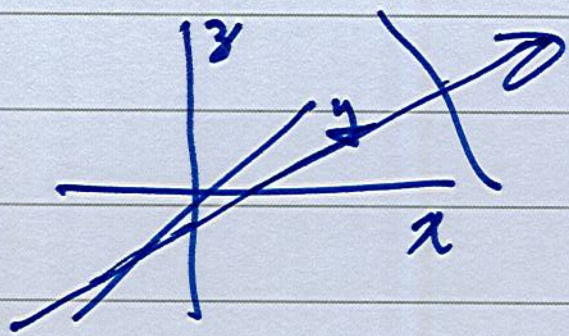
ME 513 Session 17 10/2/15

Wave Equation & Simple Solutions



$$\hat{p}(x,y,z,t) = A e^{i(\omega t \pm k_x x \pm k_y y \pm k_z z)}$$

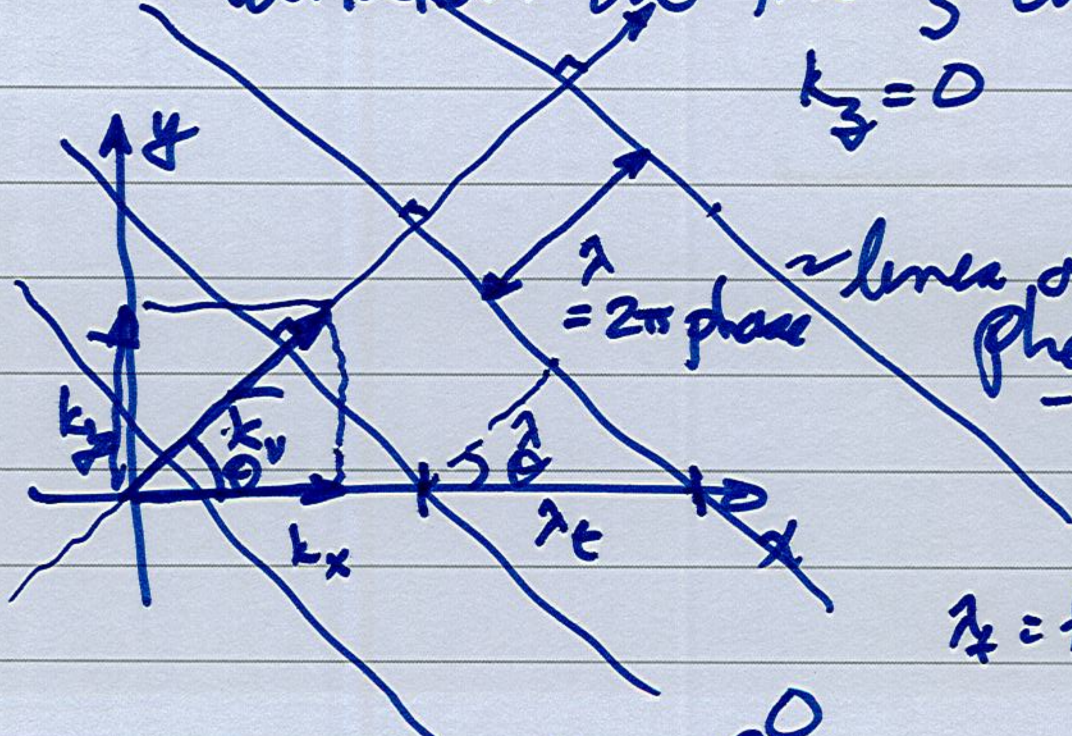
$$\bar{k}_v = k_x \bar{i} + k_y \bar{j} + k_z \bar{k}$$

$$|\bar{k}_v| = \frac{\omega}{c}$$

k_x & k_y are given

$$\underset{=}{k_z} = \sqrt{k^2 - \underbrace{k_x^2 - k_y^2}}$$

consider two-dimensional case - no variation in the z-direction



$k_z = 0$ - no - prop in the z-direction

~ lines of constant phase

- propagate at speed of sound c

λ_t : trace wavelength

$$k^2 = \underbrace{k_x^2 + k_y^2}_{} + \cancel{k_z^2}^{\,0}$$

one 1 can be chosen

$$k_y = \sqrt{k^2 - k_x^2}$$

$\lambda_t \cos\theta = \lambda$

$\lambda_t = \dfrac{\lambda}{\cos\theta}$ $\qquad \lambda_t \to \lambda \quad \theta = 0$

