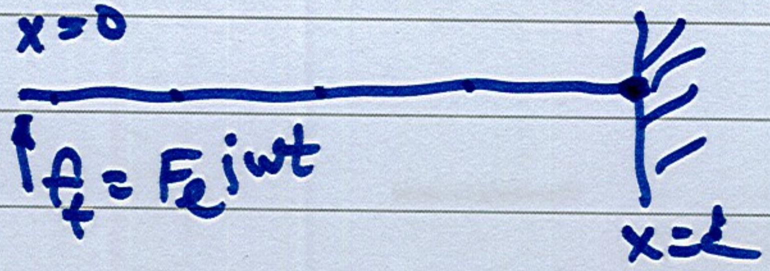
ME 513 Session 10 9/11/19 1

$$y(x,t) = A e^{i(\omega t - kx)} + B \cancel{e^{i(\omega t + kx)}}$$

B.C.'s

$$Z_{m0} = \frac{F_e\, e^{i\omega t}}{u} = \rho_L c$$ characteristic impedance

2.4.2 Finite Length String



$$y(x,t) = A e^{j(\omega t - kx)} + B e^{j(\omega t + kx)}$$

(i) $y(L,t) = 0 = A e^{-jkL} + B e^{+jkL}$ (1)

(ii) force b.c. at $x=0$

$$F e^{j\omega t} = f_t = -T \left.\frac{dy}{dx}\right|_{x=0}$$

$$F\cancel{e^{j\omega t}} = jkT(A-B)\cancel{e^{j\omega t}}$$

$$jkTA - jkTB = F \qquad (2)$$

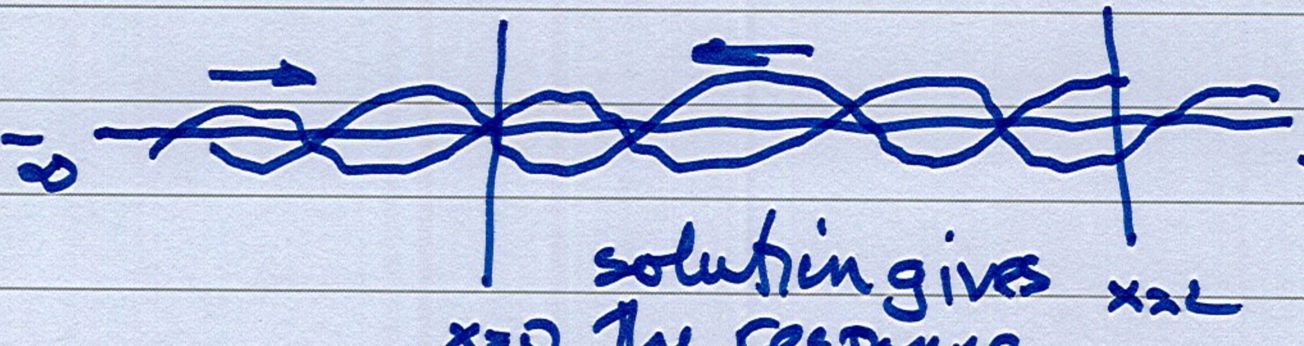
(iii) two b.c.'s & 2 unknowns

$$\begin{bmatrix} e^{-jkL} & e^{+jkL} \\ jkT & -jkT \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

$$y(x,t) = \overbrace{\frac{Fe^{jkL}}{2jkT\cos kL}}^{A} e^{j(\omega t - kx)} \overbrace{- \frac{Fe^{-jkL}}{2jkT\cos kL}}^{B} e^{j(\omega t + kx)}$$

$$= \frac{F}{2jkT\cos kL}\left[\underset{\longrightarrow}{e^{j(\omega t + k(L-x))}} - \underset{\longleftarrow}{e^{j(\omega t - k(L-x))}}\right]$$

