Chapter 2 in text

2.1 → 2.11

2.4 Forced Vibration of Strings

$Fe^{j\omega t} = f_t$

$$y(x,t) = A e^{i(\omega t - kx)}$$

Apply b.c.

$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = -\frac{f_t}{T}$$

$$y(x,t) = \frac{F}{jkT} e^{i(\omega t - kx)}$$

transverse velocity

$$u(x,t) = \frac{\partial y}{\partial t} = \frac{j\omega F}{jkT} e^{j(\omega t - kx)}$$

$$c = \sqrt{\frac{T}{\rho_L}} \rightarrow T = \rho_L c^2 \qquad k = \frac{\omega}{c}$$

$$u(x,t) = \frac{F}{\rho_L c} e^{j(\omega t - kx)}$$

$\rho_L c$ ~ characteristic impedance

Input Mechanical Impedance

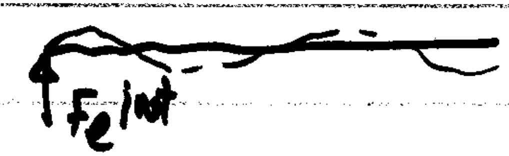
$$Z_{mo} = \frac{\text{Complex Applied Driving Force}}{\text{Velocity at the drive point}}$$

at $x = 0$
drive point in this case

$$= \frac{Fe^{j\omega t}\rho_L c}{Fe^{j\omega t}}$$

$$\boxed{Z_{mo} = \rho_L c}$$

- purely real
  resistive
- energy is carried away to infinity



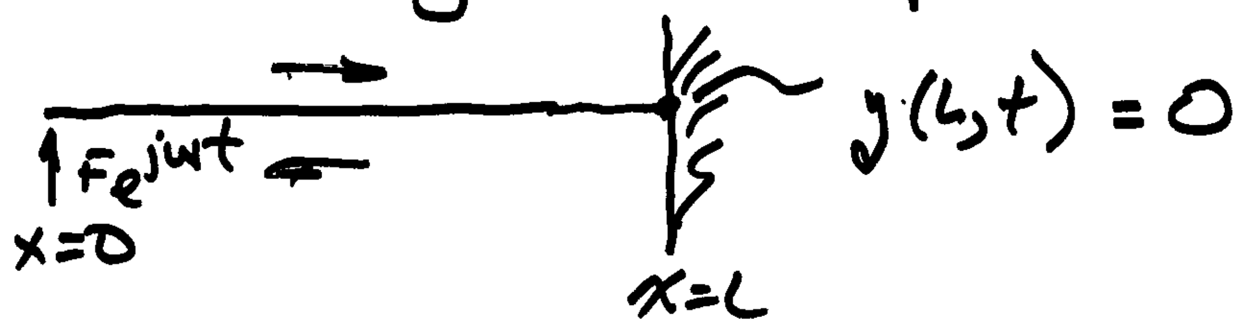
$\rho_L c$ = characteristic impedance

"characteristic" of the medium carrying the wave (and also the wave type)

and in this case, the input mechanical impedance of a semi-infinite string is $= Z_{M0} = \rho_L c$

2.4.2 Finite Length String

Forced - Fixed



$$y(x,t) = Ae^{j(\omega t - kx)} + Be^{j(\omega t + kx)}$$

(i) $y(L,t) = \boxed{0 = Ae^{-jkL} + Be^{+jkL}}$ (1)