$\lambda_t \to \infty \quad \theta = \pi/2$

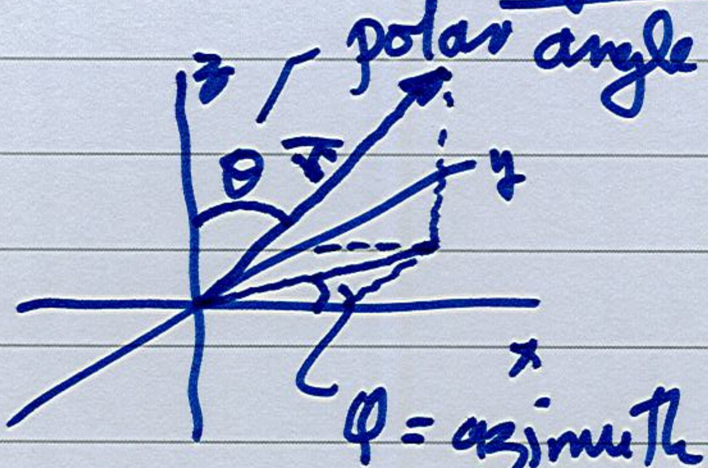
since $\lambda_t > \lambda$

$c_t = \dfrac{\lambda_t}{T} > c$

$c_t \to c$ when $\theta = 0$

$c_t \to \infty$ when $\theta = \dfrac{\pi}{2}$

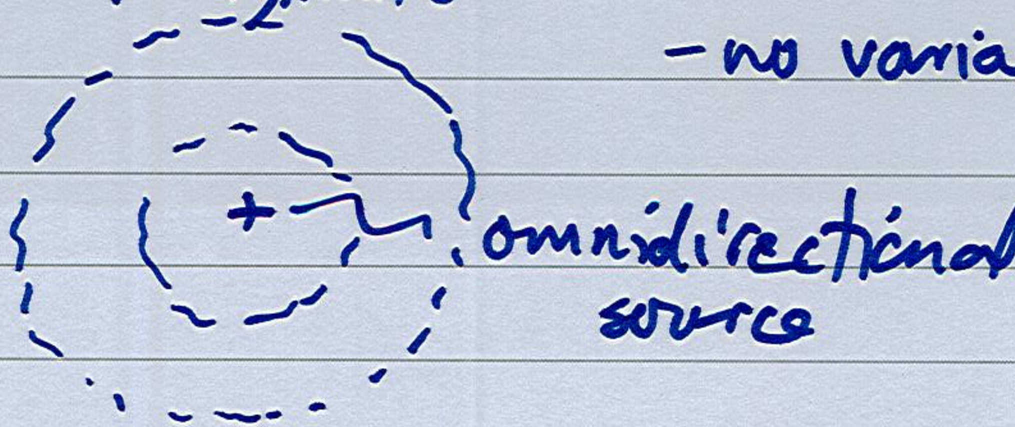
3.3.2 Spherical Waves



(r, θ, ϕ) $\quad r = |\vec{r}|$

spherically symmetric waves

- no variation with θ & ϕ



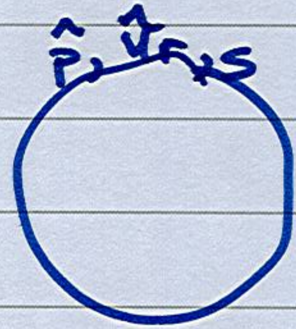
$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

In spherical coords

$$\nabla^2 = \underline{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}} + \cancelto{0}{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}}\left(\cancelto{0}{\sin\theta\frac{\partial}{\partial\phi}}\right)$$

