l}{2}\cos\theta\right)^2 \quad ②}$$

b) ① $\Rightarrow$ virtual displacements

$$\delta x_G + \left(\frac{3l}{2}\right)\cos\theta\,\delta\theta = 0 \quad ③$$

② $\Rightarrow$ virtual displacements

$$0 = 2\left(x_G - \frac{3l}{2}\sin\theta\right)\left(\delta x_G - \frac{3l}{2}\cos\theta\,\delta\theta\right) + 2\left(y_G + \frac{3l}{2}\cos\theta\right)\cdot\left(\delta y_G - \frac{3l}{2}\sin\theta\,\delta\theta\right)$$

$$\left(x_G - \frac{3l}{2}\sin\theta\right)\delta x_G + \left(y_G + \frac{3l}{2}\cos\theta\right)\delta y_G + \left[\left(x_G - \frac{3l}{2}\sin\theta\right)\left(-\frac{3l}{2}\right)\cos\theta + \left(y_G + \frac{3l}{2}\cos\theta\right)\left(-\frac{3l}{2}\right)\sin\theta\right]\delta\theta = 0$$

$$\left(x_G - \frac{3l}{2}\sin\theta\right)\delta x_G + \left(y_G + \frac{3l}{2}\cos\theta\right)\delta y_G - \left(\frac{3l}{2}\right)\left[x_G\cos\theta + y_G\sin\theta\right]\delta\theta = 0 \quad ④$$

Principle of virtual work gives

$$mg\underline{j}\cdot\delta\underline{r}_G = 0$$

$$\Rightarrow mg\underline{j}\cdot(\delta x_G\underline{i} + \delta y_G\underline{j}) = 0 \Rightarrow \boxed{\delta y_G = 0} \quad (5)$$

$$(5)\Rightarrow(4): \left(x_G - \frac{3\ell}{2}\sin\theta\right)\delta x_G - \left(\frac{3\ell}{2}\right)\left(x_G\cos\theta + y_G\sin\theta\right)\delta\theta = 0 \quad (6)$$

$$(3)\Rightarrow \delta x_G = -\left(\frac{3\ell}{2}\right)\cos\theta\,\delta\theta \quad (7)$$

$$(7)\Rightarrow(6): \left[\left(x_G - \frac{3\ell}{2}\sin\theta\right)\left(\frac{3\ell}{2}\right)\cos\theta + \left(\frac{3\ell}{2}\right)\left(x_G\cos\theta + y_G\sin\theta\right)\right]\delta\theta = 0$$

$\delta\theta$ is independent virtual displacement $= 0$

$$\left(x_G - \frac{3\ell}{2}\sin\theta\right)\cos\theta + x_G\cos\theta + y_G\sin\theta = 0$$

$$2x_G\cos\theta - \frac{3\ell}{2}\sin\theta\cos\theta + y_G\sin\theta = 0 \quad (8)$$

$$(1)\Rightarrow x_G = 2\ell - \left(\frac{3\ell}{2}\right)\sin\theta \quad (9)$$

$$(2)\Rightarrow \ell^2 = \left[\left(2\ell - \frac{3\ell}{2}\sin\theta - \frac{3\ell}{2}\sin\theta\right)^2 + \left(y_G + \frac{3\ell}{2}\cos\theta\right)^2\right]$$

$$\ell^2 = (2\ell - 3\ell\sin\theta)^2 + \left(y_G + \frac{3\ell}{2}\cos\theta\right)^2 \quad (10)$$

$$(9)\Rightarrow(8): 2\left[2\ell - \frac{3\ell}{2}\sin\theta\right]\cos\theta - \frac{3\ell}{2}\sin\theta\cos\theta + y_G\sin\theta = 0$$

$$4\ell\cos\theta - 3\ell\sin\theta\cos\theta - \frac{3\ell}{2}\sin\theta\cos\theta + y_G\sin\theta = 0$$

$$-y_G = \ell\frac{\cos\theta}{\sin\theta}\left[4 - \frac{9}{2}\sin\theta\right] \quad (11)$$

