$e^{i\omega t}$

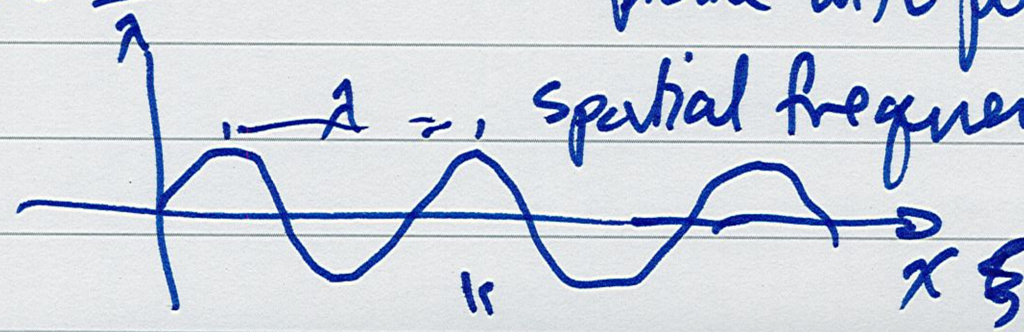
$$\frac{d^2 Y(x)}{dx^2} + k^2 Y(x) = 0$$

$$Y(x) = \left(A e^{-ikx}\right. + B e^{ikx}$$

rate of change of phase with position

$k = 2\pi$

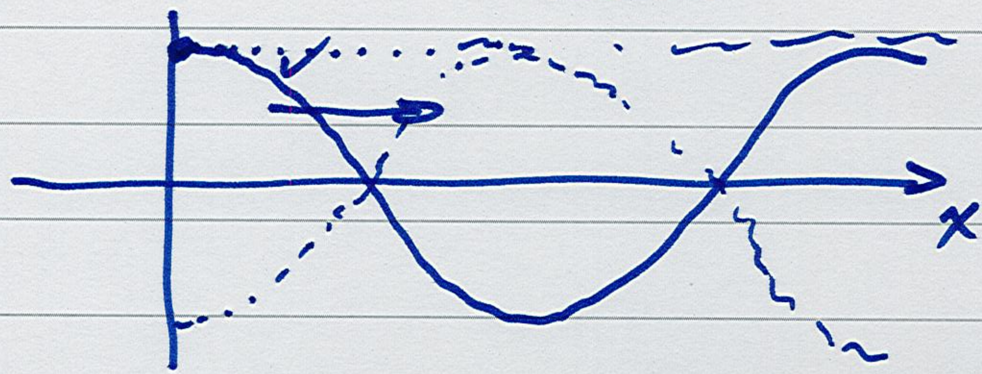
λ = spatial frequency



$$y(x,t) = A_1 e^{-jkx} e^{j\omega t} + \cancel{B}\, A_2 \underline{e^{+jkx}} e^{j\omega t}$$

$$e^{j(\omega t - kx)}$$

+ve



$t = 0 \rightarrow \frac{T}{2}$

when $t = T$ wave pattern has advanced by one wavelength.

$c = \frac{\lambda}{T}$ $\qquad f = \frac{1}{T}$

$cT = \lambda$

$c = f\lambda$

ω [rad/s]

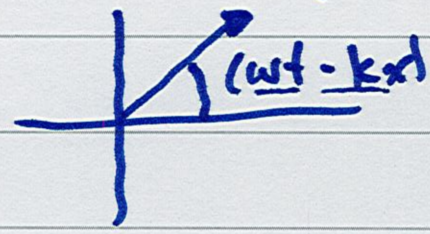
$$\boxed{\frac{\omega}{c} = \frac{2\pi}{\lambda} = k}$$

k [rad/m]

