ME 562

HW #2 SOLUTION

Summer 2010

Q1 Prove Frenet's formulas

$$\boxed{\begin{aligned} \frac{d\underline{e}_t}{ds} &= \frac{\underline{e}_n}{\rho} = K\underline{e}_n \\ \frac{d\underline{e}_b}{ds} &= -\frac{\underline{e}_n}{\tau} \\ \frac{d\underline{e}_n}{ds} &= -\frac{\underline{e}_t}{\rho} + \frac{\underline{e}_b}{\tau} \end{aligned}}$$

i) $\underline{e}_t \cdot \underline{e}_b = 0$

$$\frac{d}{ds}(\underline{e}_t \cdot \underline{e}_b) = 0$$

$$\underline{e}_t \cdot \frac{d\underline{e}_b}{ds} + \frac{d\underline{e}_t}{ds} \cdot \underline{e}_b = 0$$

$$\therefore \underline{e}_t \frac{d\underline{e}_b}{ds} = -\frac{\underline{e}_n}{\rho} \cdot \underline{e}_b = 0$$

$$\therefore \underline{e}_t \perp \frac{d\underline{e}_b}{ds}$$

$$\underline{e}_b \cdot \underline{e}_b = 1$$

$$\frac{d}{ds}(\underline{e}_b \cdot \underline{e}_b) = 0$$

$$2\frac{d\underline{e}_b}{ds} \cdot \underline{e}_b = 0$$

$\therefore \underline{e}_b \perp \frac{d\underline{e}_b}{ds}$ which means $\frac{d\underline{e}_b}{ds}$ is only along $\underline{e}_n$

$$\boxed{\frac{d\underline{e}_b}{ds} = -\frac{\underline{e}_n}{\tau} = -\tau_w \underline{e}_n}$$

ii) $\underline{e}_t \cdot \underline{e}_t = 1$

$$\frac{d}{ds}(\underline{e}_t \cdot \underline{e}_t) = 0$$

$$2\frac{d\underline{e}_t}{ds} \cdot \underline{e}_t = 0$$

$$\therefore \underline{e}_t \perp \frac{d\underline{e}_t}{ds}$$

Similarly, $\underline{e}_t \cdot \underline{e}_b = 0 \Rightarrow \frac{d}{ds}(\underline{e}_t \cdot \underline{e}_b) = 0 \Rightarrow \underline{e}_t \perp \frac{d\underline{e}_b}{ds}$ and $\frac{d\underline{e}_t}{ds} \perp \underline{e}_b$

$$\boxed{\therefore \frac{d\underline{e}_t}{ds} = \frac{\underline{e}_n}{\rho} = K\underline{e}_n}$$

iii) $\frac{d\underline{e}_n}{ds}$ is a vector which can be written in its basis as follows —

$$\frac{d\underline{e}_n}{ds} = \left(\frac{d\underline{e}_n}{ds}\cdot\underline{e}_t\right)\underline{e}_t + \left(\frac{d\underline{e}_n}{ds}\cdot\underline{e}_n\right)\underline{e}_n + \left(\frac{d\underline{e}_n}{ds}\cdot\underline{e}_b\right)\underline{e}_b$$

$\Rightarrow \underline{e}_n\cdot\underline{e}_n = 1$

$$\frac{d}{ds}(\underline{e}_n\cdot\underline{e}_n) = 0$$

$$2\,\underline{e}_n\cdot\frac{d\underline{e}_n}{ds} = 0 \quad \therefore\ \underline{e}_n \perp \frac{d\underline{e}_n}{ds}$$

Also, as $\underline{e}_n \perp \underline{e}_t$

$$\underline{e}_n\cdot\underline{e}_t = 0$$

$$\frac{d}{ds}(\underline{e}_n\cdot\underline{e}_t) = 0$$

$$\underline{e}_n\cdot\frac{d\underline{e}_t}{ds} = -\underline{e}_t\cdot\frac{d\underline{e}_n}{ds}$$

$$\underline{e}_t\cdot\frac{d\underline{e}_n}{ds} = -\frac{d\underline{e}_t}{ds}\cdot\underline{e}_n = -K\,\underline{e}_n\cdot\underline{e}_n = -K$$

