ME 513 Session 14 9/25/15

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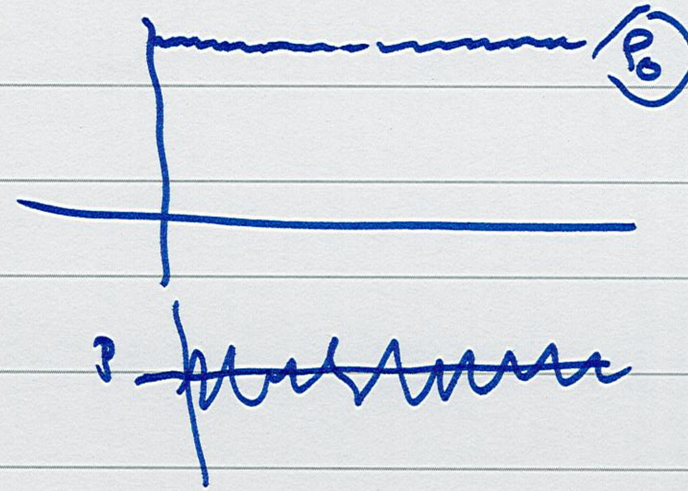
3. Wave Eqn & Simple Solutions

- sound waves



p & u [Eqn of state, Continuity]

p & u momentum eqn

→ single eqn in p → wave eqn

$p = \beta s$

$\frac{\partial s}{\partial t} + \nabla \cdot \underline{\underline{\bar{u}}} = 0$

$\nabla \cdot \bar{u}$

$\bar{u} = u_x \bar{i} + u_y \bar{j} + u_z \bar{k}$ ✓

∇ grad operator

$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

$\nabla \cdot \bar{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$

(iii) pressure vs. particle velocity expression

combine (1) + (2)

$$\boxed{\frac{1}{B}\frac{\partial p}{\partial t} + \nabla \cdot \underline{\bar{u}} = 0} \qquad (3)$$

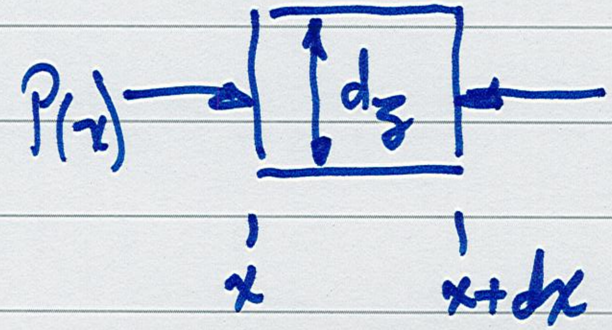
linearized continuity equation

3.2.2 Pressure - Velocity (II)

(i) Eqn of Motion

apply $F=ma$ to a fixed mass
That is moving with the fluid

1-D



$$P(x+dx) = P(x) + \left.\frac{dP}{dx}\right|_x dx + \ldots\ldots$$

Net force in the x-direction

$$df_x = P(x)\,dy\,dz - P(x+dx)\,dy\,dz$$

$$= \underbrace{-\left.\frac{dP}{dx}\right|_x}_{\text{force/unit volume}} \underbrace{dx\,dy\,dz}_{dV \text{ volume}}$$

In 3-D $\overline{df} = -\nabla P \, dV$

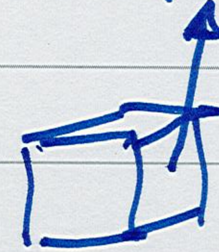
∇P = gradient of the pressure

$$= \frac{\partial P}{\partial x}\bar{i} + \frac{\partial P}{\partial y}\bar{j} + \frac{\partial P}{\partial z}\bar{k}$$

direction - steepest change in pressure.

