3. The Acoustic Wave Equation & Simple solutions

5.1 - 5.13

3.1 Introduction

- sound waves - propagating pressure fluctuations in an elastic medium
- "ideal" acoustics
  assume that the fluid is inviscid, lossless & adiabatic

- small amplitude fluctuations
  - "linear" acoustics

- wave propagation
  - inertia
  - stiffness

Derive a wave equation

$$\left.\begin{array}{ll} p+u & \left[\begin{array}{l}\text{- equation of state} \\ \text{- continuity}\end{array}\right. \\ & \\ p+u & \left[\text{- momentum}\right. \end{array}\right] \Rightarrow \begin{array}{l}\text{single} \\ \text{eqn for} \\ \text{pressure}\end{array}$$

- plane wave
- cylindrical
- spherical

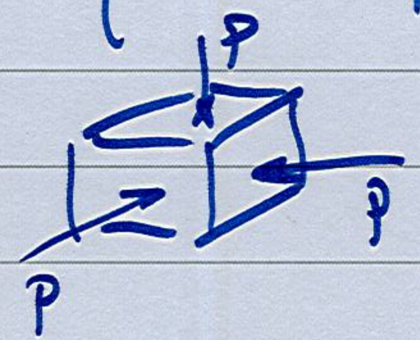
1-D

- Acoustic Intensity ✓
- specific acoustic impedance
- Decibels

## 3.2 Derivation of The Wave Equation

### 3.2.1 Pressure - Velocity (I)

(i) Equation of state




Ambient Pressure

$P_0 \simeq 1 \times 10^5$ Pa

Ambient Air Density

1.2 kg/m$^3$

for an ideal gas

- pressure is a function density
- pressure change vs density change from ambient: $P_0$, $\rho_0$

$$P = P_0 + \left(\frac{dP}{d\rho}\right)_{\rho_0}(\rho - \rho_0) + \cancel{\frac{1}{2}\frac{d^2P}{d\rho^2}(\rho - \rho_0)^2} + \dots$$

linear acoustics

- fluctuations are small compared to some ambient value.

$$(P - P_0) = \rho_0 \left(\frac{dP}{d\rho}\right)_{\rho_0} \left(\frac{\rho - \rho_0}{\rho_0}\right)$$

condensation

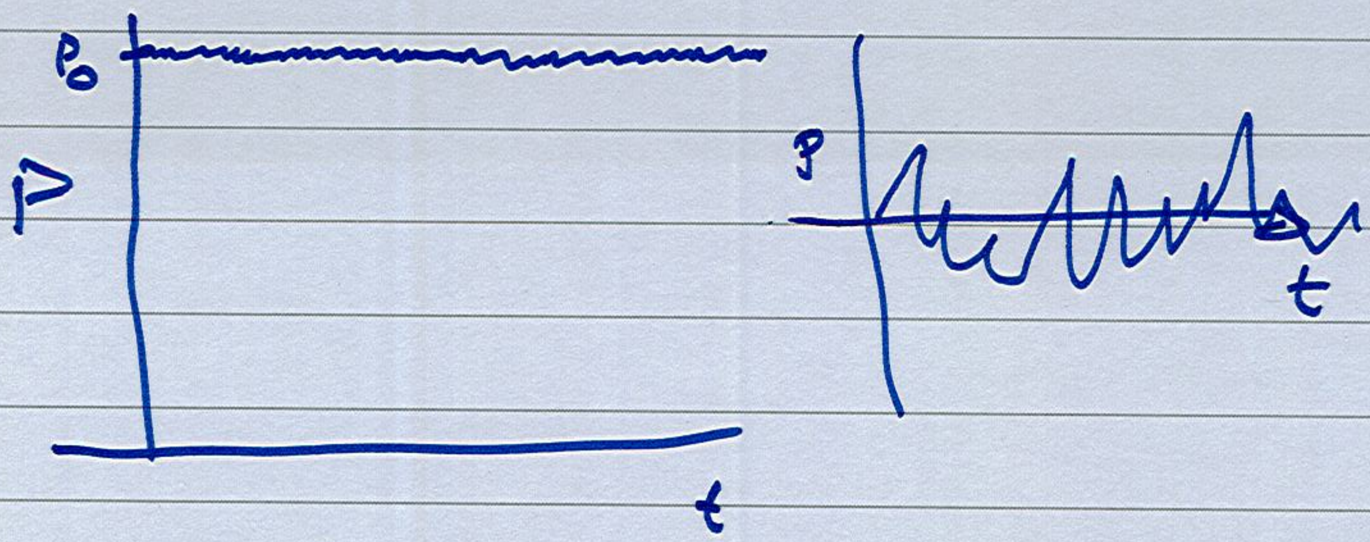
s

non-dimensional density fluctuation

$P - \underline{P_0} = p$ sound pressure

$\rho > \rho_0$ $s > 0$

$\rho < \rho_0$ $s < 0$



$$p = \rho_0 \left(\frac{dP}{d\rho}\right)_{\rho_0} s$$

Bulk modulus of elasticity

$\beta$ - adiabatic

$$\boxed{p = \beta s} \quad (1)$$

stress = modulus × strain

what is $\beta$ for adiabatic compression

Two states $P, P_0$ $\quad \rho, \rho_0$

$$\boxed{\left(\frac{P}{P_0}\right) = \left(\frac{\rho}{\rho_0}\right)^{\gamma}}$$