(ii) force b.c. at $x=0$

$$f_t = -T\left.\frac{\partial y}{\partial x}\right|_{x=0}$$

$$\frac{dy}{dx} = -jkA\,e^{j(\omega t - kx)} + jkB\,e^{j(\omega t + kx)}$$

$$F\cancel{e^{j\omega t}} = jkT(A - B)\cancel{e^{j\omega t}}$$

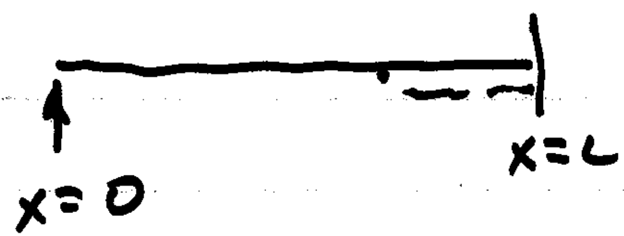
$$\boxed{jkTA - jkTB = F} \quad (2)$$

(iii) 2 simultaneous eqns in A & B

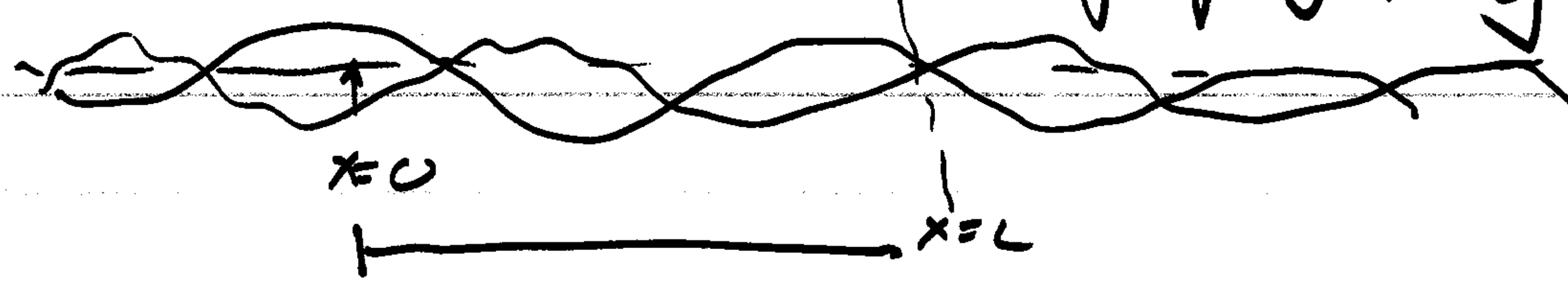
$$\begin{bmatrix} e^{-jkL} & e^{+jkL} \\ jkT & -jkT \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

$$y(x,t) = \overbrace{\frac{F e^{jkL}}{2jkT\cos kL}}^{A} e^{j(\omega t - kx)} - \overbrace{\frac{F e^{-jkL}}{2jkT\cos kL}}^{B} e^{j(\omega t + kx)}$$

$$= \frac{F}{2jkT\cos kL}\left[\underset{\longrightarrow}{e^{j(\omega t + k(L-x))}} - \underset{\longleftarrow}{e^{j(\omega t - k(L-x))}}\right]$$



soln for the displacement expressed as a superposition of two propagating waves



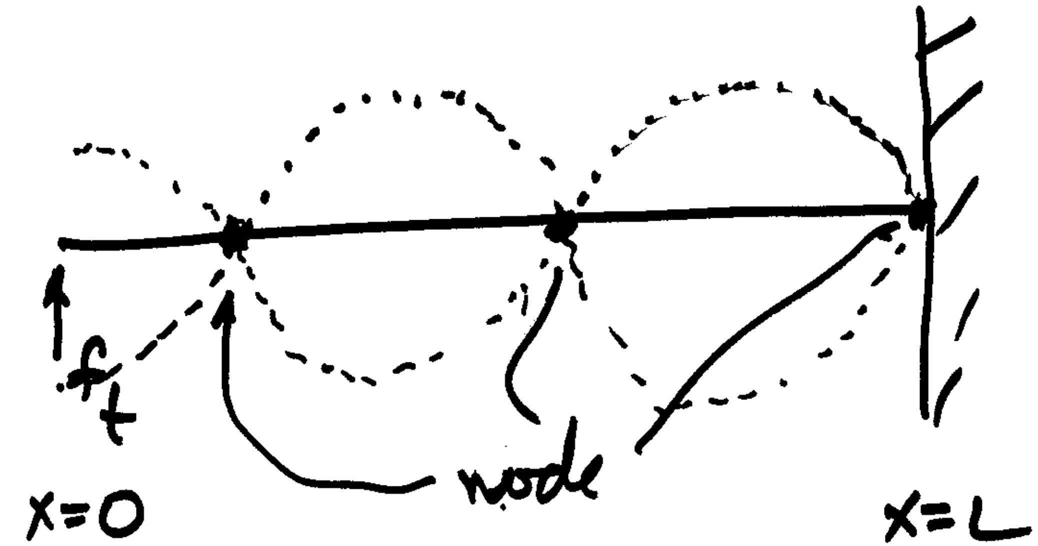
alternatively

$$y(x,t) = \frac{F e^{j\omega t} \, 2j \sin k[L-x]}{2jkT \cos kL}$$

$$= \left(\frac{F}{kT}\right) \underbrace{\left[\frac{\sin k[L-x]}{\cos kL}\right]} \underbrace{e^{j\omega t}}$$