(iii) Acceleration of a moving fluid element

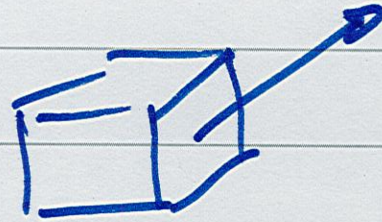
$f = m a$



$\overline{u}(x+dx, y+dy, z+dz, t+dt) = \overline{u}_f$

ξ

$\overline{u}(x, y, z, t) = \overline{u}_i$



$$\bar{u}_f = \overset{\bar{u}_i}{\underline{\bar{u}(x,y,z,t)}} + \left.\frac{\partial \bar{u}}{\partial x}\right|_{x,y,z,t} dx + \quad dy \quad + \quad dz$$

$$+ \left.\frac{\partial \bar{u}}{\partial t}\right|_{x,y,z,t} dt$$

$$dx = u_x \, dt$$

$$dy = u_y \, dt$$

$$dz = u_z \, dt$$

$$\bar{a} = \frac{\bar{u}_f - \bar{u}_i}{dt} = \left[\underbrace{u_x \frac{\partial \bar{u}}{\partial x} + u_y \frac{\partial \bar{u}}{\partial y} + u_z \frac{\partial \bar{u}}{\partial z}}_{\text{convective acceleration}} + \frac{\partial \bar{u}}{\partial t} \right]$$

- products of small quantities
- very small in linear acoustics

$$\bar{a} = \cancelto{0}{(\bar{u} \cdot \nabla)\bar{u}} + \frac{\partial \bar{u}}{\partial t}$$

neglect
because terms
are non-linear

$$d\bar{f} = -\nabla P \, dV$$

$$\bar{a} = \frac{\partial \bar{u}}{\partial t}$$

mass of the element

$$m = \rho \, dV$$

$$d\bar{f} = m\bar{a}$$

$$-\nabla P\, dV = \rho\, dV \frac{d\bar{u}}{dt}$$

(iv) Pressure-Velocity Relation

$$P = P_0 + p \qquad \nabla P = \nabla p$$

$$\rho = \rho_0 (1 + s) \qquad s = \frac{\rho - \rho_0}{\rho_0}$$

$$s << 1$$

$$\boxed{-\nabla p = \rho_0 \frac{d\bar{u}}{dt}} \quad (4)$$

linearized momentum eqn.

Euler Eqn

3.2.3 Linear Wave Eqn

(3) $\frac{1}{\beta}\frac{\partial p}{\partial t} + \nabla \cdot \bar{u} = 0 \qquad \frac{\partial}{\partial t}$

(4) $\nabla p + \rho_0 \frac{\partial \bar{u}}{\partial t} = 0 \qquad \frac{1}{\rho_0}\nabla \cdot$

Subtract (3) from (4)
to obtain

$$\nabla^2 p - \left(\frac{\rho_0}{\beta}\right)\frac{\partial^2 p}{\partial t^2} = 0$$

$$c = \sqrt{\frac{\beta}{\rho_0}}$$

$$\boxed{\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0}$$

linearized wave
Equation

$c =$ speed of sound

$= \sqrt{\frac{\beta}{\rho_0}} = \sqrt{\frac{\gamma P_0}{\rho_0}}$

- governs propagation of small amplitude pressure fluctuations in a stationary, ideal fluid

$\nexists$ mean flow

$U < \frac{1}{10} c$

3.2.4 Sound speed

$c =$ speed of wave propagation

$$= \sqrt{\frac{\gamma P_0}{\rho_0}} = 331.6 \text{ m/s at } 0^\circ C$$

$$\simeq \underline{340} \text{ m/s at room temp } 20^\circ C$$

in an isothermal atmosphere

$\left(\frac{P_0}{\rho_0}\right)$ is a constant

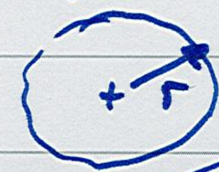
$c \neq$ function of height (temp. is constant) c = directly proportional to absolute temperature to the $\frac{1}{2}$ power

In air $\quad c = c_0 \sqrt{\frac{T_K}{273}} \qquad T_K \, [°K]$

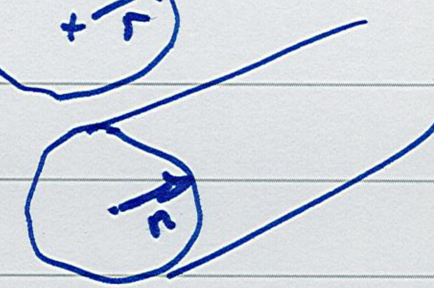
$$= c_0 \sqrt{1 + \frac{T_C}{273}} \qquad T_C \, [°C]$$

3.3 One-Dimensional Solutions

- spherical wave - spherically symmetric



- cylindrical waves



- plane