	Time	Space
frequency	ω	k
period	T	λ

General solution

$$y(x,t) = \hat{A}_1 e^{i(\omega t - kx)} + \hat{A}_2 e^{i(\omega t + kx)}$$



transverse velocity - harmonic

$$\frac{\partial y}{\partial t}(x,t) = u(x,t) = j\omega\, y(x,t)$$

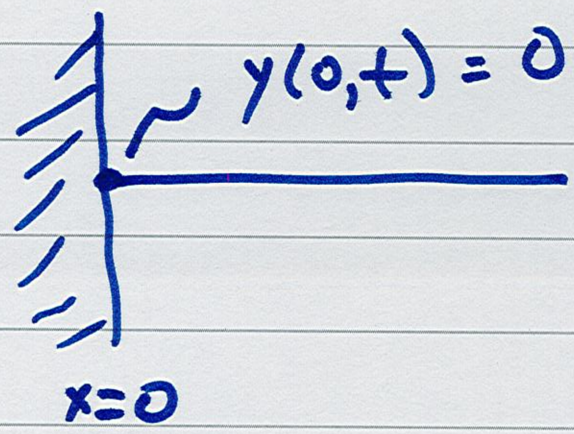
transverse acceleration

$$\frac{\partial^2 y}{\partial t^2}(x,t) = a(x,t) = \frac{\partial u}{\partial t} = (j\omega)(j\omega)y(x,t)$$
$$= -\omega^2 y(x,t)$$

2.3 Boundary Conditions

General Sol'n: $y(x,t) = \underbrace{y_1(w_1)}_{w_1 = ct - x} + \underbrace{y_2(w_2)}_{w_2 = ct + x}$

2.3.1 Fixed



$$y(0,t) = y_1\left(w_1\big|_{x=0}\right) + y_2\left(w_2\big|_{x=0}\right) = 0$$

$w_1 = ct - x$ $\qquad$ $w_2 = ct + x$

$w_1\big|_{x=0} = ct$ $\qquad$ $w_2\big|_{x=0} = ct$

$$y_2(ct) + y_1(ct) = 0$$

$$y_2(\underline{ct}) = -y_1(\underline{ct})$$

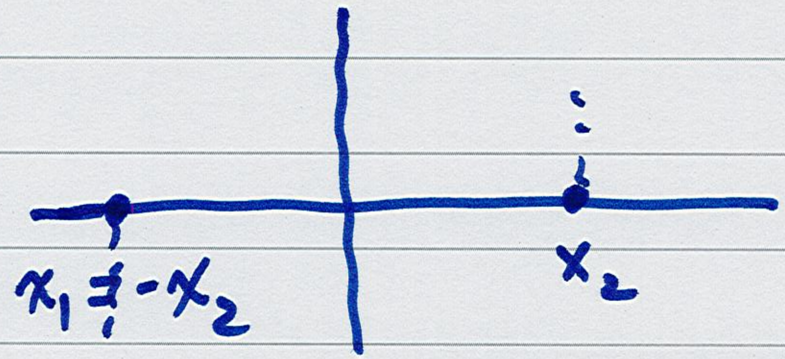
at $x=0$ $\quad$ y_1 & y_2 must be equal & opposite → cancel to create 0 displacement more generally as a result of the b.c.

y_1 + y_2 are equal & opposite whenever $w_1 = w_2$

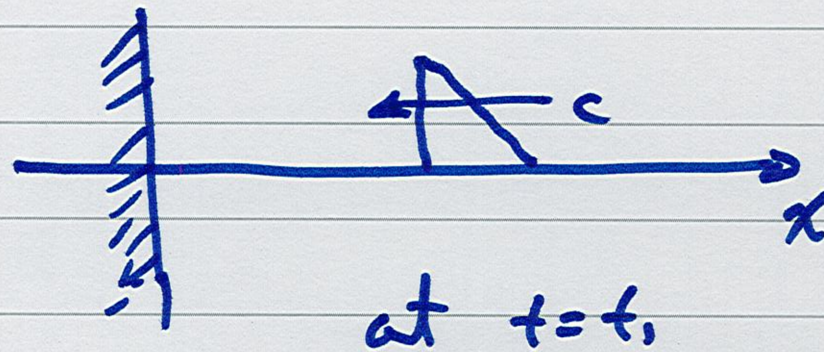
so at a particular $t = t_1$

$$\begin{array}{ccc} w_1 & & w_2 \\ \cancel{ct_1} - x_1 & = & \cancel{ct_1} + x_2 \end{array}$$

