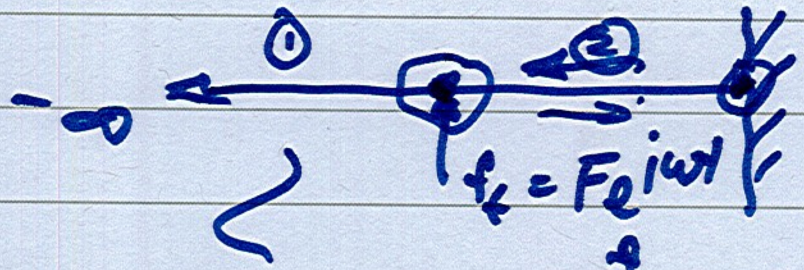
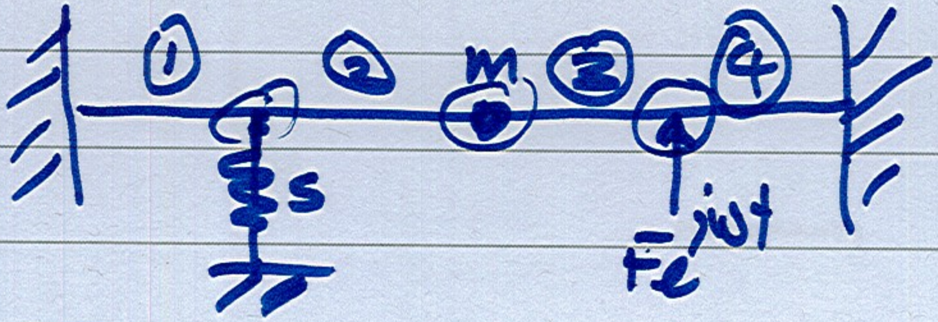
ME 513 Session 13 9/18/19




wave eqn 1 wave eqn 2

separate soln in each region

- couple the soln's by applying b.c.'s:




8 bc's → 8 unknowns

$$\begin{bmatrix} & & \\ & 8 \times 8 & \\ & & \end{bmatrix} \begin{bmatrix} A \\ \vdots \\ H \end{bmatrix} = \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{\text{forcing vector}}$$

General Approach - multisegments
- write the general sol'n for each segment
- write the b.c.'s
- sub the soln's into the b.c.'s & solve

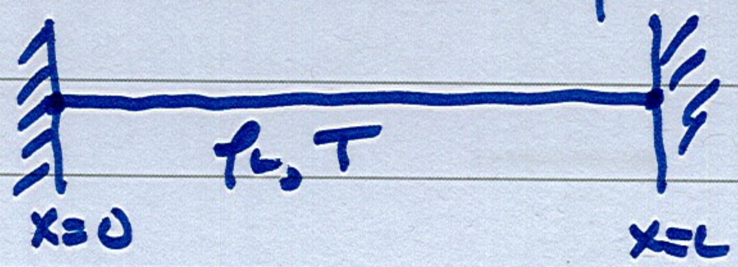
## 2.5 Normal modes of strings

$F e^{j\omega t}$ $\quad$ $y \propto e^{j\omega t}$ Forced

### Free Vibration

- string is "forced" into motion by initial conditions
- natural freqs & mode shapes

### 2.5.1 Characteristic Equation




- one wave eqn

$$y(x,t) = \underline{A}\, e^{j(\omega t - kx)} + \underline{B}\, e^{j(\omega t + kx)}$$

b.c. at $x=0$ $\quad y(0,t)=0$

$$A+B=0$$

$$B=-A \qquad (1)$$

b.c. at $x=L$ $\quad y(L,t)=0$

$$e^{j\omega t}\left(Ae^{-jkL}+Be^{+jkL}\right)=0$$

$$A\underbrace{\left(e^{-jkL}-e^{+jkL}\right)}_{-2j\sin kL}=0$$

$$\boxed{-2jA\sin kL = 0} \qquad (2)$$

if $A=0$ displacement $=0$ at all times

$$\boxed{\sin k_n L = 0}$$ characteristic eqn

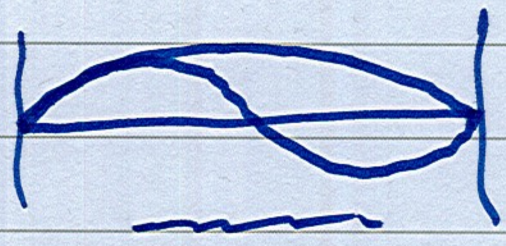
$k_n L = n\pi \qquad n = 1, 2, 3 \ldots\ldots$

