**ME 562 Advanced Dynamics**
**Summer 2010**
**HOMEWORK # 4**

**Due: June 28, 2010**

**Q1.** A massless disc of radius $R$ has an embedded particle of mass $m$ at a distance $R/2$ from the center. The disc is released from rest in the position shown and rolls without slipping down the fixed inclined plane.

Find: (a) the equation of motion of the particle in terms of the angle $\theta$ and its time derivatives;

(b) $\dot{\theta}$ as a function of $\theta$. This is really integration of the equation of motion starting with the initial condition given above in the statement. Hint: It is easier to do in terms of the energy conservation principle for the particle. **(Problem 3-10 in the text).**

**Q2.** Initially the spring has its unstretched length $l_0$ and the particle has a velocity $v_0$ in the direction shown. In the motion that follows, the spring stretches to a maximum length of $4l_0/3$. Assuming no gravity (motion in horizontal plane), find the spring stiffness $k$ as a function of $m$, $l_0$, and $v_0$. **(Problem 3-15 in the text).**

**Q3.** A spring-mass system is connected as shown in Fig. 3-27(a) of the text. There is Coulomb friction between the mass and the horizontal surface on which it slides. The mass is pulled to the right from the position in which the spring is unstretched and given an initial displacement of $x_0$. At this position, the mass has zero velocity. Now the mass is released.

(a) Derive the equations of motion for the mass for the cases of $\dot{x} > 0$ and $\dot{x} < 0$ clearly by drawing the appropriate Free Body Diagrams;

(b) Derive solutions for the two differential equation models in (a), and use the appropriate initial conditions to construct solutions in each of the velocity regions;

(c) Finally, derive the expression for displacement $x(t)$ in the $n$th half-cycle (equation (3-218) in the text), and the condition required on the initial stretch $x_0$ for this motion to be possible. **(Example 3-11 in the text, also covered in class and notes).**

**Q4.** Particle A, moving with velocity $v_0$ in a direction perpendicular to the motionless dumbbell BC, hits the particle B squarely. The impact process has a coefficient of restitution $e$. Assume that the system sits on a smooth horizontal table. Determine the velocity of particle A, and the translational and rotational velocities of the dumbbell BC immediately after impact. **(Problem 4-10 in the text).**

# ME 562

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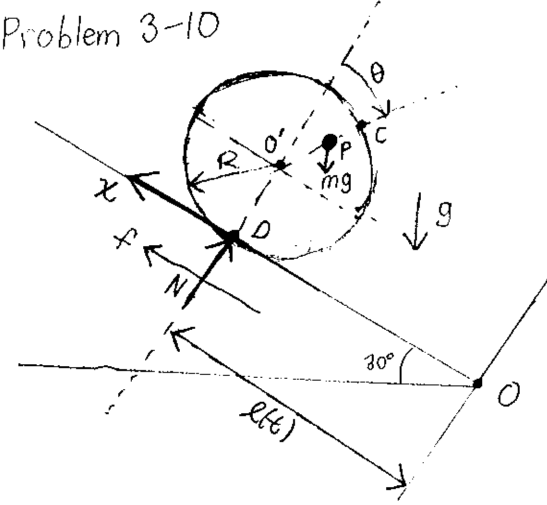
Q1 Problem 3-10






$$\underline{r}_{OP} = \ell(t)\,\underline{i} + R\,\underline{j} + \frac{R}{2}\left(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}\right)$$
$$= \left[\ell(t) - \frac{R}{2}\sin\theta\right]\underline{i} + \left[R + \frac{R}{2}\cos\theta\right]\underline{j} \quad (1)$$

$$\dot{\underline{r}}_{OP} = \left[\dot{\ell} - \frac{R\dot{\theta}}{2}\cos\theta\right]\underline{i} - \frac{R\dot{\theta}}{2}\sin\theta\,\underline{j} \quad (2)$$

$$\ddot{\underline{r}}_{OP} = \left[\ddot{\ell} - \frac{R\ddot{\theta}}{2}\cos\theta + \frac{R\dot{\theta}^2}{2}\sin\theta\right]\underline{i} - \left[\frac{R\ddot{\theta}}{2}\sin\theta + \frac{R\dot{\theta}^2}{2}\cos\theta\right]\underline{j} \quad (3)$$

