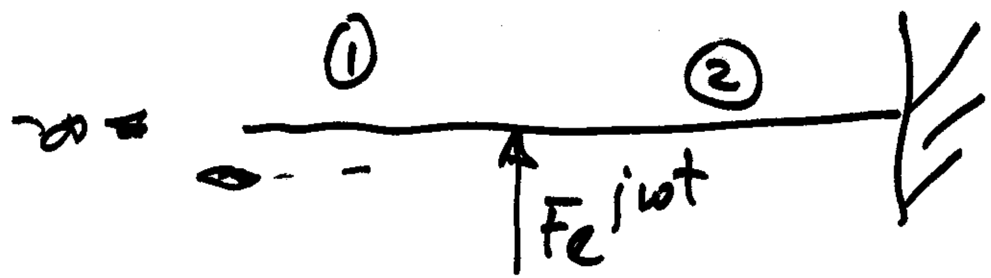
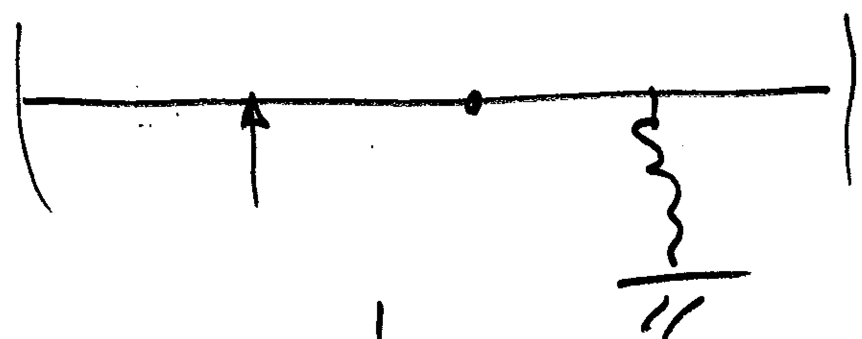
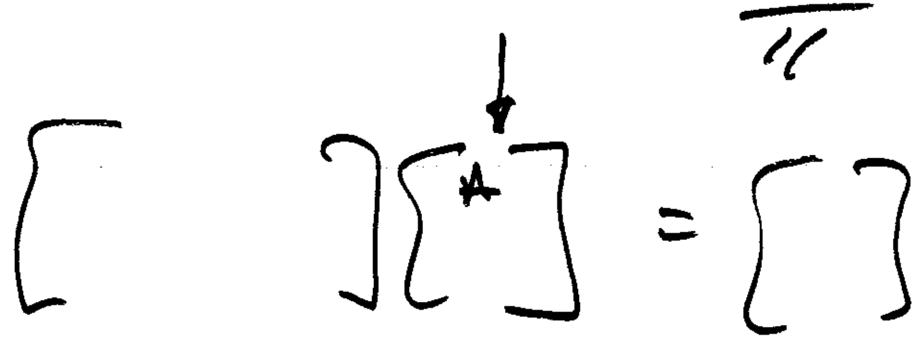


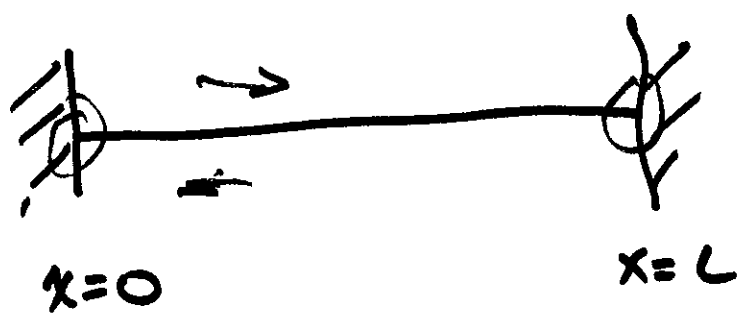
$$Z_{M0} = \rho_{L_1} c_1 - j \rho_{L_2} c_2 \cot k_2 L$$