$k_n = \frac{n\pi}{L} \qquad k = \frac{\omega}{c} = \frac{2\pi f}{c}$

$f_n = \frac{n}{2}\frac{c}{L} \qquad n = 1, 2, 3 \ldots$

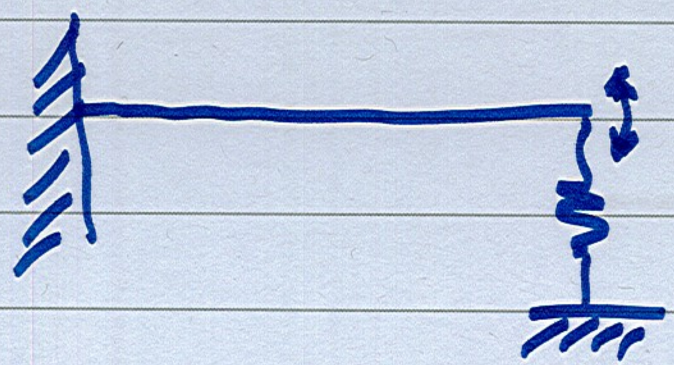
allowed natural freqs

$$L = \frac{n}{2}\frac{c}{f_n} = \frac{n}{2}\lambda_n$$

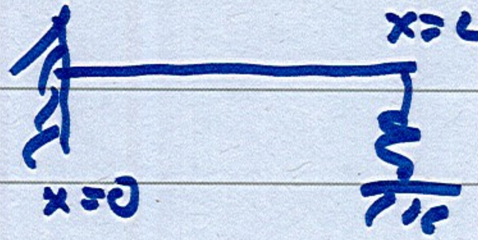


at the natural freqs, the string is an integral multiple of half wavelengths

- fixed - fixed



## 2.5.2 Mode Shapes




$$y(x,t) = \left(Ae^{-jkx} + Be^{+jkx}\right)e^{j\omega t}$$

First b.c. $\quad B = -A$

$$y_n(x) = -2jA_n \sin k_n x$$

$$y_n(x,t) = -2jA_n \sin k_n x \, e^{j\omega_n t}$$

$$= a_n \underbrace{\sin k_n x}_{\text{mode shape of the nth mode}} e^{j\omega_n t}$$

nth normal mode

modes - individual soln's that satisfy
the wave eqn <u>and</u> the b.c.'s

$k_n$'s allowed wavenumber

$\omega_n = 2\pi f_n$ - natural freqs

$a_n$ = modal amplitude

complete soln = sum of the possible

## 2.5.3 Complete Sol'n

Complete = superposition of possible soln's

$$y(x,t) = \sum_{n=1}^{\infty} \hat{a}_n \sin k_n x \, \underbrace{e^{j\omega_n t}}_{\cos\omega_n t + j\sin\omega_n t}$$

weighted sum of modes

$$\hat{a}_n = a_n + j b_n$$

Real displacement of the nth mode

$$\mathrm{Re}\{y_n(x,t)\} = (a_n \cos\omega_n t - b_n \sin\omega_n t) \sin k_n x$$

Say That $Re\{y(x,0)\}$ = known
(at $t=0$)