Rolling constraint: $\dot{\underline{r}}_{OD} = 0$ for a point D on the disc

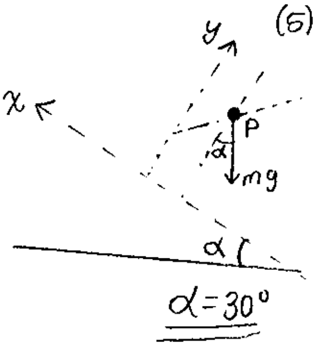
Note $\dot{\underline{r}}_{OC} = [\dot{\ell} - R\dot{\theta}\cos\theta]\underline{i} - R\dot{\theta}\sin\theta\,\underline{j}$

and $C = D$ when $\theta = \pi$

$$\dot{r}_{OD} = \left[\dot{\ell} - R\dot{\theta}\cos\pi\right]\underline{i} - R\dot{\theta}\sin\pi\,\underline{j} = 0$$

$$\dot{\ell} + R\dot{\theta} = 0 \Rightarrow \dot{\ell} = -R\dot{\theta} \quad (4)$$
$$\ddot{\ell} = -R\ddot{\theta} \quad (5)$$

(5)→(3): $$\ddot{\underline{r}}_{OP} = \left[-R\ddot{\theta} - \frac{R\ddot{\theta}}{2}\cos\theta + \frac{R\dot{\theta}^2}{2}\sin\theta\right]\underline{i} - \left[\frac{R\ddot{\theta}}{2}\sin\theta + \frac{R\dot{\theta}^2}{2}\cos\theta\right]\underline{j} \quad (6)$$




a)

Equations of motion

$$\Sigma F_x = m a_x$$
$$F - mg\sin\alpha = m\left[-R\ddot{\theta} - \frac{R\ddot{\theta}}{2}\cos\theta + \frac{R\dot{\theta}^2}{2}\sin\theta\right] \quad (7)$$

$$\Sigma F_y = m a_y$$
$$N - mg\cos\alpha = m\left[-\frac{R\ddot{\theta}}{2}\sin\theta - \frac{R\dot{\theta}^2}{2}\cos\theta\right] \quad (8)$$

$$\Sigma M_o = \frac{d}{dt}\underline{H}_o = \underline{r}_{OP} \times m\ddot{\underline{r}}_{OP}$$

$$\dot{\underline{H}}_0 = \underline{r}_{OP} \times m\ddot{\underline{r}}_{OP} = \underline{r}_{OP} \times m\ddot{\underline{r}}_{OP} = m\left[\left(\ell - \frac{R}{2}\sin\theta\right)\underline{i} + \left(R + \frac{R}{2}\cos\theta\right)\underline{j}\right] \times$$
$$\left[\left(\ddot{\ell} - \frac{R\ddot{\theta}}{2}\cos\theta + \frac{R\dot{\theta}^2}{2}\sin\theta\right)\underline{i} - \left(\frac{R\ddot{\theta}}{2}\sin\theta + \frac{R\dot{\theta}^2}{2}\cos\theta\right)\underline{j}\right]$$

(2)

$$= m\left\{\left(\ell - \frac{R}{2}\sin\theta\right)\left(-\frac{R\ddot{\theta}}{2}\sin\theta - \frac{R\dot{\theta}^2}{2}\cos\theta\right) - \left(R + \frac{R\cos\theta}{2}\right)\left(-R\ddot{\theta} - \frac{R\ddot{\theta}}{2}\cos\theta + \frac{R\dot{\theta}^2}{2}\sin\theta\right)\right\}\underline{k} \quad (9)$$

$$\Sigma \underline{M}_0 = N\ell(t)\,\underline{k} + \underline{r}_{OP} \times mg(-\sin\alpha\,\underline{i} - \cos\alpha\,\underline{j})$$

$$= N\ell(t)\,\underline{k} + mg\left[\left(\ell - \frac{R}{2}\sin\theta\right)\underline{i} + \left(R + \frac{R}{2}\cos\theta\right)\underline{j}\right] \times (-\sin\alpha\,\underline{i} - \cos\alpha\,\underline{j})$$

$$= \left[N\ell(t) - mg\left(\ell - \frac{R}{2}\sin\theta\right)\cos\alpha + mgR\sin\alpha\left(1 + \frac{1}{2}\cos\theta\right)\right]\underline{k} = \underline{\dot{H}}_0 \quad (10)$$