$\gamma$ = ratio of specific heats

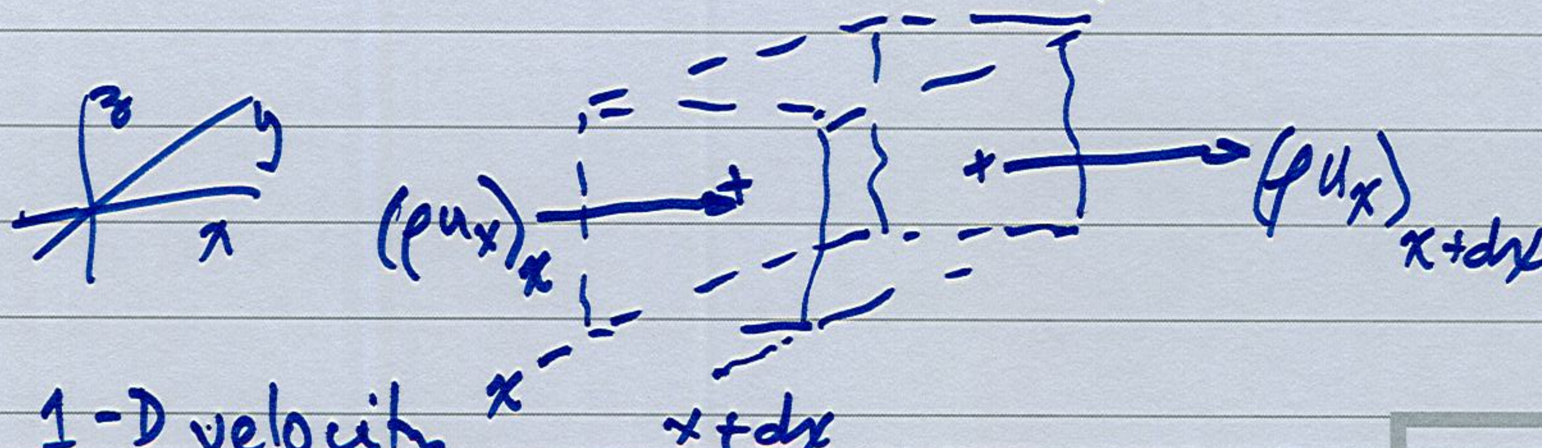
$= 1.4$

$$\underline{\underline{\beta}} = \rho_0 \left(\frac{dP}{d\rho}\right)_{\rho_0} = \underline{\underline{\gamma P_0}} = \underline{1.4 \times 10^5 \text{ Pa}}$$

$$\underline{P = \beta s}$$

(iii) Continuity Eqn (conservation of mass)

Control Volume fixed in space




1-D velocity
- motion only in the x-dir

Rate at which mass flows into the Control Volume

(a) $(\rho u_x)_x \, dy\, dz$ kg/s

Rate at which mass flows ~~of~~ out of the Control Volume

(b) $\underline{(\rho u_x)_{x+dx} \, dy\, dz}$

$$= \left[ (\rho u_x)_x + \frac{d(\rho u_x)}{\partial x}\Big|_x dx + \cdots \xrightarrow{} 0 \right] dy\, dz$$

net rate of mass inflow

$$(a) - (b) = -\left.\frac{\partial(\rho u_x)}{\partial x}\right|_x dx\,dy\,dz$$

Rate at which the mass in the control volume changes

$$\frac{d\rho}{dt}\,dx\,dy\,dz$$

$$\frac{\partial \rho}{\partial t} \cancel{dx\,dy\,dz} = -\frac{\partial(\rho u_x)}{\partial x}\bigg|_x \cancel{dx\,dy\,dz}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_x)}{\partial x} = 0}$$

Conservation of mass

$\rho$ + $u_x$ both functions of position

$$\frac{\partial \dot{\rho}}{\partial t} + u_x \frac{\partial \dot{\rho}}{\partial x} + \dot{\rho} \frac{\partial u_x}{\partial x} = 0$$

$$s = \frac{\rho - \rho_0}{\rho_0} \qquad \rho = s\rho_0 + \rho_0$$

$$= \rho_0 (s+1)$$

$$\rho_0 \frac{\partial (s+\cancelto{0}{1})}{\partial t} + \rho_0 u_x \frac{\partial (s+\cancelto{0}{1})}{\partial x} + \rho_0 (s+1) \frac{\partial u_x}{\partial x} = 0$$

$$\rho_0 \frac{\partial s}{\partial t} + \cancelto{0}{\rho_0 u_x \frac{\partial s}{\partial x}} + \rho_0 (\cancel{s}+1) \frac{\partial u_x}{\partial x} = 0$$

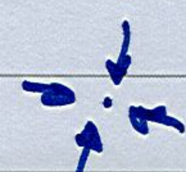
- assume that the product of small quantities is negligible → linear acoustic

$$\cancel{\rho_0} \frac{\partial s}{\partial t} + \cancel{\rho_0} \frac{\partial u_x}{\partial x} = 0$$

1-D $$\boxed{\frac{\partial s}{\partial t} + \frac{\partial u_x}{\partial x} = 0}$$

3-D $$\boxed{\frac{\partial s}{\partial t} + \nabla \cdot \overline{u} = 0} \quad (2)$$

$\nabla \cdot \overline{u}$ is the divergence of the particle velocity

$\nabla \cdot \overline{u}$ +ve  $\nabla \cdot \overline{u}$ -ve 

particle velocity

$$\bar{u} = u_x \bar{i} + u_y \bar{j} + u_z \bar{k}$$

$\nabla$ grad operator

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

$$\nabla \cdot \bar{u} = \underline{\hspace{6cm}}$$