$$Re\{y(x,0)\} = \sum_{n=1}^{\infty} a_n \sin k_n x$$

$$a_n = \frac{2}{L} \int_0^L Re\{y(x,0)\} \sin k_n x \, dx$$

from initial displacement
work out $a_n$'s

also need $b_n$'s

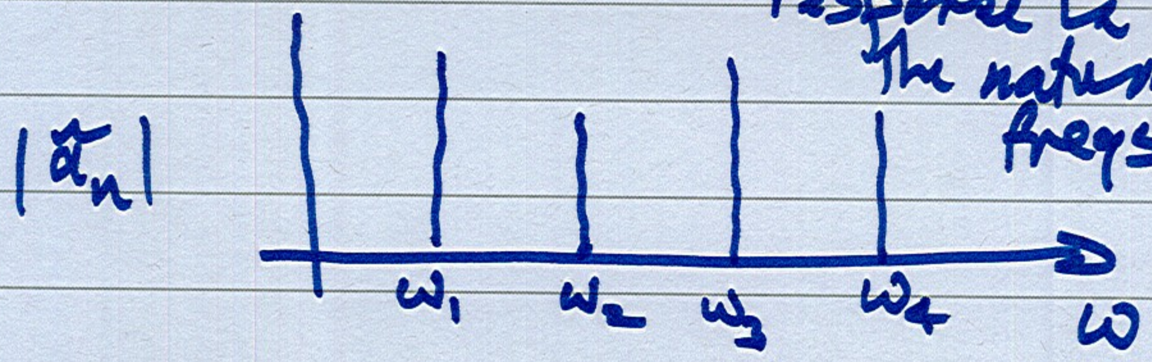
use initial velocity of the string to solve for the $b_n$'s

$$u(x,t) = \frac{\partial y}{\partial t} = \frac{\partial}{\partial t} \sum_{n=1}^{\infty} \tilde{a}_n \sin k_n x \, e^{j\omega_n t}$$

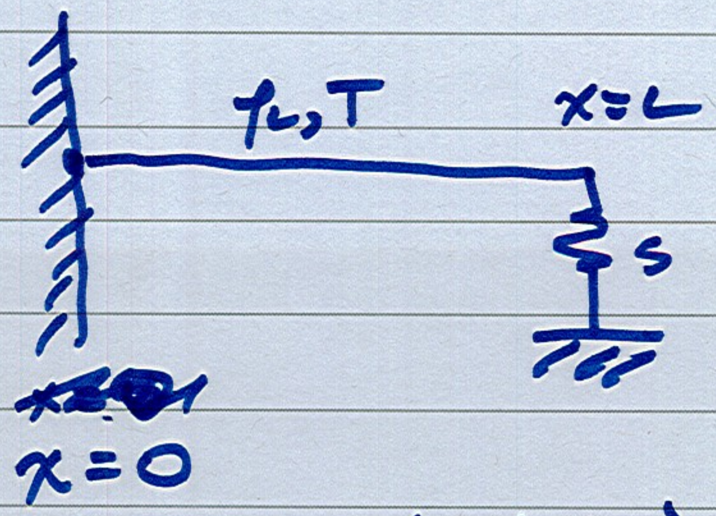
$$\mathrm{Re}\{u(x,0)\} \longrightarrow b_n$$

$\tilde{a}_n = a_n + i b_n$ now known

response is at the natural freqs



## 2.5.4 Other b.c.'s




$$k = \frac{\omega}{c}$$

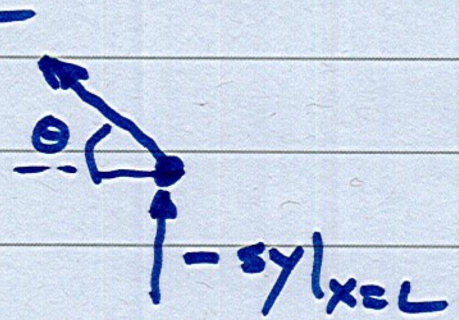
$$c = \sqrt{\frac{T}{\rho_L}}$$

$$y(x,t) = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}$$

at $x=0$ $\quad y(0,t) = 0 \Rightarrow B = -A$

$$y(x,t) = -2iA \sin kx \, e^{i\omega t}$$

at $x = L$




$\Sigma f_y = 0$

$$T\sin\theta|_{x=L} - sy|_{x=L} = 0$$

$$-T\frac{\partial y}{\partial x}\Big|_{x=L} - sy|_{x=L} = 0$$

$$\frac{\partial y}{\partial x} = -2jkA\cos kx\, e^{j\omega t}$$

$$\boxed{-(k_n L)\left(\frac{T}{sL}\right) = \tan k_n L}$$

characteristic eqn