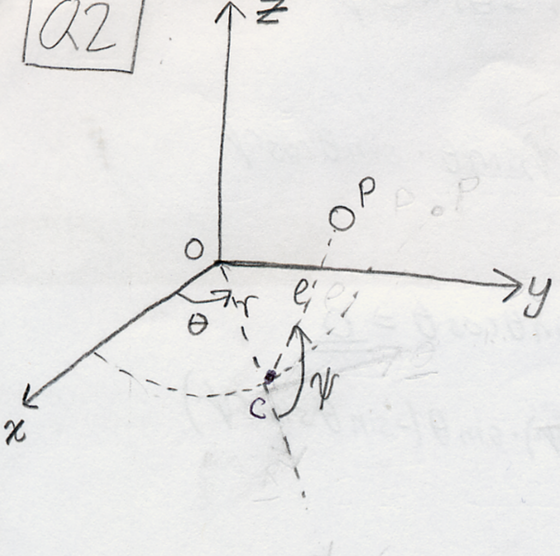
And as $\underline{e}_n \perp \underline{e}_b$

$$\frac{d}{ds}(\underline{e}_n\cdot\underline{e}_b) = 0$$

$$\frac{d\underline{e}_n}{ds}\cdot\underline{e}_b = -\underline{e}_n\cdot\frac{d\underline{e}_b}{ds} = -\underline{e}_n\cdot(-\tau\underline{e}_n) = \tau$$

$$\boxed{\frac{d\underline{e}_n}{ds} = -K\underline{e}_t + \tau\underline{e}_b}$$

Q2



Cartesian coordinates of the position P in terms of x, y, z

$$x = (r + e\cos\psi)\cos\theta$$
$$y = (r + e\cos\psi)\sin\theta$$
$$z = e\sin\psi$$

Step 1: Find $\underline{e}_e, \underline{e}_\theta, \underline{e}_\psi$ (unit vectors of $\underline{r}_{OP}$)

$$\underline{r}_{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$
$$= (r + e\cos\psi)\cos\theta\,\hat{\imath} + (r + e\cos\psi)\sin\theta\,\hat{\jmath} + e\sin\psi\,\hat{k}$$

① $$\underline{e}_e = \frac{\partial \underline{r}_{OP}/\partial e}{|\partial \underline{r}_{OP}/\partial e|} = \boxed{\cos\psi\cos\theta\,\hat{\imath} + \cos\psi\sin\theta\,\hat{\jmath} + \sin\psi\,\hat{k}}$$

where $\left|\frac{\partial \underline{r}_{OP}}{\partial e}\right| = 1$

② $$\underline{e}_\psi = \frac{\partial \underline{r}_{OP}/\partial \psi}{|\partial \underline{r}_{OP}/\partial \psi|}$$

$$\frac{\partial \underline{r}_{OP}}{\partial \psi} = -e\sin\psi\cos\theta\,\hat{\imath} - e\sin\psi\sin\theta\,\hat{\jmath} + e\cos\psi\,\hat{k}$$

$$\left|\frac{\partial \underline{r}_{OP}}{\partial \psi}\right| = e$$

$$\underline{e}_\psi = \frac{1}{e}\left[-e\sin\psi\cos\theta\,\hat{\imath} - e\sin\psi\sin\theta\,\hat{\jmath} + e\cos\psi\,\hat{k}\right]$$
$$= \frac{e}{e}\left[-\sin\psi\cos\theta\,\hat{\imath} - \sin\psi\sin\theta\,\hat{\jmath} + \cos\psi\,\hat{k}\right]$$
$$= -\sin\psi\cos\theta\,\hat{\imath} - \sin\psi\sin\theta\,\hat{\jmath} + \cos\psi\,\hat{k}$$
$$= \boxed{-\sin\psi\cos\theta\,\hat{\imath} - \sin\psi\sin\theta\,\hat{\jmath} + \cos\psi\,\hat{k}}$$

③ $$\underline{e}_\theta = \frac{\partial \underline{r}_{OP}/\partial \theta}{|\partial \underline{r}_{OP}/\partial \theta|} = \frac{-(r + e\cos\psi)\sin\theta\,\hat{\imath} + (r + e\cos\psi)\cos\theta\,\hat{\jmath}}{r + e\cos\psi}$$

$$= \boxed{-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}}$$

Step 2: Find position of point P in terms of $\underline{e}_\rho$, $\underline{e}_\theta$, & $\underline{e}_\psi$

$$\underline{r}_{OP} = (r+\rho\cos\psi)\cos\theta\,\underline{i} + (r+\rho\cos\psi)\sin\theta\,\underline{j} + \rho\sin\psi\,\underline{k}$$