$(9) \to (10)$: $N\ell(t) - mg\ell\cos\alpha + mg\frac{R}{2}\sin\theta\cos\alpha + mg\frac{R}{2}\cos\theta\sin\alpha + mgR\sin\alpha$

$$= m\left[\left(\ell - \frac{R}{2}\sin\theta\right)\left(-\frac{R\ddot{\theta}}{2}\sin\theta - \frac{R\dot{\theta}^2}{2}\cos\theta\right) - \left(R + \frac{R\cos\theta}{2}\right)\left(-R\ddot{\theta} - \frac{R\ddot{\theta}}{2}\cos\theta + \frac{R\dot{\theta}^2}{2}\sin\theta\right)\right]$$

$$N\ell(t) - mg\ell\cos\alpha + \frac{mgR}{2}\sin(\theta+\alpha) + mgR\sin\alpha = m\left[-\ell R\left(\ddot{\theta}\sin\theta + \frac{\dot{\theta}^2}{2}\cos\theta\right) + \frac{R^2}{2}\left(\frac{\ddot{\theta}}{2}\sin^2\theta + \frac{\dot{\theta}^2}{2}\sin\theta\cos\theta\right) + R^2\ddot{\theta}\left(1 + \frac{1}{2}\cos\theta\right) - \frac{R^2\dot{\theta}^2}{2}\sin\theta + \frac{R^2\ddot{\theta}}{2}\left(\cos\theta + \frac{\cos^2\theta}{2}\right) - \frac{R^2\dot{\theta}^2}{4}\sin\theta\cos\theta\right]$$

$$N\ell(t) - mg\ell\cos\alpha + \frac{mgR}{2}\sin(\theta+\alpha) + mgR\sin\alpha$$

$$= m\left[-\ell R\left(\frac{\ddot{\theta}\sin\theta}{2} + \frac{\dot{\theta}^2}{2}\cos\theta\right) + \frac{R^2}{2}\frac{\dot{\theta}^2}{2}\left(\sin\theta\cancel{\cos\theta} - \sin\theta\cancel{\cos\theta}\right) + R^2\ddot{\theta}\left(1 + \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta\right) - \frac{R^2\dot{\theta}^2}{2}\sin\theta + \frac{R^2}{2}\frac{\ddot{\theta}}{2}(\sin^2\theta + \cos^2\theta)\right]$$

$$N\ell(t) - mg\ell\cos\alpha + \frac{mgR}{2}\sin(\theta+\alpha) + mgR\sin\alpha = m\left[-\ell R\left(\frac{\ddot{\theta}\sin\theta}{2} + \frac{\dot{\theta}^2}{2}\cos\theta\right) + \frac{R^2\ddot{\theta}}{4} + R^2\ddot{\theta}(1+\cos\theta) - \frac{R^2\dot{\theta}^2}{2}\sin\theta\right]$$

$$N\ell(t) - mg\ell\cos\alpha + \frac{mgR}{2}\sin(\theta+\alpha) + mgR\sin\alpha = m\left[-\ell R\left(\frac{\ddot{\theta}}{2}\sin\theta + \frac{\dot{\theta}^2}{2}\cos\theta\right) + R^2\ddot{\theta}\left(\frac{5}{4} + \cos\theta\right) - \frac{R^2\dot{\theta}^2}{2}\sin\theta\right] \quad (11)$$

(7), (8), (11) are the required equations. Using (8), N can be eliminated from (11)

$$\cancel{\ell mg\cos\alpha} + \cancel{m\ell\left[-\frac{R\ddot{\theta}}{2}\sin\theta - \frac{R\dot{\theta}^2}{2}\cos\theta\right]} - \cancel{mg\ell\cos\alpha} + \frac{mgR}{2}\sin(\theta+\alpha) + mgR\sin\alpha$$

$$= \cancel{m\ell\left[-\frac{R\ddot{\theta}}{2}\sin\theta - \frac{R\dot{\theta}^2}{2}\cos\theta\right]} + mR^2\ddot{\theta}\left(\frac{5}{4} + \cos\theta\right) - \frac{mR^2\dot{\theta}^2}{2}\sin\theta$$

$$mR^2\left[\ddot{\theta}\left(\frac{5}{4}+\cos\theta\right)-\frac{\dot{\theta}^2}{2}\sin\theta\right]=\underbrace{\frac{mgR}{2}\sin(\theta+\alpha)+mgR\sin\alpha}_{mgR\left[\frac{1}{2}\sin(\theta+30^\circ)+\frac{1}{2}\right]=\frac{mgR}{2}\left[\sin(\theta+30^\circ)+1\right]} \quad (12)$$

$$\boxed{mR^2\left[\ddot{\theta}\left(\frac{5}{4}+\cos\theta\right)-\frac{\dot{\theta}^2}{2}\sin\theta\right]=\frac{mgR}{2}\left[1+\sin\left(\theta+\frac{\pi}{6}\right)\right]} \quad (13)$$