$$(10)\ y_G = -\frac{3\ell}{2}\cos\theta + \sqrt{\ell^2 - (4\ell^2 + 9\ell^2\sin^2\theta - 12\ell^2\sin\theta)}$$

$$= -\frac{3\ell}{2}\cos\theta + \ell\sqrt{-3 - 9\sin^2\theta + 12\sin\theta} \quad (12)$$

$$(11) = (12)\quad \frac{\ell\cos\theta}{\sin\theta}\left[-4 + \frac{9}{2}\sin\theta\right] = -\frac{3\ell}{2}\cos\theta + \ell\sqrt{-3 - 9\sin^2\theta + 12\sin\theta}$$

$$\cos\theta\left(-4 + \frac{9}{2}\sin\theta\right) = -\frac{3}{2}\sin\theta\cos\theta + \sin\theta\sqrt{-3 - 9\sin^2\theta + 12\sin\theta}$$

$$-4\cos\theta+\frac{9}{2}\sin\theta\cos\theta+\frac{3}{2}\sin\theta\cos\theta=\sin\theta\sqrt{-3-9\sin^2\theta+12\sin\theta}$$

$$-4\cos\theta+6\sin\theta\cos\theta=\sin\theta\sqrt{-3-9\sin^2\theta+12\sin\theta}$$

$$16\cos^2\theta+36\sin^2\theta\cos^2\theta-48\sin\theta\cos^2\theta=\sin^2\theta\left(-3-9\sin^2\theta+12\sin\theta\right)$$

$$16(1-\sin^2\theta)+36\sin^2\theta(1-\sin^2\theta)-48\sin\theta(1-\sin^2\theta)$$
$$=\sin^2\theta(-3-9\sin^2\theta+12\sin\theta)$$

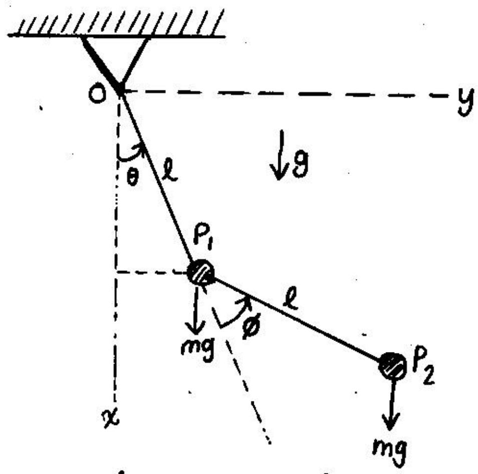
$$16-16\sin^2\theta+36\sin^2\theta-36\sin^4\theta-48\sin\theta+48\sin^3\theta$$
$$=-3\sin^2\theta-9\sin^4\theta+12\sin^3\theta$$

$$-27\sin^4\theta+36\sin^3\theta+23\sin^2\theta-48\sin\theta+16=0$$

$$\boxed{27\sin^4\theta-36\sin^3\theta-23\sin^2\theta+48\sin\theta-16=0}$$

$$(\sin\theta=0.9216\Rightarrow\underline{\underline{\theta=67.17^\circ}})$$

Q4 Problem 6-7



$$\underline{r}_{P_1} = l(\cos\theta\,\underline{i} + \sin\theta\,\underline{j})$$

$$\dot{\underline{r}}_{P_1} = l\dot{\theta}(\sin\theta\,\underline{i} + \cos\theta\,\underline{j})$$

$$\ddot{\underline{r}}_{P_1} = l\ddot{\theta}(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}) + l\dot{\theta}^2(-\cos\theta\,\underline{i} - \sin\theta\,\underline{j})$$

$$\delta\underline{r}_{P_1} = l(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j})\,\delta\theta$$

$$\underline{r}_{P_2} = l(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}) + l[\cos(\theta+\phi)\underline{i} + \sin(\theta+\phi)\underline{j}]$$

$$\dot{\underline{r}}_{P_2} = l\dot{\theta}(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}) + l(\dot{\theta}+\dot{\phi})[-\sin(\theta+\phi)\underline{i} + \cos(\theta+\phi)\underline{j}]$$

$$\delta\underline{r}_{P_2} = l\delta\theta(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}) + l(\delta\theta+\delta\phi)[-\sin(\theta+\phi)\underline{i} + \cos(\theta+\phi)\underline{j}]$$