$\boxed{1}$ $\underline{r}_{OP}\cdot\underline{e}_\rho = (r+\rho\cos\psi)\cos\theta\cdot\cos\theta\cos\psi + (r+\rho\cos\psi)\sin\theta\cdot\sin\theta\cos\psi$
$+\rho\sin\psi\sin\psi = \underline{r\cos\psi+\rho}$

$\boxed{2}$ $\underline{r}_{OP}\cdot\underline{e}_\theta = (r+\rho\cos\psi)\cos\theta\cdot(-\sin\theta) + (r+\rho\cos\psi)\sin\theta\cos\theta = \underline{\underline{0}}$

$\boxed{3}$ $\underline{r}_{OP}\cdot\underline{e}_\psi = (r+\rho\cos\psi)\cos\theta\cdot(-\cos\theta\sin\psi) + (r+\rho\cos\psi)\cdot\sin\theta(-\sin\theta\sin\psi)$
$+\rho\sin\psi\cos\psi) = \underline{\underline{-r\sin\psi}}$

$$\boxed{\underline{r}_{OP} = (r\cos\psi+\rho)\,\underline{e}_\rho + (-r\sin\psi)\,\underline{e}_\psi}$$

Step 3: Find all derivatives

$$\underline{e}_\rho = \cos\theta\cos\psi\,\underline{i} + \sin\theta\sin\psi\,\underline{j} + \sin\psi\,\underline{k}$$

$$\frac{d\underline{e}_\rho}{\partial\rho} = 0$$

$$\frac{\partial\underline{e}_\rho}{\partial\theta} = -\cos\psi\sin\theta\,\underline{i} + \cos\psi\cos\theta\,\underline{j}$$

$$\frac{\partial\underline{e}_\rho}{\partial\psi} = -\sin\psi\cos\theta\,\underline{i} - \sin\psi\sin\theta\,\underline{j} + \cos\psi\,\underline{k}$$

$$\underline{e}_\theta = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}$$

$$\frac{\partial\underline{e}_\theta}{\partial\rho} = 0$$

$$\frac{\partial\underline{e}_\theta}{\partial\theta} = -\cos\theta\,\underline{i} - \sin\theta\,\underline{j}$$

$$\frac{\partial\underline{e}_\theta}{\partial\psi} = 0$$

$\underline{e}_\psi = -\cos\theta\sin\psi\,\underline{i} - \sin\theta\sin\psi\,\underline{j} + \cos\psi\,\underline{k}$

$$\frac{\partial \underline{e}_\psi}{\partial e} = 0$$

$$\frac{\partial \underline{e}_\psi}{\partial \theta} = \sin\theta\sin\psi\,\underline{i} - \cos\theta\sin\psi\,\underline{j}$$

$$\frac{\partial \underline{e}_\psi}{\partial \psi} = -\cos\theta\cos\psi\,\underline{i} - \sin\theta\sin\psi\,\underline{j} - \sin\psi\,\underline{k}$$

$\boxed{Q3}$ Find velocity of point P in terms of e, θ, ψ & $\underline{e}_e, \underline{e}_\theta, \underline{e}_\psi$

$$\underline{r}_{OP} = (r\cos\psi + e)\,\underline{e}_e + (-r\sin\psi)\,\underline{e}_\psi$$

$$\dot{\underline{r}}_{OP} = \frac{d}{dt}\left[(r\cos\psi + e)\,\underline{e}_e + (-r\sin\psi)\,\underline{e}_\psi\right]$$

$$\frac{d}{dt}(r\cos\psi + e) = -r\sin\psi\,\dot{\psi} + \dot{e}$$

$$\frac{d}{dt}(-r\sin\psi) = -r\cos\psi\,\dot{\psi}$$

$$\frac{d}{dt}(\underline{e}_e) = \frac{\partial \underline{e}_e}{\partial e}\frac{\partial e}{\partial t} + \frac{\partial \underline{e}_e}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial \underline{e}_e}{\partial \psi}\frac{\partial \psi}{\partial t}$$