2.5 Normal Modes of Finite Strings

2.5.1 Characteristic Equation



One segment

$$y(x,t) = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}$$

b.c. at $x=0$ $\quad y(0,t) = 0$

$$A + B = 0$$

$$B = -A \qquad (1)$$

b.c. at $x=L$ $\quad y(L,t) = 0$

$$\left(A e^{-jkL} + B e^{+jkL}\right) \cancel{e^{j\omega t}} = 0$$

$$A\left(e^{-jkL} - e^{+jkL}\right) = 0$$

$$\boxed{-2jA\sin kL = 0} \quad (2)$$

if $A = 0$ disp $= 0$ at all times

$$\sin kL = 0$$

$$\boxed{\sin k_n L = 0}$$ characteristic equation

$$k_n L = n\pi$$

$$k_n = \frac{n\pi}{L} \qquad n = 1, 2, 3 \ldots$$

$$k_n = \frac{n\pi}{L} \qquad k = \frac{\omega}{c} = \frac{2\pi f}{c}$$

$$f_n = \frac{n}{2}\frac{c}{L} \qquad n = 1, 2, 3 \ldots\ldots$$

allowed frequencies (natural frequencies)

$$L = \frac{n}{2}\frac{c}{f_n} = \frac{n}{2}\lambda_n \qquad c = \sqrt{\frac{T}{\rho_L}}$$

$c = f_n \lambda_n$ at the natural frequencies
The string is an integral
multiple of $\lambda/2$
(for fixed boundarys)

2.5.2 Mode shapes

$$y(x,t) = \left(Ae^{-jkx} + Be^{jkx}\right)e^{j\omega t}$$

1st b.c. $\quad B = -A$

$$y(x) = -2jA\sin kx$$

$$y_n(x,t) = \underbrace{\underbrace{-2jA_n}_{C_n}\,\underbrace{\sin k_n x}_{\text{mode shape}}\,e^{j\omega_n t}}_{\text{normal mode}}$$

modes

- individual solutions of the wave equation
- each satisfies the b.c.'s of the problem

k_n's are the allowed wave numbers

ω_n's $= 2\pi f_n$'s natural freqs

$c_n =$ modal amplitude

Complete solution = sum of the possible solutions

Final solution $\Rightarrow$ apply initial conditions

2.5.3 Total Solution

Total solution = superposition of possible solutions

$$y(x,t) = \sum_{n=1}^{\infty} \tilde{a}_n \sin k_n x \, \underbrace{e^{j\omega_n t}}_{\cos\omega_n t + i \sin \omega_n t}$$

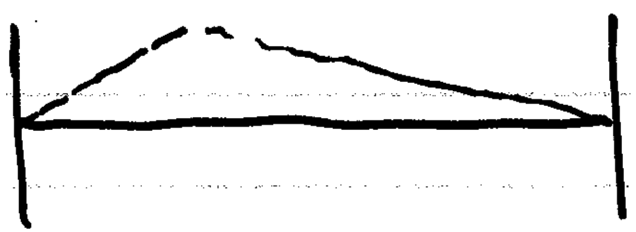
$$\tilde{a}_n = a_n + i b_n$$

Real displacement of the nth mode

$$\mathrm{Re}\{y_n(x,t)\} = (a_n \cos \omega_n t - b_n \sin \omega_n t) \sin k_n x$$

Initial Conditions

Say $\mathrm{Re}\{y(x,0)\}$ = known



at $t=0$

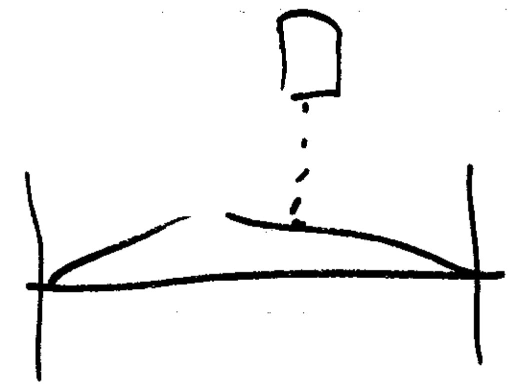
$$\underbrace{\mathrm{Re}\{y(x,0)\}}_{\text{known}} = \sum_{n=1}^{\infty} a_n \sin k_n x$$