Expressed as a superposition of two propagating waves



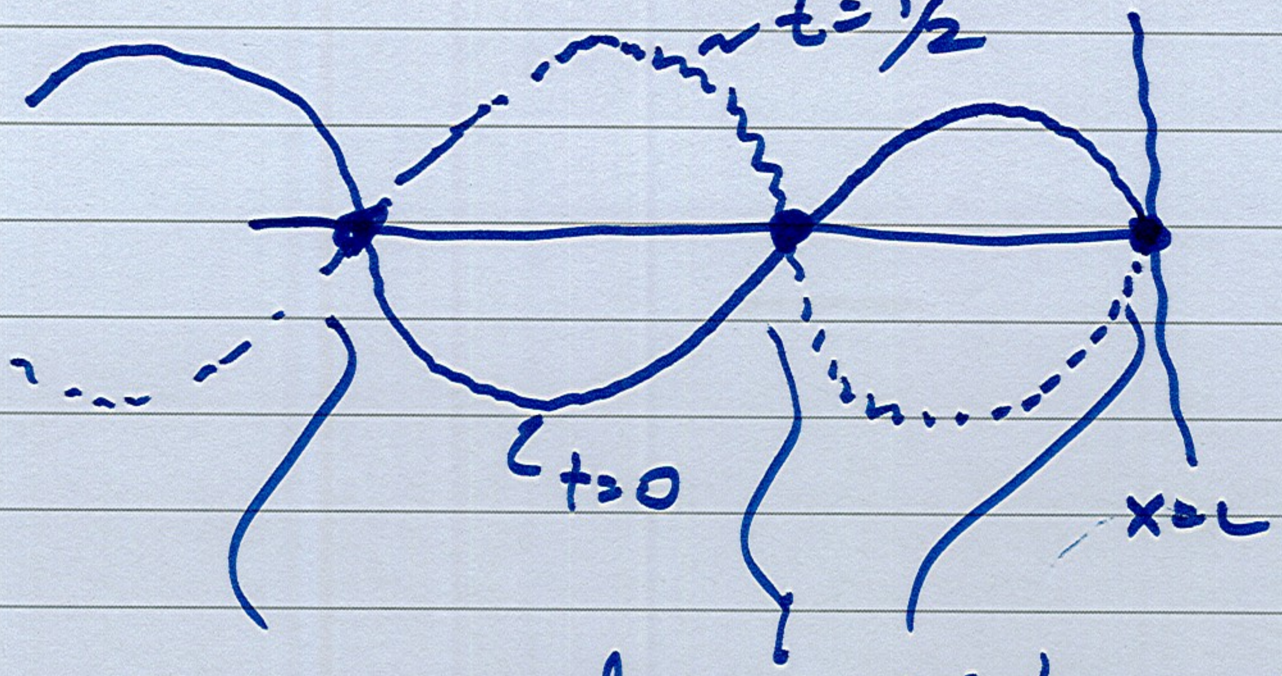
solution gives the response in this region alternatively

$$y(x,t) = \frac{F e^{i\omega t}\, 2j \sin k[L-x]}{2jkT \cos kL}$$

$$= \left(\frac{F}{kT}\right)\left(\frac{\sin k[L-x]}{\cos kL}\right) e^{i\omega t}$$

standing wave representation
(separation of time & space)

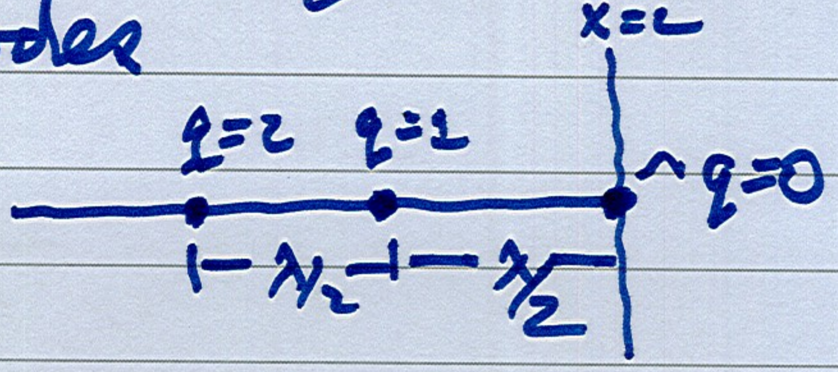
standing waves - superposition
of propagating waves that
interfere with each other