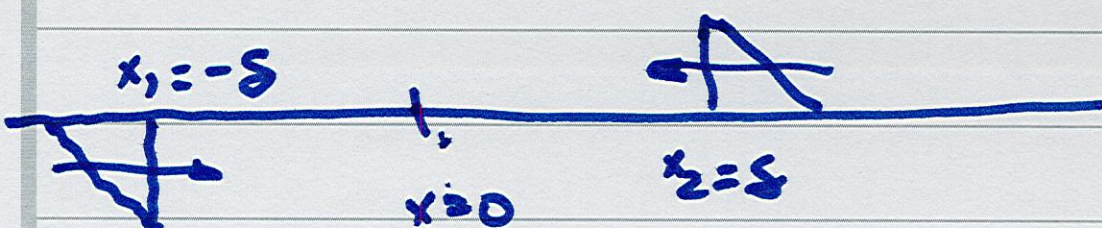
$$x_1 = -x_2$$



hard boundary causes a reflection that is upside down + backwards

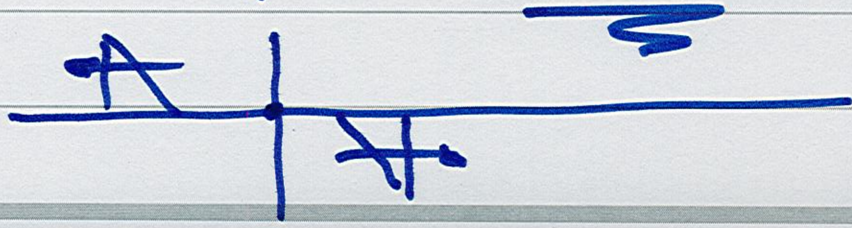


at $t = t_1$

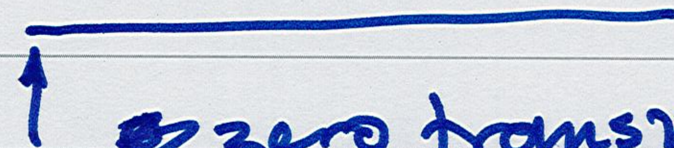


image

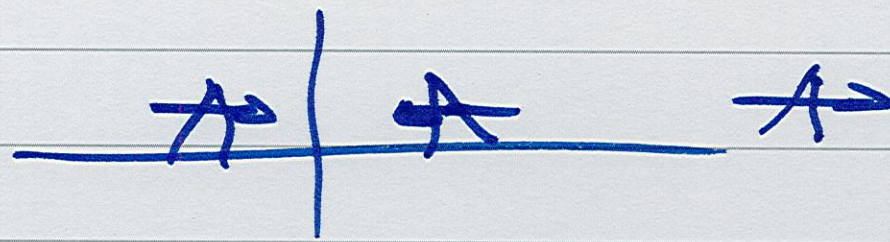
→ real world



free end



⇒ zero transverse force at the free end of a string



$$\frac{\partial y}{\partial x} = 0$$

frequency domain

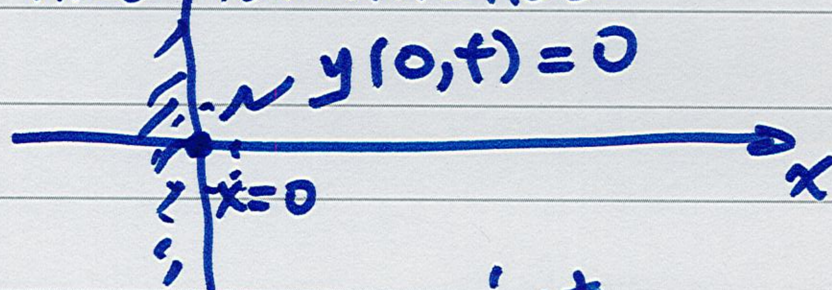
$$y(x,t) = \underset{\sim}{A}\, e^{j(\omega t - kx)} + \underset{\sim}{B}\, e^{j(\omega t + kx)}$$