$\tilde{p}$ is a function only of r

<u>1-D</u>



All acoustic quantities are instantaneously constant on an expanding spherical surface

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

$$\underbrace{\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r}}_{\frac{1}{r}\frac{\partial^2 (rp)}{\partial r^2}} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

$$\frac{\partial^2 (rp)}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 (rp)}{\partial t^2} = 0$$

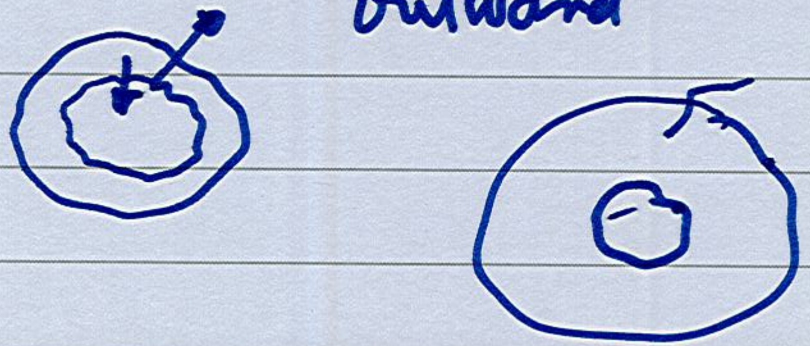
In harmonic case $e^{j\omega t}$

$$\frac{d^2(r\tilde{p})}{dr^2} + k^2 (r\tilde{p}) = 0 \qquad k = \frac{\omega}{c}$$

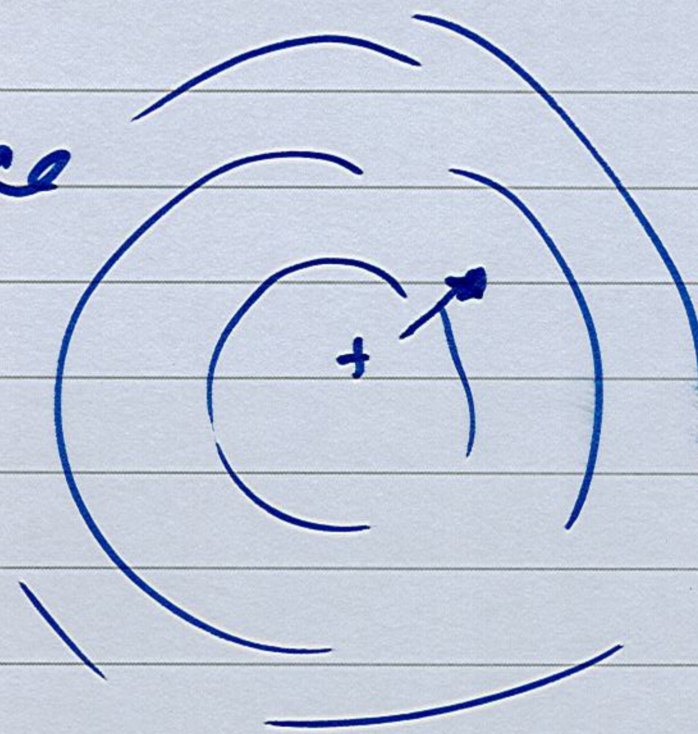
$$r\tilde{p} = Ae^{-jkr} + Be^{+jkr}$$

$$\tilde{p} = \underbrace{\frac{A}{r} e^{-jkr}}_{\text{outward}} + \underbrace{\frac{B}{r} e^{+jkr}}_{\text{inward}}$$