Equation of motion in terms of angle $\theta$

b) Find $\dot{\theta}$ in terms of a function of $\theta$. Integrate equation of motion as function of $\theta$ (12).

$$\Rightarrow \text{Let } \ddot{\theta}=\frac{d\dot{\theta}}{d\theta}\dot{\theta}=\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)$$

From (12) $$mR^2\frac{d}{d\theta}\left[\frac{1}{2}\dot{\theta}^2\left(\frac{5}{4}+\cos\theta\right)\right]=\frac{mgR}{2}\sin(\theta+\alpha)+mgR\sin\alpha$$

$$\cancel{m}R^{\cancel{2}}\int d\left[\frac{1}{2}\dot{\theta}^2\left(\frac{5}{4}+\cos\theta\right)\right]=\int_0^\theta\left[\frac{\cancel{m}g\cancel{R}}{2}\sin(\theta+\alpha)+\cancel{m}g\cancel{R}\sin\alpha\right]d\theta$$

$$R\left[\frac{1}{2}\dot{\theta}^2\left(\frac{5}{4}+\cos\theta\right)\right]=g\left[-\frac{1}{2}\cos(\theta+\alpha)+\sin\alpha\cdot\theta\right]_0^\theta$$

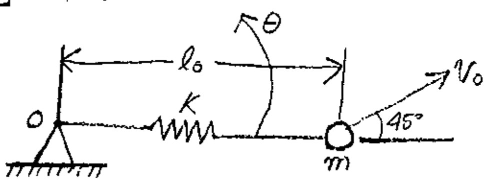
$$\frac{R}{\cancel{2}}\left[\frac{\dot{\theta}^2}{4}(5+4\cos\theta)\right]=\frac{g}{\cancel{2}}\left[-\cos(\theta+\alpha)+2\sin\alpha\cdot\theta+\cos\alpha\right]$$

$$\dot{\theta}^2=\frac{4g}{R}\left[\frac{-\cos(\theta+\alpha)+2\theta\sin\alpha+\cos\alpha}{5+4\cos\theta}\right] \quad \text{for } \alpha=30^\circ$$

$$\dot{\theta}^2=\frac{g}{R}\left[\frac{-4\cos(\theta+30^\circ)+4\theta+2\sqrt{3}}{5+4\cos\theta}\right]$$

$$\boxed{\dot{\theta}=\sqrt{\frac{g}{R}\left[\frac{2\sqrt{3}+4\theta-4\cos(\theta+30^\circ)}{5+4\cos\theta}\right]}}$$

22 Problem 3-15




$r_{max} = r = \frac{4l_0}{3}$

$\dot{r} = 0$

Assumption: No gravity (motion in horizontal plane)

① Conservation of energy $(T_1 + V_1 = T_2 + V_2)$

$T_2 = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} I \omega^2$ where $\omega = \dot{\theta}$ and $I = mr^2$

$T_2 = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 = \frac{1}{2} m (0)^2 + \frac{1}{2} m \left(\frac{4l_0}{3}\right)^2 \dot{\theta}^2 = \frac{8}{9} m l_0^2 \dot{\theta}^2$

$V_2 = \frac{1}{2} K (\Delta l)^2 = \frac{1}{2} k (r - l_0)^2 = \frac{1}{2} k \left(\frac{4l_0}{3} - l_0\right)^2 = \frac{1}{2} k \left(\frac{l_0}{3}\right)^2 = \frac{1}{18} k l_0^2$

$T_1 = \frac{1}{2} m V_0^2$

$V_1 = 0$ (spring is unstretched so potential energy due to the spring is zero).

$\Rightarrow \frac{1}{2} m V_0^2 = \frac{8}{9} m l_0^2 \dot{\theta}^2 + \frac{1}{18} k l_0^2$ ①

② Conservation of angular momentum $(\Sigma H_0 = m \times r v)$

$m r^2 \dot{\theta} = m l_0 (V_0 \sin 45°)$

$r^2 \dot{\theta} = l_0 V_0 \frac{\sqrt{2}}{2}$

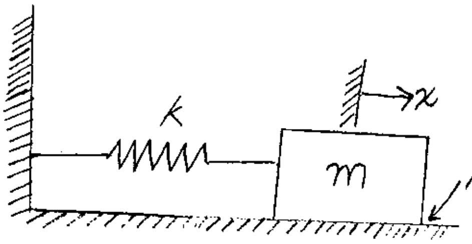
$\frac{16 l_0^2}{9} \dot{\theta} = l_0 V_0 \frac{\sqrt{2}}{2} \Rightarrow \dot{\theta} = \frac{9 V_0}{16\sqrt{2}\, l_0}$ ② when $r = r_{max} = \frac{4l_0}{3}$