Fourier Analysis

$$a_n = \frac{2}{L}\int_0^L \mathrm{Re}\{y(x,0)\} \sin k_n x \, dx$$

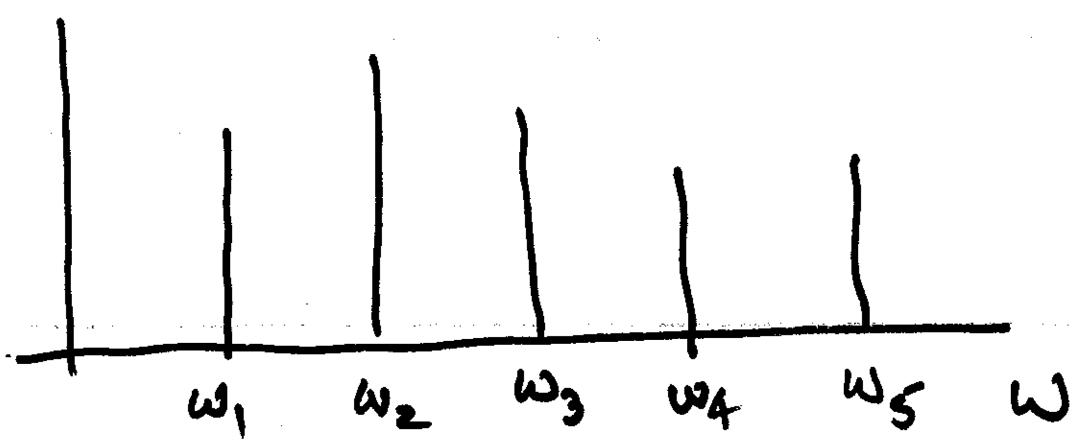
$$\tilde{a}_n = a_n + j b_n$$

use the initial velocity of the string
to solve for the b_n's

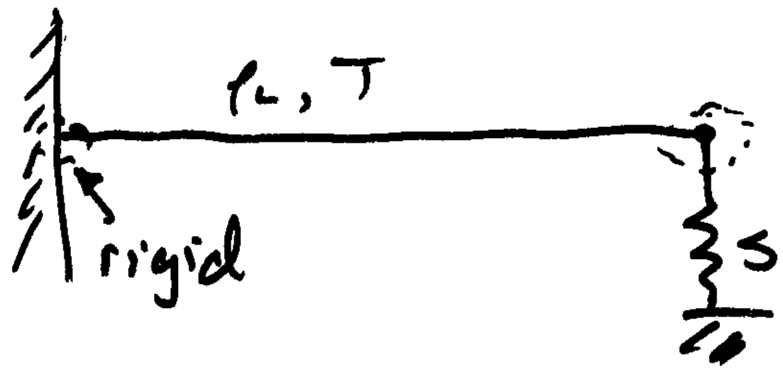


$$\hat{a}_n = a_n + j b_n$$

$$y(x,t) = \sum_{n=1}^{\infty} \hat{a}_n \sin k_n x \, e^{j\omega_n t}$$



2.5.4 other b.c.'s



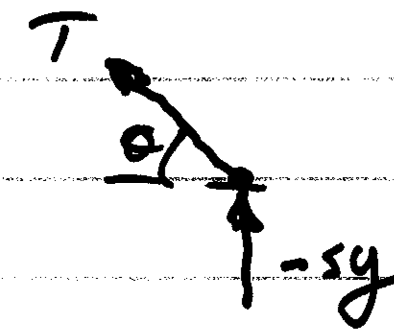
$$y(x,t) = A e^{i(\omega t - kx)} + B e^{i(\omega t + kx)}$$

at $x=0$ $\quad y(0,t) = 0$

$$B = -A$$

$$y(x,t) = -2j A \sin kx \, e^{j\omega t}$$

at $x = L$ $\quad \Sigma f_y = 0$



$$T\sin\theta\Big|_{x=L} - sy\Big|_{x=L} = 0$$

$$-T\frac{\partial y}{\partial x}\Big|_{x=L} - sy\Big|_{x=L} = 0$$

$$\frac{\partial y}{\partial x} = -2jkA\cos kx\, e^{j\omega t}$$

$$+\cancel{2j}\,kT\,\cancel{A}\,\cancel{e^{j\omega t}}\cos kL + \cancel{2j}\,s\,\cancel{A}\,\cancel{e^{j\omega t}}\sin kL = 0$$