$$\ddot{\underline{r}}_{P_2} = l\ddot{\theta}(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}) + l\dot{\theta}^2(-\cos\theta\,\underline{i} - \sin\theta\,\underline{j})$$
$$+ l(\ddot{\theta}+\ddot{\phi})[-\sin(\theta+\phi)\underline{i} + \cos(\theta+\phi)\underline{j}] + l(\dot{\theta}+\dot{\phi})^2[-\cos(\theta+\phi)\underline{i} - \sin(\theta+\phi)\underline{j}]$$

D'Alembert's principle

$$(\underline{F}_1 - m\ddot{\underline{r}}_{P_1})\cdot\delta\underline{r}_{P_1} + (\underline{F}_2 - m\ddot{\underline{r}}_{P_2})\cdot\delta\underline{r}_{P_2} = 0$$

$$\underline{F}_1 = mg\,\underline{i}$$

$$\underline{F}_1\cdot\delta\underline{r}_{P_1} = -mgl\sin\theta\,\delta\theta \quad (1)$$

$$\underline{F}_2 = mg\,\underline{i}$$

$$\underline{F}_2\cdot\delta\underline{r}_{P_2} = -mgl\sin\theta\,\delta\theta - mgl\sin(\theta+\phi)(\delta\theta+\delta\phi)$$
$$= -mgl[\sin\theta + \sin(\theta+\phi)]\delta\theta - mgl\sin(\theta+\phi)\delta\phi \quad (2)$$

$$m\ddot{\underline{r}}_{P_1}\cdot\delta\underline{r}_{P_1} = m[l\ddot{\theta}(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}) + l\dot{\theta}^2(-\cos\theta\,\underline{i} - \sin\theta\,\underline{j})]\cdot l(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j})\delta\theta$$
$$= m l^2[\ddot{\theta}\sin^2\theta + \dot{\theta}^2\sin\theta\cos\theta + \ddot{\theta}\cos^2\theta - \dot{\theta}^2\sin\theta\cos\theta]\delta\theta$$
$$= ml^2\ddot{\theta}\,\delta\theta \quad (3)$$

$$m\ddot{\underline{r}}_{P_2}\cdot\delta\underline{r}_{P_2} = m\Big[\ell\ddot\theta(-\sin\theta\,\underline{i}+\cos\theta\,\underline{j}) + \ell\dot\theta^2(-\cos\theta\,\underline{i}-\sin\theta\,\underline{j}) + \ell(\ddot\theta+\ddot\phi)\{-\sin(\theta+\phi)\underline{i}+\cos(\theta+\phi)\underline{j}\}$$
$$+\ell(\dot\theta+\dot\phi)^2\{-\cos(\theta+\phi)\underline{i}-\sin(\theta+\phi)\underline{j}\}\Big]\cdot\Big[\ell(-\sin\theta\,\underline{i}+\cos\theta\,\underline{j})\delta\theta$$
$$+\ell\{-\sin(\theta+\phi)\underline{i}+\cos(\theta+\phi)\underline{j}\}(\delta\theta+\delta\phi)\Big]$$

$$= m\ell^2\Big[\{\ddot\theta\sin^2\theta + \dot\theta^2\cos\theta\cancel{\sin\theta} + (\ddot\theta+\ddot\phi)\sin\theta\sin(\theta+\phi) + (\dot\theta+\dot\phi)^2\cos(\theta+\phi)\sin\theta$$
$$+\ddot\theta\cos^2\theta - \dot\theta^2\sin\theta\cancel{\cos\theta} + (\ddot\theta+\ddot\phi)\cos(\theta+\phi)\cos\theta - (\dot\theta+\dot\phi)^2\sin(\theta+\phi)\cos\theta\}\delta\theta$$
$$+\{\ddot\theta\sin\theta\sin(\theta+\phi) + \dot\theta^2\sin(\theta+\phi)\cos\theta + (\ddot\theta+\ddot\phi)\sin^2(\theta+\phi) + (\dot\theta+\dot\phi)^2\cancel{\sin(\theta+\phi)}\cos(\theta+\phi)$$
$$+\ddot\theta\cos\theta\cos(\theta+\phi) - \dot\theta^2\sin\theta\cos(\theta+\phi) + (\ddot\theta+\ddot\phi)\cos^2(\theta+\phi) - (\dot\theta+\dot\phi)^2\cancel{\sin(\theta+\phi)}\cos(\theta+\phi)$$
$$\}(\delta\theta+\delta\phi)\Big]$$