General Sol'n for spherically symmetric waves

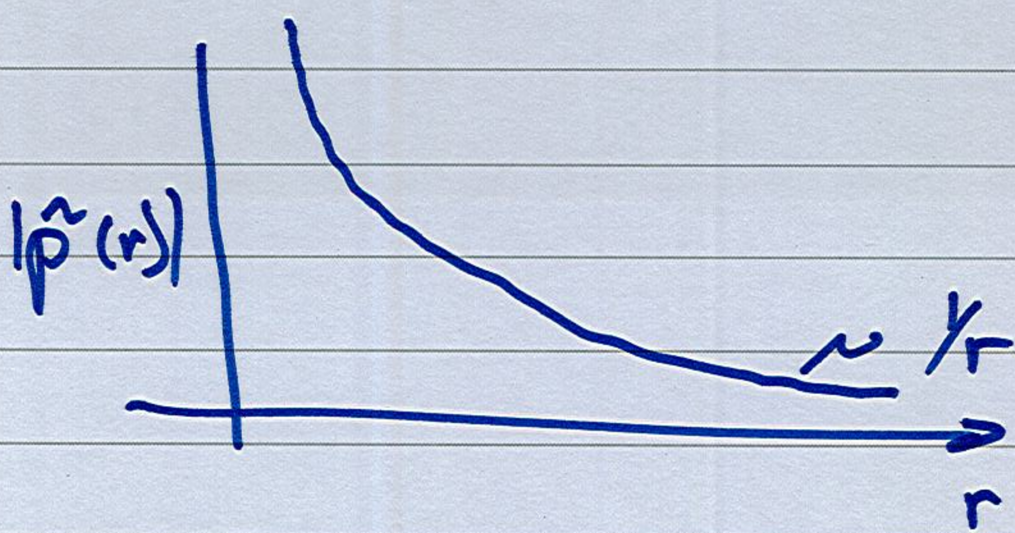


Solution in free space



$$\tilde{p}(r) = \frac{A}{r} e^{-ikr}$$

$$|\tilde{p}(r)| \propto \frac{1}{r}$$



Acoustic Particle Velocity

$$-\nabla p = \rho_0 \frac{\partial \bar{u}}{\partial t}$$

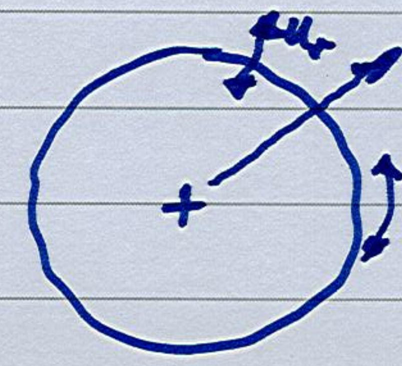
spherical symmetry

$$\nabla = \left(\frac{\partial}{\partial r}\right)\bar{r}$$ unit vector in the radial direction

harmonic case

$$p(r,t) = \hat{p}(r)e^{j\omega t}$$

$$\underline{\underline{u_r(r,t)}} = \hat{u}_r(r)e^{j\omega t}$$



1-D version momentum

$$-\frac{d\hat{p}}{dr} = j\omega\rho_0\hat{u}_r$$

$$\hat{u}_r = -\frac{1}{j\omega\rho_0}\frac{d\hat{p}}{dr}$$

$$\tilde{u}_r = -\frac{1}{j\omega\rho_0}\frac{d\tilde{p}}{dr}$$

$$\tilde{p}_+ = \frac{A}{\underline{\underline{r}}}\, e^{-jk\underline{r}} \quad \text{outward}$$

$$\frac{d\tilde{p}_+}{dr} = -\frac{A}{r}\left(\frac{1}{r} + jk\right) e^{-jkr}$$

$$\tilde{u}_{r+} = \frac{jk}{j\omega\rho_0}\left(1 + \frac{1}{jkr}\right)\frac{A}{r}e^{-jkr}$$

$$= \frac{1}{\underline{\rho_0 c}}\underline{\left(1 + \frac{1}{jkr}\right)}\,\tilde{p}_+$$

$$\tilde{u}_{r+} = \frac{1}{\rho c}\left(1 + \underbrace{\frac{1}{jkr}}\right)\tilde{p}_+$$

nearfield term
- significant close to the source

kr ?

