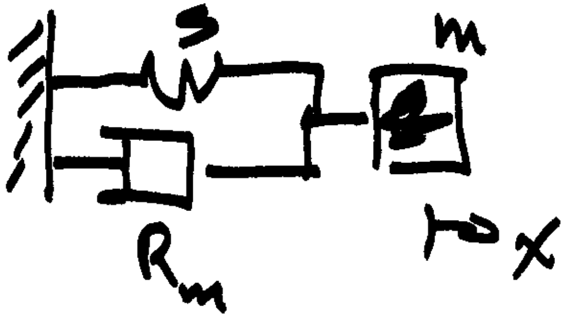
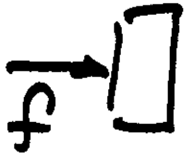
- governing equations
- wave equation
- possible solutions
- boundary solutions



Restoring force



$$f = -s x - R_m \frac{dx}{dt}$$

$$f = m \frac{d^2x}{dt^2} \qquad \omega_0 = \sqrt{\frac{s}{m}}$$

$$\frac{d^2x}{dt^2} + \left(\frac{R_m}{m}\right)\frac{dx}{dt} + \omega_0^2 x = 0$$

$$\gamma = -\frac{\left(\frac{R_m}{m}\right) \pm \sqrt{\left(\frac{R_m}{m}\right)^2 - 4\omega_0^2}}{2}$$

$$\underline{\hat{x} = \hat{A} e^{\gamma t}}$$

$$\beta = \left(\frac{R_m}{2m}\right)$$

$$\gamma = -\beta \pm \sqrt{\underline{\beta^2 - \omega_0^2}}$$

Usually - underdamped

$$\beta < \omega_0$$

$$\gamma = -\beta \pm j\sqrt{\underbrace{\omega_0^2 - \beta^2}_{}} \text{ positive}$$

$$\omega_0^2 - \beta^2 = \omega_d^2$$

damped natural frequency.

$\gamma = -\beta \pm j\omega_d$ $\tilde{A}e^{\gamma t}$

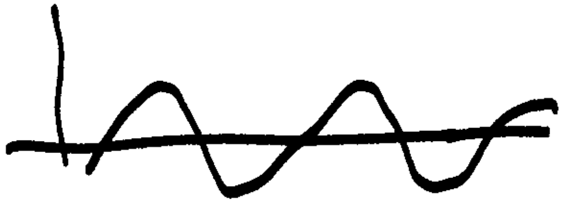
Complete solution

$$\tilde{x} = \tilde{A}_1 \underset{\checkmark}{e^{-\beta t}} e^{+j\omega_d t} + \tilde{A}_2 \underset{\checkmark}{e^{-\beta t}} e^{-j\omega_d t}$$

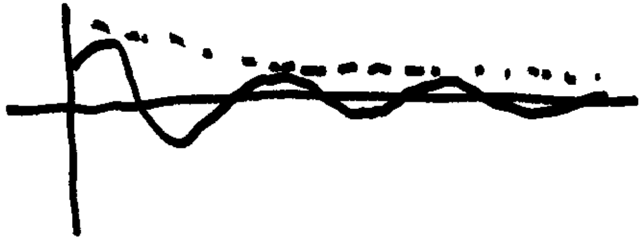
$\beta = \frac{R_m}{2m}$

$$= \underbrace{e^{-\beta t}}_{\text{exponential decay}} \Big\{ \underbrace{\hat{A}_1 e^{+j\omega_d t} + \hat{A}_2 e^{-j\omega_d t}}_{\text{oscillatory term}} \Big]$$

