3. The Acoustic Wave Equation and simple Solutions

5.1 → 5.13

3.1 Introduction

- sound waves – propagating pressure fluctuations in an elastic medium

- "ideal" acoustics assume that the fluid is inviscid, lossless, adiabatic

- small amplitude fluctuations
  - "linear" acoustics

- wave propagation
  - inertia
  - stiffness

Derive the wave equation

p & u $\left\{ \begin{array}{l} \text{- Equation of state} \\ \text{- Continuity Eqn} \end{array} \right.$

p & u $\left\{ \text{- momentum eqns} \right.$

- plane waves
- cylindrical waves
- spherical

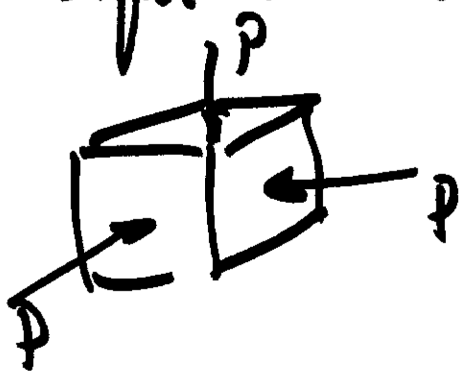
1-D

- Acoustic Intensity
- specific acoustic impedance
- decibels

3.2 Derivation of the Wave Equation

3.2.1 Pressure - Velocity (I)

(i) Equation of state



Ambient Pressure

$P_0 \simeq 1 \times 10^5$ Pa

Ambient Density

$\rho_0 \simeq 1.2$ kg/m^3

for an ideal gas

- pressure is a function of density
- pressure change vs. density change

Ambient state: P_0, ρ_0

$$P = P_0 + \left(\frac{dP}{d\rho}\right)_{\rho_0}(\rho-\rho_0) + \cancel{\frac{1}{2}\left(\frac{d^2P}{d\rho^2}\right)}(\rho-\rho_0)^2 + \cancel{\cdots}$$

linear acoustics

- fluctuations are assumed small

Rearrange

$$(P - P_0) = \rho_0 \left(\frac{dP}{d\rho}\right)_{\rho_0} \left(\frac{\rho - \rho_0}{\rho_0}\right)$$

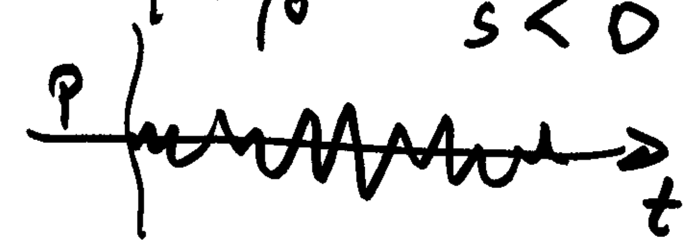
condensation s

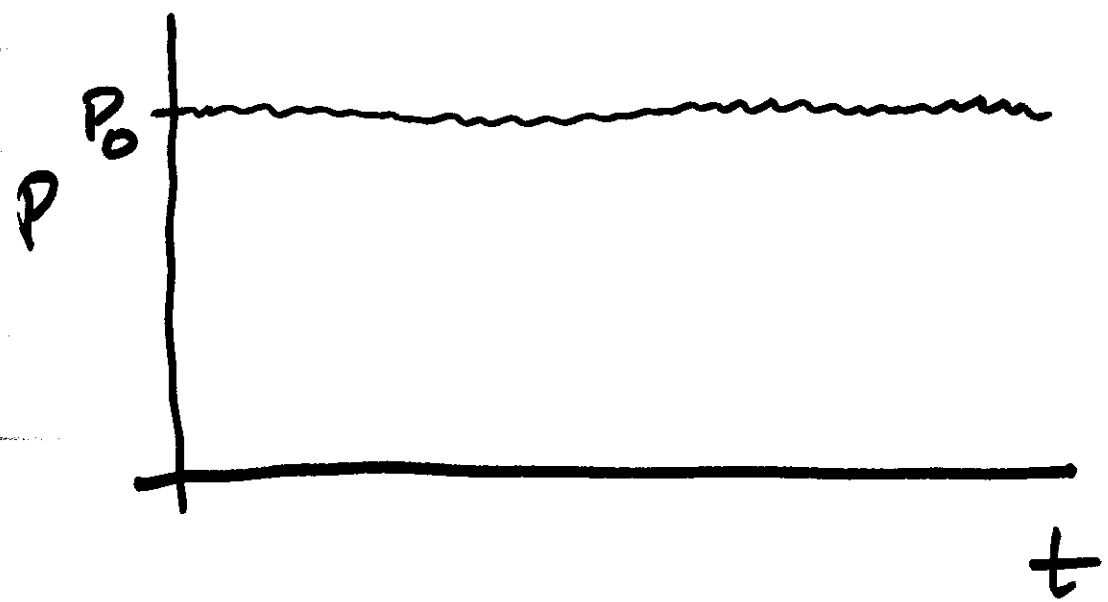
- non-dimensional density fluctuation

$\rho > \rho_0$ $s > 0$

$\rho < \rho_0$ $s < 0$

$$\underbrace{P - P_0} = p \quad \text{sound pressure}$$





$$p = \rho_0 \left(\frac{dP}{d\rho}\right)_{\rho_0} s$$

Bulk modulus of elasticity of the fluid

β adiabatic

$$\boxed{p = \beta s} \qquad (\underline{1})$$

stress = modulus × strain

Two states $\quad P$ & $P_0 \quad \rho$ & ρ_0

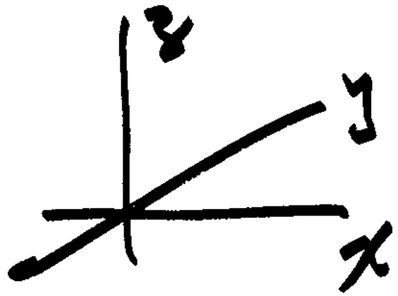
$$\left(\frac{P}{P_0}\right) = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