② → ①: $\frac{1}{2} m V_0^2 = \frac{8}{9} m \cancel{l_0^2} \left(\frac{81 V_0^2}{512 \cancel{l_0^2}}\right) + \frac{1}{18} K l_0^2$

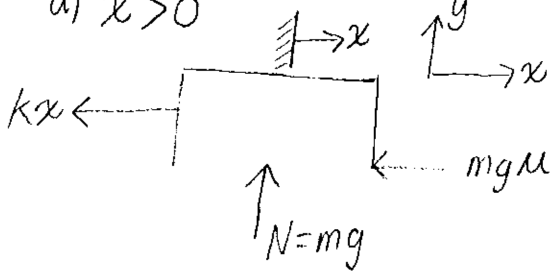
$m V_0^2 \left[\frac{1}{2} - \frac{9}{64}\right] = \frac{1}{18} k l_0^2 \Rightarrow \frac{23}{64} m V_0^2 = \frac{1}{18} k l_0^2$

$$\boxed{k = 18\left(\frac{23}{64}\right) m \left(\frac{V_0}{l_0}\right)^2 = 6.4688 \frac{m V_0^2}{l_0^2}}$$

Q3 



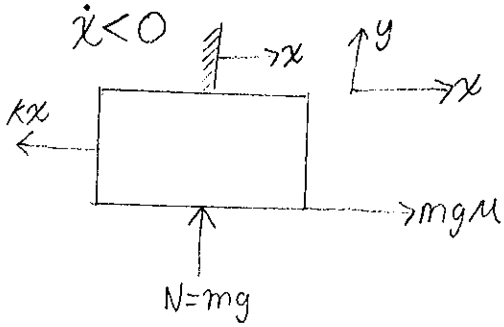

a) $\dot{x} > 0$




$m\ddot{x} = -kx - mg\mu$

$\boxed{m\ddot{x} + kx = -mg\mu}$

$\dot{x} < 0$




$m\ddot{x} = mg\mu - kx$

$\boxed{m\ddot{x} + kx = mg\mu}$

b) $x(0) = x_0, \ \dot{x}(0) = 0$

$x(t) = x_h(t) + x_p(t)$

For $\dot{x} < 0$: $kx_p = mg\mu \Rightarrow x_p = \frac{mg\mu}{k}$

$x_h = A\cos\omega_n t + B\sin\omega_n t \quad \left(\omega_n = \sqrt{\frac{k}{m}}\right)$

$x = \frac{mg\mu}{k} + A\cos\omega_n t + B\sin\omega_n t$

$\dot{x} = -\omega_n A\sin\omega_n t + \omega_n B\cos\omega_n t$

$x(0) = x_0 = \frac{mg\mu}{k} + A \Rightarrow A = x_0 - \frac{mg\mu}{k}$

$\dot{x}(0) = 0 = \omega_n B\omega_n \Rightarrow B = 0$

$$\boxed{x(t) = \frac{mg\mu}{k} + \left(x_0 - \frac{mg\mu}{k}\right)\cos\omega_n t}$$

$\dot{x} = 0 = -\omega_n\left(x_0 - \frac{mg\mu}{k}\right)\sin\omega_n t \Rightarrow \sin\omega_n t = 0$

$\Rightarrow t_1 = \frac{\pi}{\omega_n} \Rightarrow x\left(\frac{\pi}{\omega_n}\right) = \frac{mg\mu}{k} + \left(x_0 - \frac{mg\mu}{k}\right)\cos\pi$

$$\boxed{x\left(\frac{\pi}{\omega_n}\right) = \frac{mg\mu}{k} - x_0 + \frac{mg\mu}{k} = \frac{2mg\mu}{k} - x_0 \text{ where } \dot{x}(\pi/\omega_n) = 0}$$