nodes - displacement is always = 0

$c = f\lambda$

(iv) location of the nodes



$$\sin k(L - x_q) = 0$$

$$k(L - x_q) = q\pi \qquad q = 0, 1, 2, 3 \ldots$$

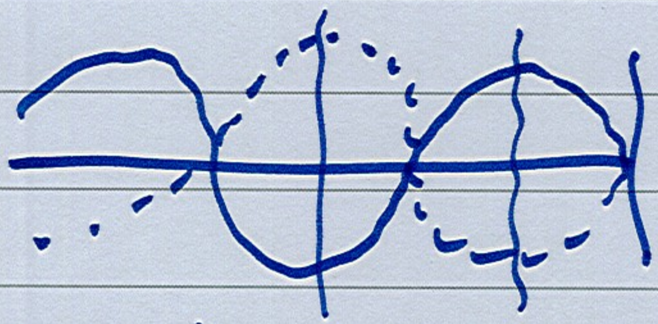
$$x_q = L - q\left(\frac{\pi}{k}\right) \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$= L - q\left(\frac{\lambda}{2}\right)$$

nodal points are at

L − integer number of $(\lambda/2)$'s away from the termination

as the frequency increases
the nodes move to the right
since λ is decreasing

(v) Antinodes 

The points of maximum
displacement
- halfway between the
nodes

(vi) $y(x,t) = \left(\dfrac{F}{kT\cos kL}\right) \sin k(L-x)\, e^{j\omega t}$

response of the system is largest at the frequency at which $\cos kL \to 0$

resonance condition

$\cos kL = 0$

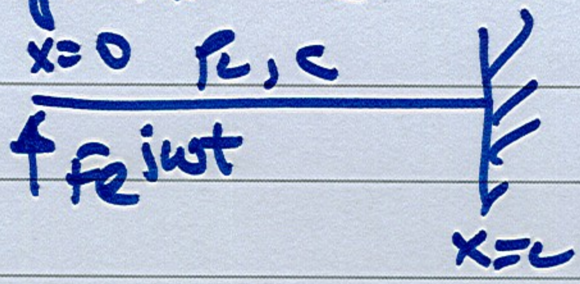
$$k_n L = \left(\frac{2n-1}{2}\right)\pi \quad n = 1, 2, 3\ldots$$

$$k_n = \frac{\omega_n}{c} = \frac{2\pi f_n}{c}$$

$$f_n = \left(\frac{2n-1}{4}\right)\left(\frac{c}{L}\right) \quad n = 1, 2, 3\ldots$$

b

(vii) Input impedance



$$Z_{mo} = \frac{\text{input force}}{\text{velocity at the drive point}} = \frac{Fe^{j\omega t}}{(j\omega)\frac{Fe^{j\omega t}}{kT}\frac{\sin k(L-\cancel{x})}{\cos kL}} \quad x \to 0$$