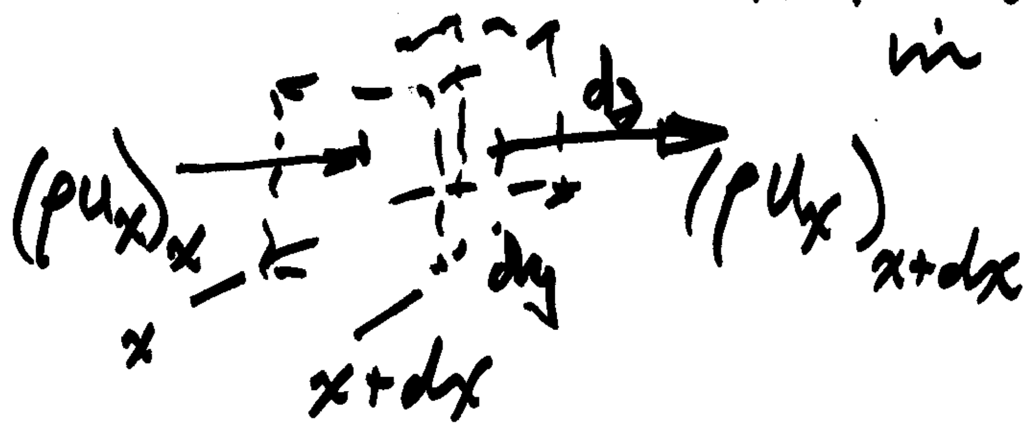
γ = ratio of specific heats $= 1.4$

$$\beta = \rho_0 \left(\frac{dP}{d\rho}\right)_{\rho_0} = \gamma P_0 = \underbrace{1.4 \times 10^5 \text{ Pa}}_{\text{stiffness of air}}$$

(ii) Continuity Eqn (Conservation of Mass)

Control Volume fixed in space.





1-D velocities are only in the x-direction

Rate at which mass flows into the control volume

(a) $(\rho u_x)\, dy\, dz$ kg/s

Rate at which mass flows out of the control volume

(b) $(\rho u_x)\big|_{x+dx}\, dy\, dz$

$$= \left[(\rho u_x)_x + \frac{\partial(\rho u_x)}{\partial x}\Big|_x dx + \cancelto{0}{\ldots\ldots} \right] dy\, dz$$

net rate of mass inflow

$$(a) - (b) = -\left.\frac{\partial(\rho u_x)}{\partial x}\right|_x dx\,dy\,dz$$

Rate at which the mass in the control volume changes

$$\frac{\partial \rho}{\partial t}\,dx\,dy\,dz$$

$$\frac{d\rho}{dt}\cancel{dx\,dy\,dz} = -\left.\frac{\partial(\rho u_x)}{\partial x}\right|_x \cancel{dx\,dy\,dz}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_x)}{\partial x} = 0}$$

Conservation of mass

ρ & u_x are themselves function of x

$$\frac{\partial \rho}{\partial t} + u_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial u_x}{\partial x} = 0$$

$$s = \frac{\rho - \rho_0}{\rho_0} \qquad \rho = s\rho_0 + \rho_0 = \rho_0 (s+1)$$

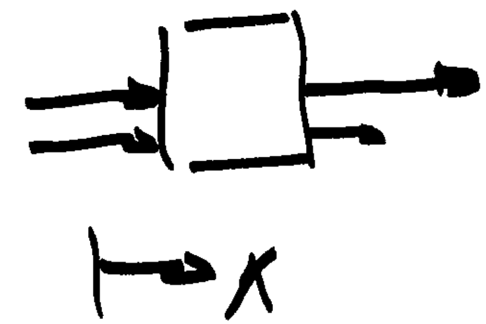
$$\rho_0 \frac{\partial (s+\cancelto{0}{1})}{\partial t} + \rho_0 u_x \frac{\partial (s+\cancelto{0}{1})}{\partial x} + \rho_0 (s+1) \frac{\partial u_x}{\partial x} = 0$$

$$\rho_0 \frac{\partial s}{\partial t} + \cancelto{0}{\rho_0 u_x \frac{\partial s}{\partial x}} + \rho_0 (\cancelto{0}{s}+1) \frac{\partial u_x}{\partial x} = 0$$

- linear acoustics
  - setting products of small quants $\to 0$

$$\cancel{\rho_0}\frac{\partial s}{\partial t} + \cancel{\rho_0}\frac{\partial u_x}{\partial x} = 0$$

(1 - D) $$\boxed{\frac{\partial s}{\partial t} + \frac{\partial u_x}{\partial x} = 0}$$



(3 - D) $$\boxed{\frac{\partial s}{\partial t} + \nabla \cdot \bar{u} = 0} \quad (2)$$

$\nabla \cdot \bar{u}$ divergence of the particle velocity

$\nabla \cdot \bar{u}$ = divergence

Velocity

$$\bar{u} = \cancel{x}\ u_x \bar{i} + u_y \bar{j} + u_z \bar{k}$$

∇ grad operator

$$\nabla = \bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z}$$

$\nabla \cdot \bar{u}$ = ______________