For $\dot{x}>0$: $kx_p = -mg\mu \Rightarrow x_p = -\frac{mg\mu}{k}$

$$x_h = A\cos\omega_n t + B\sin\omega_n t$$

$$x(t) = A\cos\omega_n t + B\sin\omega_n t - \frac{mg\mu}{k}$$

$$\dot{x}(t) = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t$$

At $t = \pi/\omega_n$

$$x(\pi/\omega_n) = A\cos\pi + B\sin\pi - \frac{mg\mu}{k} = -A - \frac{mg\mu}{k} = \frac{2mg\mu}{k} - x_0$$

$$A = x_0 - \frac{3mg\mu}{k}$$

$$\dot{x}(\pi/\omega_n) = -A\omega_n \sin\pi + B\omega_n\cos\pi = -B\omega_n = 0 \Rightarrow B = 0$$

$$\boxed{x(t) = \left(x_0 - \frac{3mg\mu}{k}\right)\cos\omega_n t - \frac{mg\mu}{k} \quad \text{for} \quad \pi/\omega_n < t < 2\pi/\omega_n}$$

$$\dot{x}(t) = -\omega_n\left(x_0 - \frac{3mg\mu}{k}\right)\sin\omega_n t = 0 \Rightarrow \omega_n t = 2\pi \Rightarrow t_2 = \frac{2\pi}{\omega_n}$$

$$\boxed{x\left(\frac{2\pi}{\omega_n}\right) = \left(x_0 - \frac{3mg\mu}{k}\right)\cos 2\pi - \frac{mg\mu}{k} = x_0 - \frac{4mg\mu}{k}}$$

c) $$x(t) = (-1)^{n-1}\frac{\mu mg}{k} + \left\{x_0 - (2n-1)\frac{\mu mg}{k}\right\}\cos\omega_n t \; ; \left[\frac{(n-1)\pi}{\omega_n} < t < \frac{n\pi}{\omega_n}\right]$$

First half-cycle: $x_0 > \frac{\mu mg}{k}$

Second half-cycle: $x_0 > \frac{3\mu mg}{k}$

⋮

$n^{th}$ half-cycle: $\boxed{x_0 > (2n-1)\frac{\mu mg}{k}}$ ✓

Equation of motion: $m\ddot{x} + kx = (-1)^{n-1} mg\mu$ for any $n^{th}$ cycle

$$x_p = \frac{(-1)^{n-1} mg\mu}{k}$$

$$x_h = A\cos\omega_n t + B\sin\omega_n t$$

$$x(t) = \frac{(-1)^{n-1} mg\mu}{k} + A\cos\omega_n t + B\sin\omega_n t$$

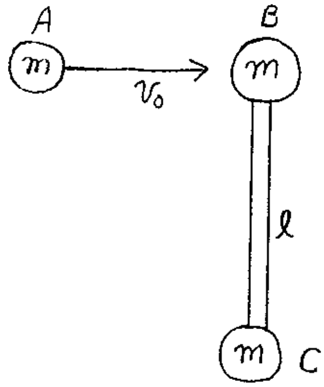
$$x(0) = x_0 = \frac{(2n-1)mg\mu}{k} + A \Rightarrow A = x_0 - \frac{(2n-1)mg\mu}{k}$$

$$\dot{x}(t) = -\omega_n A \sin \omega_n t + B\omega_n \cos \omega_n t$$

$$\dot{x}(0) = 0 = B\omega_n \Longrightarrow B = 0$$

$$\boxed{x(t) = (-1)^{n-1}\frac{\mu m g}{k} + \left\{x_0 - (2n-1)\frac{\mu m g}{k}\right\}\cos \omega_n t}$$

Q4




Assumption: System sits on a smooth horizontal table

$$I = \frac{1}{3}(2m)\ell^2 = \frac{2}{3} m\ell^2$$

$$v_A' = \frac{m_A - e m_B}{m_A + m_B} v_A + \frac{(1-e)m_B}{m_A + m_B} \cancel{v_B}^{\,0}$$

$v_B = 0$ and $v_A = v_0$

$$v_A' = \frac{m - em}{2m} v_0 = \boxed{\left(\frac{1-e}{2}\right) v_0}$$

$$v_B' = \frac{(1+e)m_A}{m_A + m_B} v_A + \frac{m_B - e m_A}{m_A + m_B} \cancel{v_B}^{\,0} \quad (v_B = 0 \;\&\; v_A = v_0)$$

$$= \frac{(1+e)m}{2m} = \left(\frac{1+e}{2}\right) v_0$$

$$\boxed{v_{cm} = \frac{1}{2} v_B' = \left(\frac{1+e}{4}\right) v_0}$$ translational velocity

$$v_B' = \ell\omega' \Rightarrow \boxed{\omega' = \frac{v_B'}{\ell} = \left(\frac{1+e}{2\ell}\right) v_0}$$ rotational velocity