$$= (-\cos\psi\sin\theta\,\underline{i} + \cos\psi\cos\theta\,\underline{j})\,\dot{\theta} + (-\sin\psi\cos\theta\,\underline{i} - \sin\psi\sin\theta\,\underline{j} + \cos\psi\,\underline{k})\,\dot{\psi}$$

$$\frac{d}{dt}(\underline{e}_\psi) = \frac{\partial \underline{e}_\psi}{\partial e}\cdot\frac{\partial e}{\partial t} + \frac{\partial \underline{e}_\psi}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial \underline{e}_\psi}{\partial \psi}\frac{\partial \psi}{\partial t}$$

$$= (\sin\theta\sin\psi\,\underline{i} - \cos\theta\sin\psi\,\underline{j})\,\dot{\theta} + (-\cos\theta\cos\psi\,\underline{i} - \sin\theta\cos\psi\,\underline{j} - \sin\psi\,\underline{k})\,\dot{\psi}$$

Simplifying,

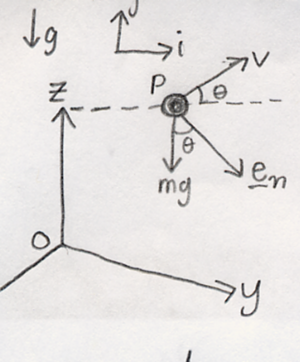
$$\frac{d}{dt}(\underline{e}_e) = \cos\psi\,\dot{\theta}\,\underline{e}_\theta + \dot{\psi}\,\underline{e}_\psi$$

$$\frac{d}{dt}(\underline{e}_\psi) = -\sin\psi\,\dot{\theta}\,\underline{e}_\theta - \dot{\psi}\,\underline{e}_e$$

Substituting all,

$$\boxed{\underline{v}_{OP} = (\dot{e})\,\underline{e}_e + (e\dot{\psi})\,\underline{e}_\psi + (r\dot{\theta} + e\dot{\theta}\cos\psi)\,\underline{e}_\theta}$$

Q4



Velocity of Point P

$$\underline{V}_p = v\cos\theta\, i + v\sin\theta\, j$$

$$\boxed{\begin{cases} \underline{a}_n = (g\cos\theta)\cdot \underline{e}_n \\ \underline{a}_t = -g\sin\theta \cdot \underline{e}_t \end{cases}}$$

$$\underline{a}_p = g\cos\theta\, \underline{e}_n + (-g\sin\theta)\, \underline{e}_t$$

where $a_n = \dfrac{v^2}{\rho}$ $\boxed{\therefore \rho = \dfrac{v^2}{g\cos\theta}}$

$$\frac{d\underline{v}_p}{dt} = \underline{a}_p = (\dot{v}\cos\theta - v\dot{\theta}\sin\theta)\, i + (\dot{v}\sin\theta + v\dot{\theta}\cos\theta)\, j$$

$$\dot{v}\cos\theta - v\dot{\theta}\sin\theta = 0$$

$$\dot{v}\sin\theta + v\dot{\theta}\cos\theta = -g$$

$$\therefore \boxed{\dot{\theta} = \frac{-g\cos\theta}{v}}$$

and $\dot{v} = -g\sin\theta$

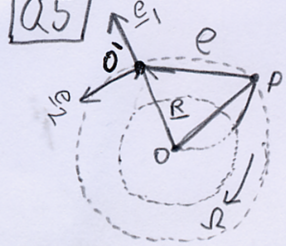
$$\dot{\rho} = (2v\cdot\dot{v}\cos\theta + v^2\dot{\theta}\sin\theta)/(g\cos^2\theta)$$

$$\therefore \boxed{\dot{\rho} = -3v\tan\theta}$$

$$\ddot{\theta} = -\frac{g}{v^2}(-v\dot{\theta}\sin\theta - \dot{v}\cos\theta)$$

$$\boxed{\therefore \ddot{\theta} = -\left(\frac{g}{v}\right)^2 \sin 2\theta}$$

Q5



i) $\Omega = 1200\,rpm = 40\pi\ (rad/s)$

$\underline{V}_p = \dot{\underline{R}} + (\dot{\underline{e}})_{rel} + \underline{\omega} \times \underline{e}\ ,\quad \underline{\omega} = -\Omega\,\underline{e}_3$