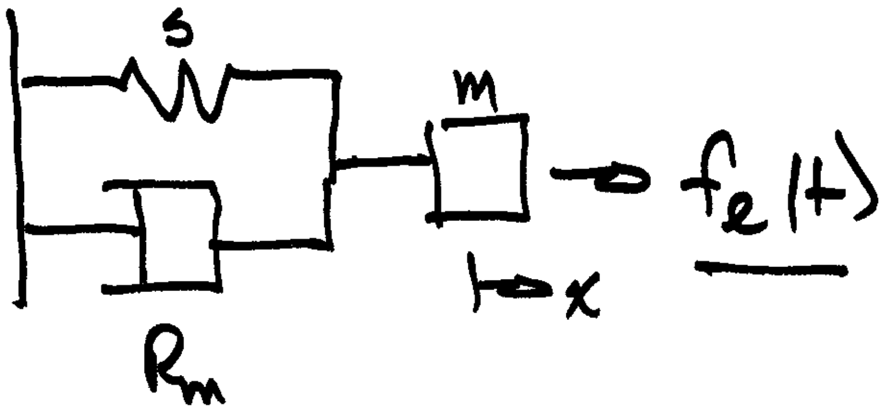
×



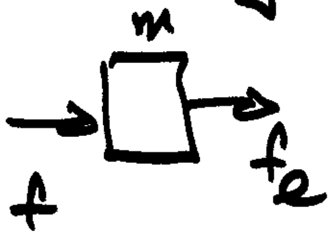
$\text{Re}\{\tilde{x}(t)\}$



1.3 Forced Oscillation



1.3.1 Governing Equations



$$f = -sx - R_m \frac{dx}{dt} \qquad (1)$$

EOM

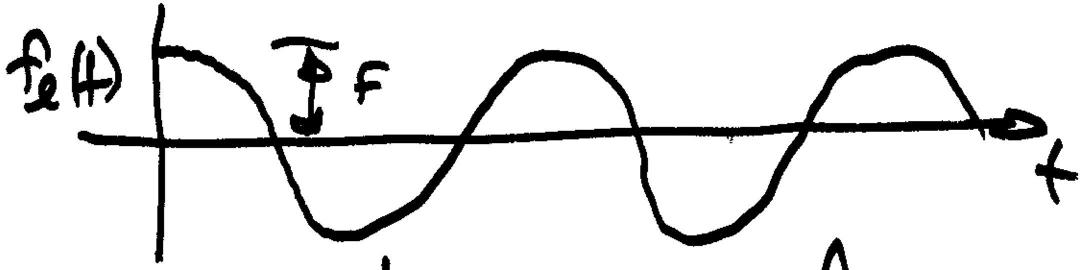
$$f + f_e = ma = m \frac{d^2x}{dt^2} \qquad (2)$$

Combine (1) + (2)

$$\boxed{m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + sx = f_e(t)} \qquad (3)$$

inhomogeneous ODE

__Example__ $f_e(t) = F\cos\omega t \quad t \geq 0$



Laplace Transform

soln of the homogenous eqn

$$x = x_{\text{transient}} + x_{\text{steady-state}}$$

since $R_m > 0$ always

as t increases transient sol'n decays
and becomes negligible

$e^{-\beta t} \to 0$ when $\beta t \gg 0$

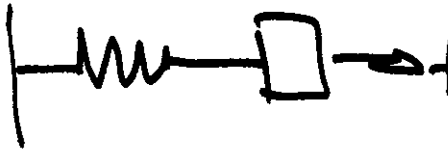
1.3.2 Steady State Solutions

Assume complex form for the driving force

$$\hat{f}_e = Fe^{j\underset{\substack{\uparrow \\ \text{driving} \\ \text{frequency}}}{\omega} t}$$

$$\text{Re}\{\hat{f}_e(t)\} = F\cos\omega t$$

if F is real.



$\hat{f}_e(t) = Fe^{j\omega t}$

Linear Systems

- in steady state respond at the driving frequency

Assumed solution

$$\underbrace{\tilde{f}_e = F e^{j\omega t} \qquad \hat{x} = \hat{A}\, e^{j\omega t}}_{\text{substitute into (3)}}$$

$$-\omega^2 m \hat{A} \cancel{e^{j\omega t}} + j\omega R_m \hat{A} \cancel{e^{j\omega t}} + s \hat{A} \cancel{e^{j\omega t}} = F \cancel{e^{j\omega t}}$$

$$\hat{A} = \frac{F}{-\omega^2 m + j\omega R_m + s}$$

$$= \frac{F}{j\omega \left(R_m + i\left(\omega m - \frac{s}{\omega}\right)\right)}$$

$$\tilde{x} = \tilde{A} e^{j\omega t}$$

$$= \frac{F e^{j\omega t}}{j\omega\left(R_m + j\left(\omega m - \frac{s}{\omega}\right)\right)}$$