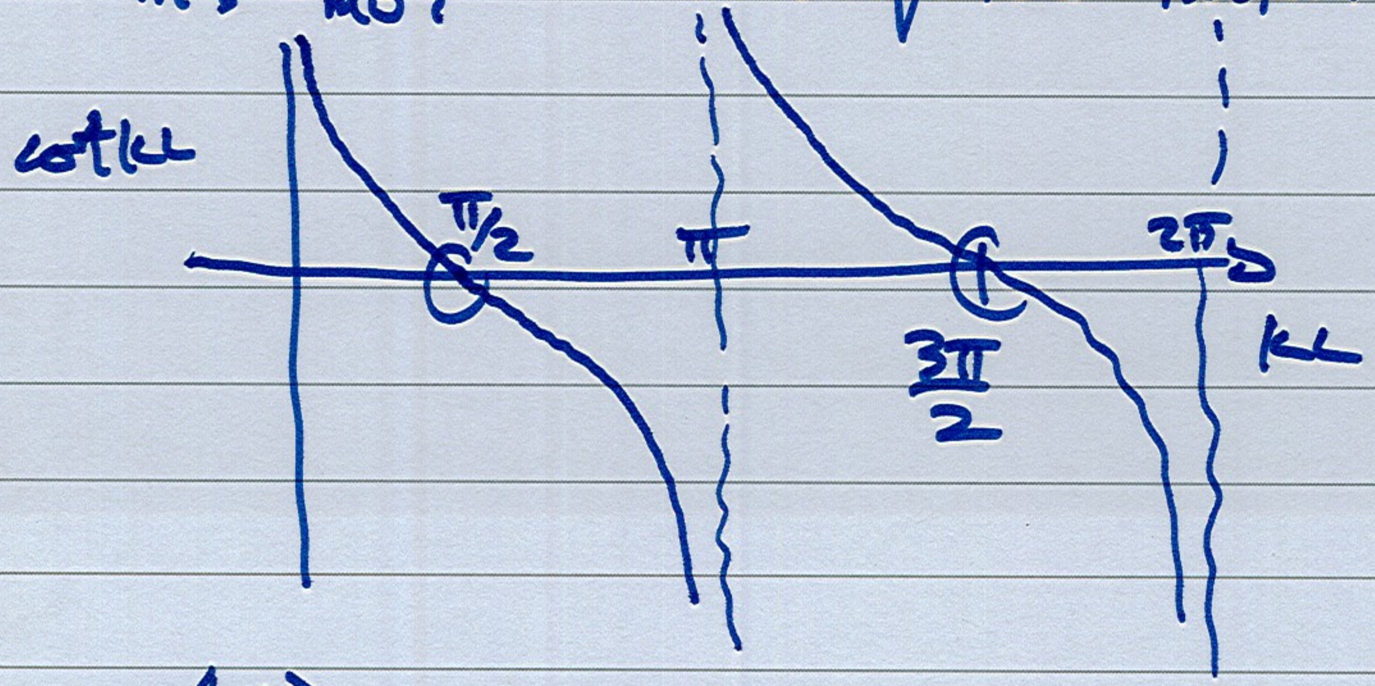
$k = \omega/c$, $T = \rho_L c^2$

$$\boxed{Z_{mo} = -j\,\underbrace{\rho_L c}_{\text{medium}}\,\underbrace{\cot kL}_{\text{geometry}}}$$

purely imaginary

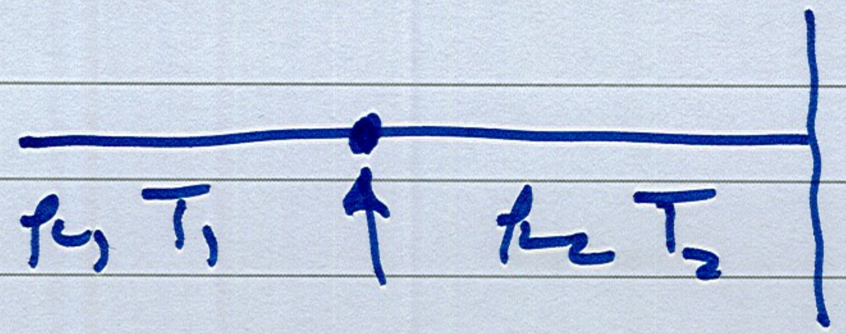
$Z_{m0} = -j\rho_L c \cot kL$

$\mathrm{Im}\{Z_{m0}\} = 0$ defines natural frequency



$(kL)_1 = \frac{\pi}{2}$ $\quad (kL)_2 = \frac{3\pi}{2}$

→ natural frequencies



$\rho_1\ T_1$
$\rho_2\ T_2$