$\underline{R} = 0.5 \times (-\underline{e}_2 \sin(\pi/3) + \underline{e}_1 \cdot \cos(\pi/3))$

$= \frac{1}{4}\underline{e}_1 - \frac{\sqrt{3}}{4}\underline{e}_2$

$\dot{\underline{R}} = \frac{1}{4}\dot{\underline{e}}_1 - \frac{\sqrt{3}}{4}\dot{\underline{e}}_2$ where $\begin{cases} \dot{\underline{e}}_1 = \underline{\omega} \times \underline{e}_1 = (-\Omega\,\underline{e}_3) \times \underline{e}_1 = -\Omega\,\underline{e}_2 \\ \dot{\underline{e}}_2 = \underline{\omega} \times \underline{e}_2 = (-\Omega\,\underline{e}_3) \times \underline{e}_2 = \Omega\,\underline{e}_1 \end{cases}$

$\therefore\ \dot{\underline{R}} = -\Omega\left(\frac{\sqrt{3}}{4}\underline{e}_1 + \frac{1}{4}\underline{e}_2\right)$

$\underline{e} = -0.5\,\underline{e}_1\ ,\ (\dot{e})_{rel} = V \cdot (-e_2) = -20\,\underline{e}_2$

$\underline{\omega} \times \underline{e} = (-\Omega\,\underline{e}_3) \times \underline{e} = -\Omega\,\underline{e}_3 \times (-0.5\,\underline{e}_1) = 0.5\Omega\,\underline{e}_2$

$\therefore\ \underline{\omega} \times \underline{e} = 20\pi\,\underline{e}_2$

$\therefore\ \underline{V}_p = -\Omega\left(\frac{\sqrt{3}}{4}\underline{e}_1 + \frac{1}{4}\underline{e}_2\right) - 20\underline{e}_2 + 20\pi\,\underline{e}_2$

$\boxed{\underline{V}_p = -54.41\,\underline{e}_1 + 11.41\,\underline{e}_2}$

ii) $\underline{a}_p = \ddot{\underline{R}} + \dot{\underline{\omega}} \times \underline{e} + (\ddot{\underline{e}})_{rel} + 2\,\underline{\omega} \times (\dot{\underline{e}})_{rel} + \underline{\omega} \times (\underline{\omega} \times \underline{e})$

$\ddot{\underline{R}} = -\Omega\left(\frac{\sqrt{3}}{4}\dot{\underline{e}}_1 + \frac{1}{4}\dot{\underline{e}}_2\right)$

$= -\Omega\left(\frac{\sqrt{3}}{4}(-\Omega\,\underline{e}_3) \times \underline{e}_1 + \frac{1}{4}(-\Omega\,\underline{e}_3) \times \underline{e}_2\right)$

$= \Omega^2\left(-\frac{1}{4}\underline{e}_1 + \frac{\sqrt{3}}{4}\underline{e}_2\right) = -400\pi^2\,\underline{e}_1 + 400\pi^2\sqrt{3}\,\underline{e}_2$

$\dot{\underline{\omega}} \times \underline{e} = 0$

$(\ddot{e})_{rel} = \frac{v^2}{0.5}\underline{e}_1 = \frac{400}{0.5}\underline{e}_1 = 800\,\underline{e}_1$

$\underline{\omega} \times (\dot{e})_{rel} = (-\Omega)\,\underline{e}_3 \times (-20)\,\underline{e}_2 = -20\Omega\,\underline{e}_1 = -800\pi\,\underline{e}_1$

$\underline{\omega} \times (\underline{\omega} \times \underline{e}) = (-\Omega\,\underline{e}_3) \times (-\Omega\,\underline{e}_3 \times (-0.5)\,\underline{e}_1)$

$= (-\Omega\,\underline{e}_3) \times (0.5\Omega \cdot \underline{e}_2)$

$= 0.5\Omega^2\,\underline{e}_1 = 800\pi^2\,\underline{e}_1$

$\underline{a}_p = -400\pi^2\,\underline{e}_1 + 400\pi^2\sqrt{3}\,\underline{e}_2 + 800\,\underline{e}_1 - 1600\pi\,\underline{e}_1 + 800\pi^2\,\underline{e}_1$

$\therefore\ \boxed{\underline{a}_p = -278.71\,\underline{e}_1 + 6837.86\,\underline{e}_2}$