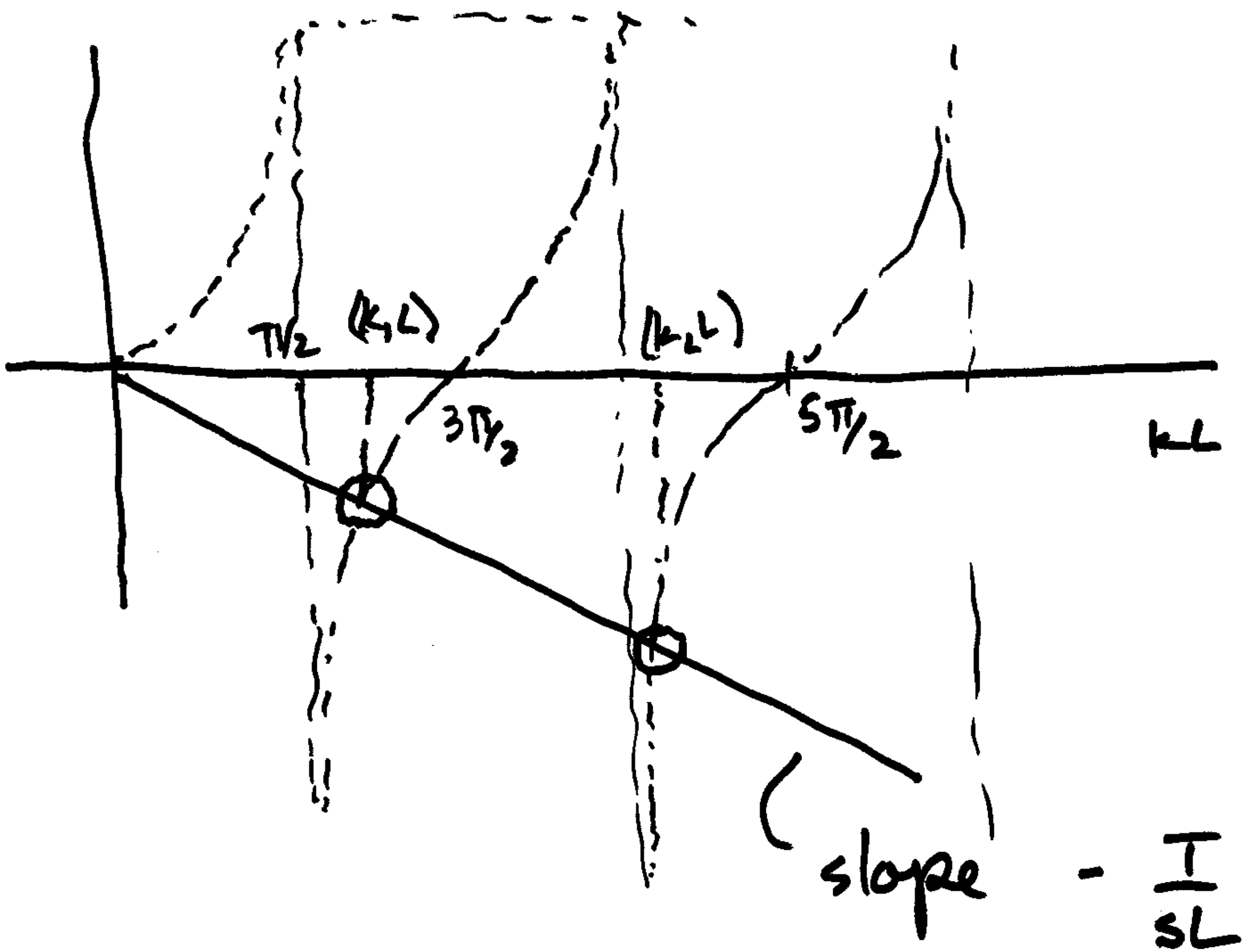
$$kT\cos kL = -s\sin kL$$

$$-\frac{k_n T}{s} = \tan(k_n L)$$

$$\boxed{-(k_n L)\left(\frac{T}{sL}\right) = \tan(k_n L)}$$

characteristic equation

$$-(k_n L)\left(\frac{I}{sL}\right) = \tan kL$$



$k_1 \to \omega_1 \to f_1$