ME 513 session 15 9/28/15 1

$$\frac{1}{\beta}\frac{\partial \bar{p}}{\partial t} + \nabla \cdot \bar{u} = 0 \qquad \frac{\partial}{\partial t}$$

Conservation of mass

- given velocity

$$\nabla p + \rho_0 \frac{\partial \bar{u}}{\partial t} = 0 \qquad \frac{1}{\rho_0}\nabla \cdot$$

EOM - given pressure field $\rightarrow \bar{u}$

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \qquad c = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

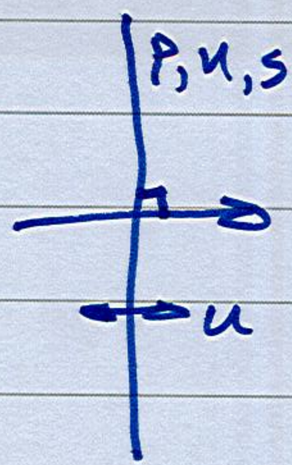
$$c = c_0 \sqrt{1 + \frac{T_c}{273}}$$

331.6 m/s $c$ @ $0^\circ C$

# One-Dimensional Solutions

## 3.3.1 Plane Waves

- properties are instantaneously uniform over an infinite plane $\perp$ to the direction of wave propagation

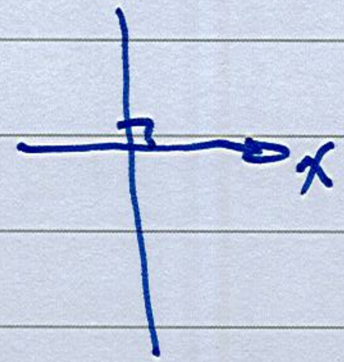



propagation

longitudinal wave

- particle motion is $\parallel$ to the direction of wave prop

3.3.1.1 Propagation in the <u>x-direction</u>




- no variation in the y or z directions

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

$$p(x,t) = \underbrace{p_1(ct - x)}_{+ve} + \underbrace{p_2(ct + x)}_{-ve}$$

complex harmonic form

$$p(x,t) = \tilde{p}(x)\, e^{j\omega t}$$

separable ⇒ sub into wave eqn

$$\frac{d^2\tilde{p}}{dx^2} + k^2\tilde{p} = 0$$

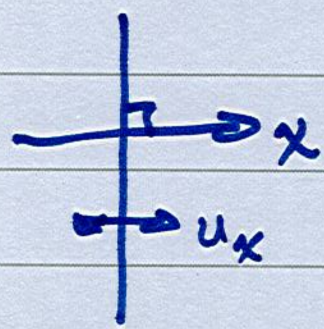
scalar Helmholtz Eqn

$$k = \frac{\omega}{c}$$

$$\tilde{p}(x) = A_1 e^{-jkx} + B_1 e^{+jkx}$$

steady state

$$p(x,t) = \tilde{p}(x)\, e^{j\omega t}$$




$u_x(x,t)$

linearized momentum eqn.

$$-\nabla p = \rho_0 \frac{\partial \bar{u}}{\partial t} \qquad \bar{u} = u_x \bar{i} \cancel{+ u_y \bar{j}} \cancel{+ u_z \bar{k}}$$

1-D $$-\frac{\partial p}{\partial x} \cancel{\bar{i}} = \rho_0 \frac{\partial u_x}{\partial t} \cancel{\bar{i}}$$

$$p(x,t) = \tilde{p}(x)e^{j\omega t}$$

$$u_x(x,t) = \tilde{u}_x(x)e^{j\omega t}$$

$$-\frac{d\tilde{p}}{dx} = (j\omega)\rho_0 u_x$$

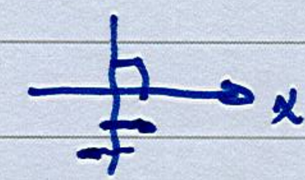
$$\boxed{u_x = -\frac{1}{j\omega\rho_0}\frac{d\tilde{p}}{dx}}$$

one component of the particle velocity

$$u_y = -\frac{1}{j\omega\rho_0}\frac{d\tilde{p}}{dy}$$

$$u_z = -\frac{1}{j\omega\rho_0}\frac{d\tilde{p}}{dz}$$

Particle Velocity associated with



$$\tilde{p}_+ = A_1 e^{-jkx}$$

$$\tilde{u}_{x_+} = -\frac{1}{j\omega\rho_0}\frac{d\tilde{p}}{dx} = -\frac{1}{j\omega\rho_0}(-jk)A_1 e^{-jkx}$$

