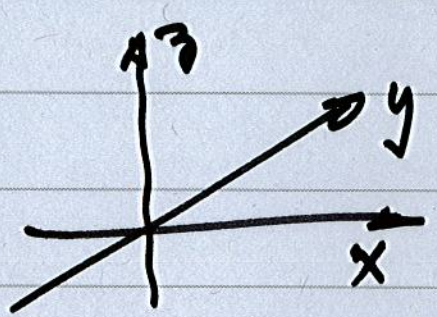
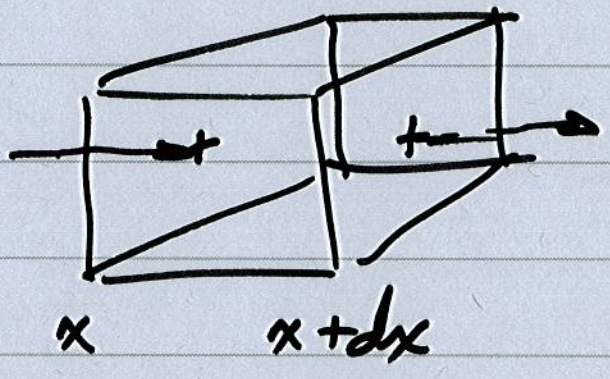
Wave Eqn - governs propagation of small amplitude fluctuations of pressure in an elastic medium

p+u - eqn of state
- continuity
→ $p = \beta s$ $\quad s = \frac{\rho - \rho_0}{\rho_0}$

⇕

p+u - equation of motion





in $(\rho u_x)_x \, dy dz$ (a) kg/s

out $(\rho u_x)_{x+dx} \, dy dz$

$$= \left[(\rho u_x)_x + \frac{\partial (\rho u_x)}{\partial x}\bigg|_x dx + \ldots\ldots \right] dy dz$$

(b)

net rate of mass inflow

$$(a) - (b) = -\left.\frac{\partial(\rho u_x)}{\partial x}\right|_x dx\,dy\,dz$$

We also know the rate at which the mass in the control volume changes

$$\frac{\partial \rho}{\partial t} dx\,dy\,dz \qquad kg/s$$

$$\frac{\partial \rho}{\partial t} dx dy dz = -\frac{\partial (\rho u_x)}{\partial x}\bigg|_x dx dy dz$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_x)}{\partial x} = 0}$$

conservation of mass

ρ & u_x are functions of x

$$\frac{\partial \rho}{\partial t} + u_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial u_x}{\partial x} = 0$$

$$s = \frac{\rho - \rho_0}{\rho_0} \qquad \rho = s\rho_0 + \rho_0 = \rho_0(s+1)$$

$$\rho_0 \frac{\partial (s+1)}{\partial t} + \rho_0 u_x \frac{\partial (s+1)}{\partial x} + \rho_0 (s+1) \frac{\partial u_x}{\partial x} = 0$$

$$\rho_0 \frac{\partial s}{\partial t} + \cancel{\rho_0 u_x \frac{\partial s}{\partial x}} + \rho_0 (s+1) \frac{\partial u_x}{\partial x} = 0$$

assume that the product of small quantities is negligible (linear acoustics)

$$\cancel{\rho_0}\frac{\partial s}{\partial t} + \cancel{\rho_0}\frac{\partial u_x}{\partial x} = 0$$

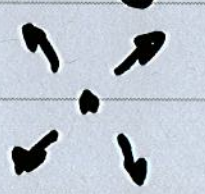
1-D

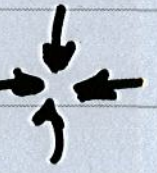
$$\boxed{\frac{\partial s}{\partial t} + \frac{\partial u_x}{\partial x} = 0}$$

3-D

$$\boxed{\frac{\partial s}{\partial t} + \nabla \cdot \bar{u} = 0} \quad (2)$$

$\nabla \cdot \bar{u}$ = divergence of the particle velocity

$\nabla \cdot \bar{u}$ is +ve 

$\nabla \cdot \bar{u}$ is -ve 

particle velocity

$$\bar{u} = u_x \bar{i} + u_y \bar{j} + u_z \bar{k}$$

∇ grad operator

$$\nabla = \bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z}$$

$$\nabla \cdot \bar{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

(iii) pressure vs particle velocity

combining (1) & (2)