$$k = \frac{\omega}{c}$$

two b.c.'s to solve for A & B

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Fixed termination



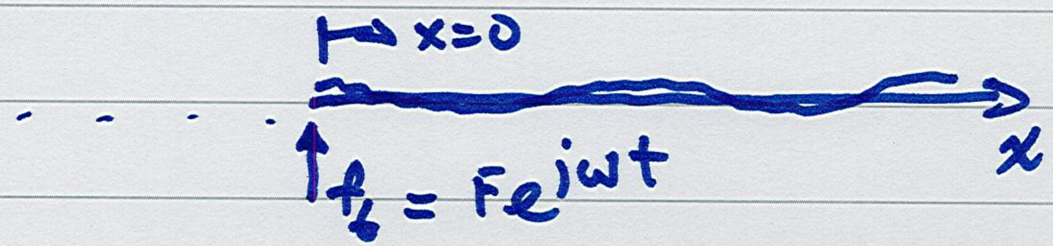
$$y(0,t) = A\cancel{e^{j\omega t}} + B\cancel{e^{j\omega t}} = 0$$

$$B = -A$$

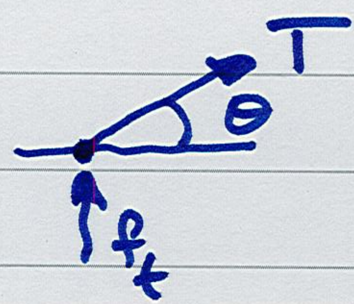
$$y(x,t) = A e^{j\omega t}\left(e^{-jkx} - e^{+jkx}\right)$$

$$= -2jA e^{j\omega t} \sin(kx)$$

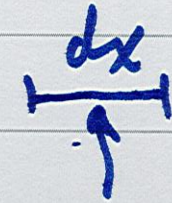
2.3.2 Force b.c. at $x=0$



FBD



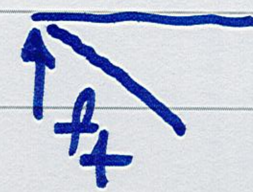
$\Sigma f_y = 0$



$$f_t + T\underbrace{\sin\theta}_{\frac{\partial y}{\partial x}\big|_{x=0}}\Big|_{x=0} = 0$$

$$f_t + T \frac{\partial y}{\partial x}\bigg|_{x=0} = 0$$

$$\frac{\partial y}{\partial x}\bigg|_{x=0} = -\frac{f_t}{T}$$



in special case

unconstrained end $\rightarrow f_t = 0$

$$\rightarrow \frac{\partial y}{\partial x} = 0$$

no lateral constraint

In the harmonic case

$$f_t = F e^{j\omega t}$$

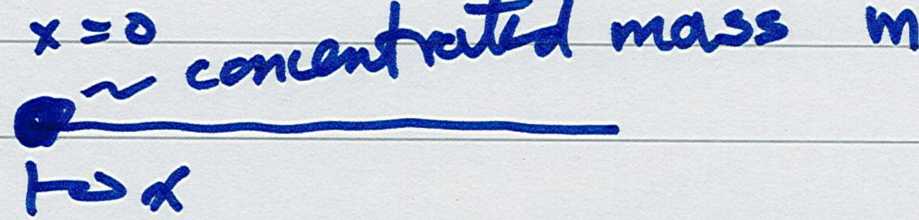
$$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{f_t}{T} \qquad y(x,t) = A e^{j(\omega t - kx)} + B e^{j(\omega t + kx)}$$

$$\frac{dy}{dx} = -jkA e^{j(\omega t - kx)} + jkB e^{j(\omega t + kx)}$$

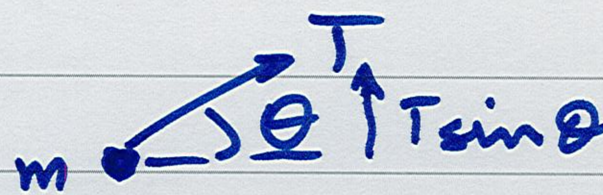
$$\left.\frac{dy}{dx}\right|_{x=0} = -jkA\cancel{e^{j\omega t}} + jkB\cancel{e^{j\omega t}} = \frac{F\cancel{e^{j\omega t}}}{T}$$

$$\boxed{B = -\frac{F}{jkT} + A}$$

2.3.3 Mass at $x=0$



FBD



$$\Sigma f_y = ma$$

$$T \underbrace{\sin\theta}_{\frac{\partial y}{\partial x}}\Big|_{x=0} = m\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial t^2}\Big|_{x=0} = \frac{T}{m}\frac{\partial y}{\partial x}\Big|_{x=0}$$