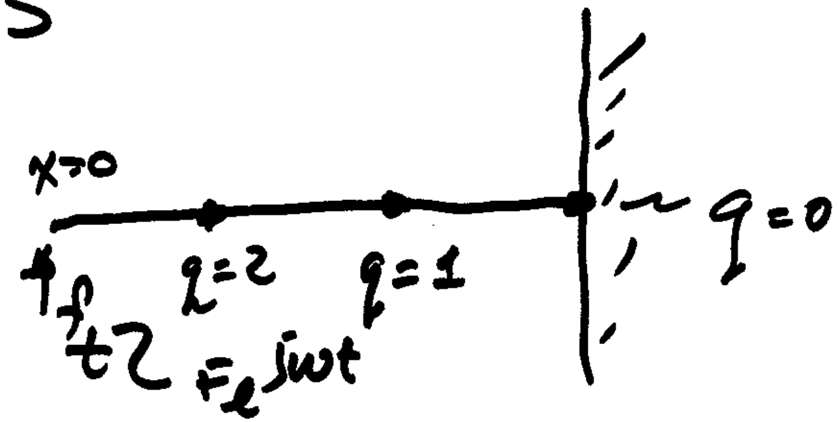
standing wave representation
(separation of time & space dependence)

Standing wave = superposition of two
propagating waves
(interfere)



node displacement always = 0

(iv) locations of the nodes



$$\sin k(L - x_q) = 0$$

$$k(L - x_q) = q\pi \qquad q = 0, 1, 2 \ldots\ldots$$

$$c = f\lambda$$

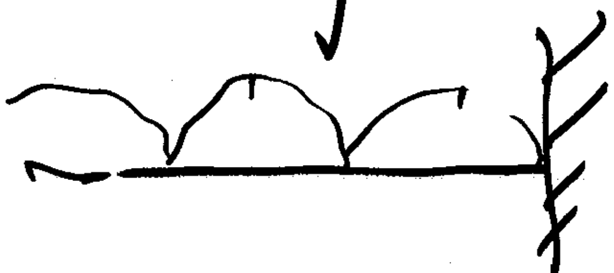
$$x_q = L - q\frac{\pi}{k} \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$x_q = L - q\left(\frac{\lambda}{2}\right)$$

nodal points are at
L − integer number of $(\lambda/2)$'s

As the frequency increases, the nodes move to the right since λ is decreasing

(v) Antinodes



The points of maximum displacement half way between the nodes.

(vi) $$y(x,t) = \underbrace{\left(\frac{F}{KT\cos kL}\right)} \sin k(L-x)\, e^{i\omega t}$$

$$k = \frac{\omega}{c}$$

response is largest at the frequency at which $\cos kL \to 0$

resonance

$\cos kL = 0$

$$k_n L = \left(\frac{2n-1}{2}\right)\pi \qquad n = 1, 2, 3, \ldots$$

$$k_n = \frac{\omega_n}{c} = \frac{2\pi f_n}{c} \qquad f_n = \frac{2n-1}{4}\left(\frac{c}{L}\right)$$

natural freqs

$n = 1, 2, 3 \ldots$

(vii) Input Impedance

$$z_{mo} = \frac{\text{input force}}{\text{velocity at the drive point}} = \frac{Fe^{j\omega t}}{(j\omega)\,\frac{Fe^{j\omega t}}{kT}\,\frac{\sin k[L - \cancel{x}^{\,0}]}{\cos kL}}$$

$\frac{\omega}{c}$ $\quad T = \rho_L c^2$

$$\boxed{z_{mo} = -j\,\rho_L c \cot kL}$$

purely imaginary