Steady-state response

Physical Displacement (steady-state)

$$x(t) = \mathrm{Re}\{\tilde{x}(t)\}$$

Velocity: $\hat{u} = \dfrac{d\hat{x}}{dt} = j\omega\hat{x}$

ω
$\uparrow$
driving
freq

ω_0
$\uparrow$
natural
freq

Acceleration: $\hat{a} = \dfrac{d^2\hat{x}}{dt^2} = \dfrac{d\hat{u}}{dt} = -\omega^2\hat{x} = j\omega\hat{u}$

$$\hat{u} = \frac{F e^{j\omega t}}{R_m + j\left(\omega m - \frac{s}{m}\right)}$$

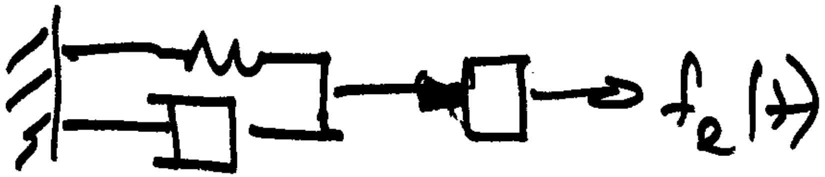
1.3.3 Mechanical Impedance

Define:

$$\hat{z}_m = \frac{\text{complex driving force}}{\text{complex velocity at the driven point}}$$

$$= \frac{\hat{f}_e}{\tilde{u}}$$

Damped SDOF

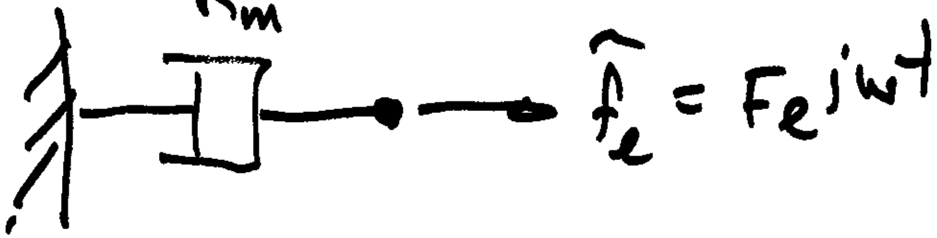


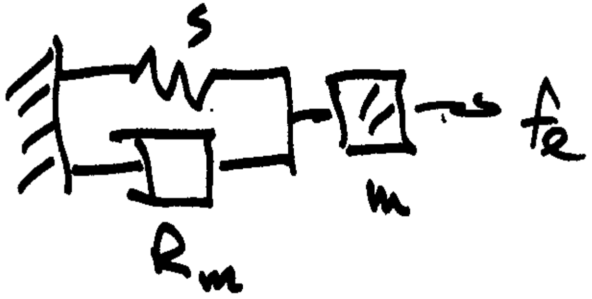
$$\hat{z}_m = \frac{Fe^{j\omega t}}{\dfrac{Fe^{j\omega t}}{R_m + i(\omega m - \frac{s}{\omega})}}$$

$$= \underbrace{R_m}_{\substack{\text{mechanical}\\ \text{resistance}}} + \underbrace{i\left(\omega m - \frac{s}{\omega}\right)}_{\substack{\text{mechanical}\\ \cancel{\text{reactiv}}\\ \text{reactance}}}$$

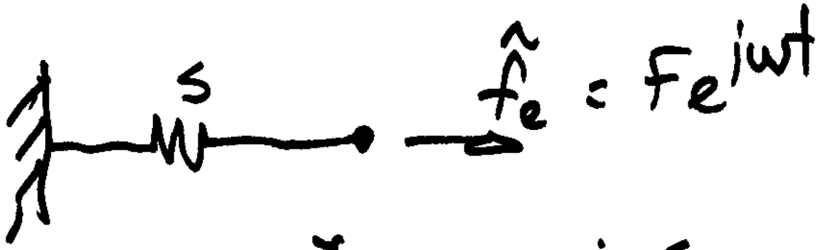
Impedance is a property of the system

$$\hat{z}_m = R + jX \qquad \begin{aligned} R &= Re\{\tilde{z}_m\} \\ X &= Im\{\tilde{z}_m\} \end{aligned}$$

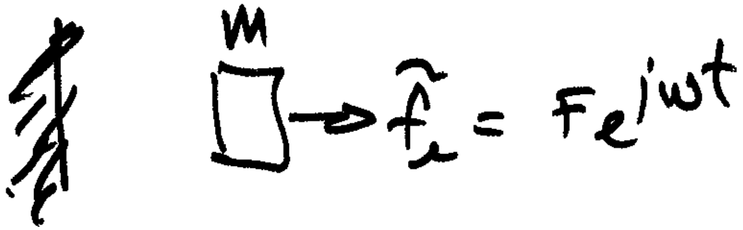




$\tilde{z}_m = R_m$ pure resistance



$\tilde{z}_m = -j\frac{s}{\omega}$ pure stiffness



$\tilde{z}_m = j\omega m$ pure mass