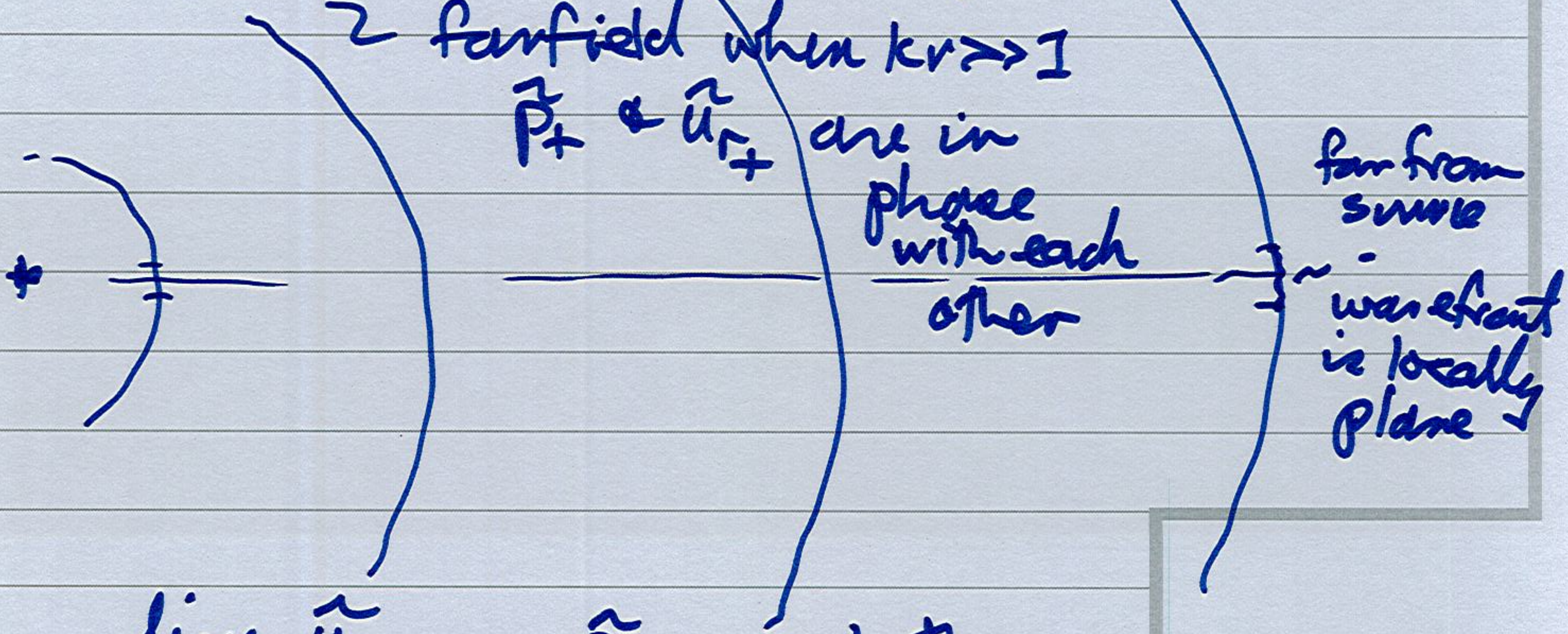
$$\underline{\underline{kr}} = \frac{2\pi r}{\lambda} = 2\pi\left(\frac{r}{\lambda}\right)$$

non-dimensional radius

$$\lim_{kr \to \infty} \hat{u}_{r_+} = \frac{\hat{p}_+}{\rho_0 c}$$ same as for plane waves

$\sim$ farfield when $kr \gg 1$

$\tilde{p}_+$ & $\hat{u}_{r_+}$ are in phase with each other



$$\lim_{kr \to 0} \hat{u}_{r_+} = \frac{\hat{p}_+}{jkr\rho_0 c}$$

in the ~~the~~ 'near' field

$\tilde{p}_+$ & $\hat{u}_{r_+}$ are out-of-phase

$|\tilde{u}_{r_+}|$
$\sim \frac{1}{r^2}$
$\sim 1/r$
r
nearfield
farfield
$kr \simeq 1$

3. 3.3 Cylindrical Waves