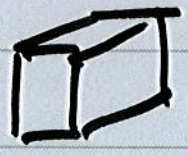
$$\frac{1}{\beta}\frac{\partial p}{\partial t} + \nabla \cdot \bar{u} = 0 \qquad (3)$$

linearized continuity equation

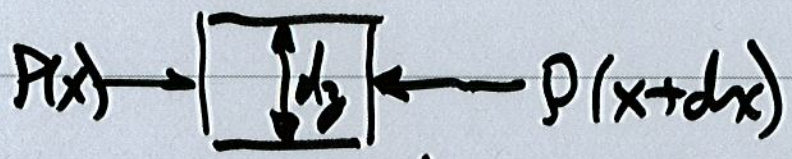
3.2.2. Pressure - Velocity (II)

(1) Eqn of Motion

apply $\Sigma f = ma$ to a fixed mass that is moving with the fluid



1-D



$$= P(x) + \left.\frac{dP}{dx}\right|_x dx + \ldots$$

Net force in the x-direction

$$df_x = P(x)\,dy\,dz - P(x+dx)\,dy\,dz$$

$$df_x = -\left.\frac{dP}{dx}\right|_x \frac{dx\,dy\,dz}{\text{2 volume} = dV}$$

(force/volume

In 3-D $\quad d\bar{f} = -\nabla P\, dV$

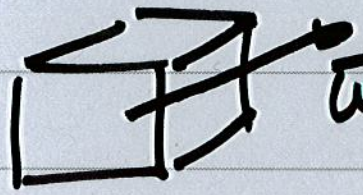
∇P = gradient of the pressure

$$= \frac{dP}{dx}\bar{i} + \frac{dP}{dy}\bar{j} + \frac{dP}{dz}\bar{k}$$

direction - steepest or fastest change in pressure with position

(iii) Acceleration of a moving fluid element

$F = ma$



$\vec{u}(x, y, z, t)$

u_i

$\vec{u}(x+dx, y+dy, z+dz, t+dt)$

$= u_f$

$$\bar{u}_i$$

$$\bar{u}_f = \bar{u}(x,y,z,t) + \left.\frac{\partial \bar{u}}{\partial x}\right|_{x,y,z,t} dx + \ldots + \ldots$$

$$+ \left.\frac{\partial \bar{u}}{\partial t}\right|_{x,y,z,t} dt$$

$$dx = u_x dt$$

$$dy = u_y dt$$

$$dz = u_z dt$$

$$\bar{a} = \frac{\bar{u}_f - \bar{u}_i}{dt} = \left[u_x \frac{\partial \bar{u}}{\partial x} + u_y \frac{\partial \bar{u}}{\partial y} + u_z \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{u}}{\partial t} \right]$$

convective acceleration

- products of small quantities are very small in linear acoustics

$$\bar{a} = \cancel{(\bar{u} \cdot \nabla)\bar{u}}^{\,0} + \frac{\partial \bar{u}}{\partial t}$$

neglect non-linear terms

$$d\bar{f} = -\nabla P \, dV$$

mass of the element

$$m = \rho \, dV$$

$$\bar{a} = \frac{\partial \bar{u}}{\partial t}$$

$$d\bar{f} = m\bar{a}$$

$$-\nabla P dV = \rho dV \frac{d\bar{u}}{dt}$$

(iv) Pressure - Velocity Relation

$$P = P_0 + p \qquad \nabla P = \nabla p$$

$$\rho = \rho_0(s+1) \qquad s = \frac{\rho - \rho_0}{\rho_0} \ll 1$$

$$\boxed{-\nabla p = \rho_0 \frac{d\bar{u}}{dt}}$$

(4) linearized momentum eqn.
Euler Eqn.

3.2.3 Linear Wave Eqn

(3) $\frac{1}{\beta_0}\frac{\partial p}{\partial t} + \nabla \cdot \bar{u} = 0$ $\frac{\partial}{\partial t}$

(4) $\nabla p + \rho_0 \frac{\partial \bar{u}}{\partial t} = 0$ $\frac{1}{\rho_0}\nabla \cdot$

subtract (3) from (4)

$$\nabla^2 p - \left(\frac{\rho_0}{\beta}\right)\frac{\partial^2 p}{\partial t^2} = 0$$

$$c = \sqrt{\frac{\beta}{\rho_0}}$$

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

linear wave eqn

c = speed of sound

$$= \sqrt{\frac{\beta}{\rho_0}} = \sqrt{\frac{\gamma p_0}{\rho_0}}$$

stationary & ideal fluid