$$= \frac{k}{\omega\rho_0} A_1 e^{-jkx} \qquad k = \frac{\omega}{c}$$

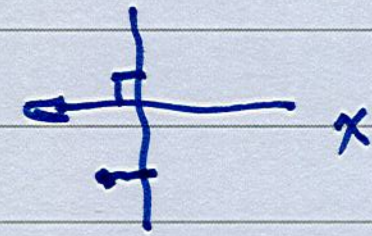
$$= \frac{A_1}{\rho_0 c} e^{-jkx} \quad \tilde{p}_+ \qquad\qquad \frac{\tilde{p}_+}{\tilde{u}_{x_+}} = \rho_0 c$$

$$= \frac{\tilde{p}_+}{\rho_0 c}$$

$\rho_0 c \Rightarrow$ characteristic impedance

$\rho_0 c = 415$ [Rayls]

$$\tilde{p}_- = A_2 e^{+ikx}$$




$$\tilde{u}_{x-} = -\frac{1}{j\omega\rho_0}\frac{\partial \tilde{p}_-}{\partial x} = -\frac{1}{j\omega\rho_0}(jk)\, A_2 e^{+jkx}$$

$$= -\frac{A_2}{\rho_0 c} e^{+ikx}\, \tilde{p}_- \qquad k = \frac{\omega}{c}$$

$$= -\frac{\tilde{p}_-}{\rho_0 c}$$

$$\frac{\tilde{p}_-}{\tilde{u}_{x-}} = - \rho_0 c$$

sign of impedance depends on direction of wave prop.

use these results to calculate the particle velocity for any given sound field

$$\bar{u} = -\frac{1}{j\omega\rho_0}\nabla\hat{p}$$

$$\hat{p}(x) = \underbrace{A_1 e^{-ikx}}_{\hat{p}_+} + \underbrace{A_2 e^{+ikx}}_{\hat{p}_-}$$

$$\tilde{u}_x = \frac{\hat{p}_+}{\rho_0 c} - \frac{\tilde{p}_-}{\rho_0 c}$$

true for plane waves in a lossless medium

$$= \frac{A_1}{\rho_0 c} e^{-ikx} - \frac{A_2}{\rho_0 c} e^{+ikx}$$

# Typical Velocities

- plane wave prop in the +ve x-dir

$$|\hat{\tilde{p}}_+| = 1 \text{ Pa} \qquad 94 \text{ dB}$$

$$|\hat{\tilde{u}}_{x+}| = \frac{|\hat{\tilde{p}}_+|}{\rho_0 c} \simeq 2.4 \text{ mm/s}$$

$$c = 340 \frac{m}{s} \qquad |\hat{u}_{x+}| \ll c$$

$\xi$ displacement

for harmonic fields

$$j\omega\hat{\xi} = \hat{u}$$

$$|\hat{\xi}| = \frac{|\hat{u}|}{\omega}$$

particle displacement

$|\tilde{\xi}| = \dfrac{|\hat{u}|}{\omega}$ at 1 kHz $\omega \simeq 6000$

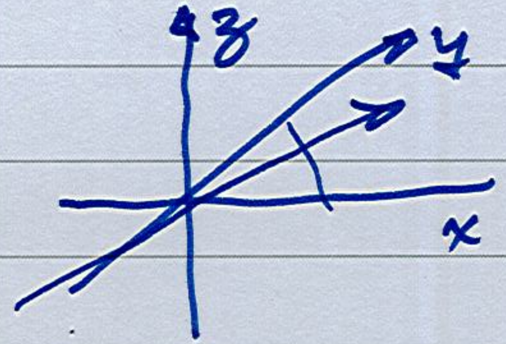
94 dB $|\tilde{\xi}| = \dfrac{2.4 \times 10^{-3}}{6 \times 10^{3}}$ $O(10^{-6})$ microns

74 dB $|\hat{\xi}|$ $O(10^{-7})$

54 dB $|\tilde{\xi}|$ $O(10^{-8})$

34 $|\hat{\xi}|$ $O(10^{-9})$ <u>nanometers</u>

## 3.3.1.2 Arbitrary Direction wrt coordinate system




$$\nabla^2 \tilde{p} + k^2 \tilde{p} = 0$$

$$\tilde{p}(x,y,z,t) = A e^{j(\omega t - \underbrace{k_x}x - \underbrace{k_y}y - \underbrace{k_z}z)}$$

component wave numbers

if this is to a solution of the wave eqn
- has to be true that

$$k^2 = k_x^2 + k_y^2 + k_z^2 \qquad k = \frac{\omega}{c}$$

~~for disturbance to be~~ sound

$$\underset{\sim}{k}^2 = k_x^2 + k_y^2 + k_z^2 \qquad k = \frac{\omega}{c}$$

Only 2 of the $k_x$, $k_y$ & $k_z$ can be chosen independently if the solution is "sound"