$$= m\ell^2\Big[\{\ddot\theta + (\ddot\theta+\ddot\phi)\cos\phi + (\dot\theta+\dot\phi)^2(-\sin\phi)\}\delta\theta + \{\ddot\theta\cos\phi + \dot\theta^2\sin\phi + (\ddot\theta+\ddot\phi)\}(\delta\theta+\delta\phi)\Big]$$

④

Using ①, ②, ③, ④: $-mg\ell\sin\theta\,\delta\theta - mg\ell\{\sin\theta+\sin(\theta+\phi)\}\delta\theta - mg\ell\sin(\theta+\phi)\delta\phi$

$$-m\ell^2\ddot\theta\,\delta\theta - m\ell^2\{\ddot\theta + (\ddot\theta+\ddot\phi)\cos\phi - (\dot\theta+\dot\phi)^2\sin\phi\}\delta\theta$$
$$-m\ell^2\{\ddot\theta\cos\phi + \dot\theta^2\sin\phi + (\ddot\theta+\ddot\phi)\}(\delta\theta+\delta\phi) = 0$$

$$\Big[-mg\ell\sin\theta - mg\ell\{\sin\theta+\sin(\theta+\phi)\} - m\ell^2\ddot\theta - m\ell^2\{\ddot\theta + (\ddot\theta+\ddot\phi)\cos\phi - (\dot\theta+\dot\phi)^2\sin\phi\}$$
$$-m\ell^2\{\ddot\theta\cos\phi + \dot\theta^2\sin\phi + (\ddot\theta+\ddot\phi)\}\Big]\delta\theta$$
$$+\Big[-mg\ell\sin(\theta+\phi) - m\ell^2\{\ddot\theta\cos\phi + \dot\theta^2\sin\phi + (\ddot\theta+\ddot\phi)\}\Big]\delta\phi = 0$$

$\delta\theta$ & $\delta\phi$ are independent, thus

$\delta\theta$: $-2mg\ell\sin\theta - mg\ell\sin(\theta+\phi) - m\ell^2(3\ddot\theta) - m\ell^2\ddot\phi - m\ell^2\ddot\theta(2\cos\phi) - m\ell^2\ddot\phi\cos\phi$

$$+(\dot\theta+\dot\phi)^2\sin\phi(m\ell^2) - m\ell^2\dot\theta^2\sin\phi = 0$$

$$\Rightarrow \boxed{m\ell^2[(3+2\cos\phi)\ddot\theta + (1+\cos\phi)\ddot\phi - (\dot\phi^2+2\dot\theta\dot\phi)\sin\phi + mg\ell[2\sin\theta+\sin(\theta+\phi)]] = 0}$$

$\delta\phi$: $-mg\ell\sin(\theta+\phi) - m\ell^2\ddot\theta\cos\phi - m\ell^2\dot\theta^2\sin\phi - m\ell^2(\ddot\theta+\ddot\phi) = 0$

$$\Rightarrow \boxed{m\ell^2[(1+\cos\phi)\ddot\theta + \ddot\phi + \dot\theta^2\sin\phi] + mg\ell\sin(\theta+\phi) = 0}$$

$\phi \,\$\, \theta \approx \text{small} \Rightarrow \ddot{\theta} = \ddot{\phi} = \dot{\theta} = \dot{\phi} = 0$

$\underline{\underline{\theta}}$: $m\ell^2[(3+2)(0) + (1+1)(0) - (0+0)\phi] + mg\ell[2\theta + \theta + \phi] = 0$

$$\boxed{mg\ell(3\theta+\phi) = 0}$$

$\underline{\underline{\phi}}$: $\boxed{mg\ell(\theta+\phi) = 0}$