ME 513
Fall 2017
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Homework 5 Solutions

57 Points

Problem 7.1.4 (15 points)

Given

Simple sound source in air radiates an acoustic power of 10 mW at 400 Hz

Find

At 0.5 m from the source:

a) Intensity
b) Pressure amplitude
c) Particle speed amplitude
d) Particle displacement amplitude
e) Condensation amplitude

Solution

For air at 20 ℃: $c = 343\ \mathrm{m/s}$ $\qquad \rho_0 c = 415\ \mathrm{Pa \cdot s/m}$ $\qquad \gamma = 1.402$ [Appendix A10c]

$$\lambda = \frac{c}{f} = \frac{343\ \mathrm{m/s}}{400\ \mathrm{s^{-1}}} = 0.858\ \mathrm{m}$$

Equation 7.2.16: $\Pi = \frac{1}{2}\pi\rho_0 c\left(\frac{Q}{\lambda}\right)^2$

$$Q = \lambda\sqrt{\frac{2\Pi}{\pi\rho_0 c}} = 0.858\ \mathrm{m}\sqrt{\frac{2(10\cdot 10^{-3}\ \mathrm{W})}{\pi(415\ \mathrm{Pa\cdot s/m})}} = 3.36\cdot 10^{-3}\ \mathrm{m^3}/s$$

a) Equation 7.2.15: $I = \frac{1}{8}\rho_0 c\left(\frac{Q}{\lambda r}\right)^2 = \frac{1}{8}(415\ \mathrm{Pa\cdot s/m})\left[\frac{3.36\cdot 10^{-3}\mathrm{m^3/s}}{0.858\ \mathrm{m}\ (0.5\ \mathrm{m})}\right]^2 = \boxed{3.18\cdot 10^{-3}\ \frac{\mathrm{W}}{\mathrm{m^2}}}$

b) Equation 7.2.14: $P = \frac{1}{2}\rho_0 c\left(\frac{Q}{\lambda r}\right) = \frac{1}{2}(415\ \mathrm{Pa\cdot s/m})\left[\frac{3.36\cdot 10^{-3}\mathrm{m^3/s}}{0.858\ \mathrm{m}\ (0.5\ \mathrm{m})}\right] = \boxed{1.63\ \mathrm{Pa}}$

c) Equation 7.2.13: $\tilde{p}(r,t) = \frac{1}{2}j\rho_0 c\left(\frac{Q}{\lambda r}\right)e^{j(\omega t - kr)}$

$$\tilde{u}_r = -\frac{1}{j\omega\rho_0}\frac{\partial\tilde{p}}{\partial r} = -\frac{1}{j\omega\rho_0}\left(\frac{1}{2}j\rho_0 c\frac{Q}{\lambda}\right)e^{j\omega t}\frac{\partial}{\partial r}\left(\frac{1}{r}e^{-jkr}\right)$$

$$= -\left(\frac{1}{2}\frac{Q}{k\lambda}\right)e^{j\omega t}\left(-\frac{1}{r^2}e^{-jkr} - \frac{jk}{r}e^{-jkr}\right) = \frac{Q}{4\pi}\left(\frac{1}{r^2} + \frac{jk}{r}\right)e^{j(\omega t - kr)}$$

$$= \frac{Q}{4\pi r^2}\left(1 + \frac{j2\pi r}{\lambda}\right)e^{j(\omega t - kr)}$$

$$|\tilde{u}_r| = \frac{Q}{4\pi r^2}\sqrt{1 + \left(\frac{2\pi r}{\lambda}\right)^2} = \frac{3.36\cdot 10^{-3}\mathrm{m^3/s}}{4\pi(0.5\ \mathrm{m})^2}\sqrt{1 + \left[\frac{2\pi(0.5\ \mathrm{m})}{0.858\ \mathrm{m}}\right]^2} = \boxed{4.06\cdot 10^{-3}\ \mathrm{m/s}}$$

d) $\tilde{\xi}_r = \frac{1}{j\omega}\tilde{u}_r = \frac{1}{j\omega}\left[\frac{Q}{4\pi r^2}\left(1 + \frac{j2\pi r}{\lambda}\right)e^{j(\omega t - kr)}\right] = \frac{Q}{4\pi r^2}\left(\frac{1}{j\omega} + \frac{r}{c}\right)e^{j(\omega t - kr)}$

$$= \frac{Q}{4\pi r^2}\left(\frac{1}{j2\pi f} + \frac{r}{c}\right)e^{j(\omega t - kr)}$$

$$\left|\tilde{\xi}_r\right| = \frac{Q}{4\pi r^2}\sqrt{\left(\frac{r}{c}\right)^2 + \left(\frac{1}{2\pi f}\right)^2} = \frac{3.36\cdot 10^{-3}\ \mathrm{m}^3/s}{4\pi(0.5\ \mathrm{m})^2}\sqrt{\left(\frac{0.5\ \mathrm{m}}{343\ \mathrm{m/s}}\right)^2 + \left[\frac{1}{2\pi(400\ \mathrm{s}^{-1})}\right]^2}$$

$$\boxed{\left|\tilde{\xi}_r\right| = 1.62\cdot 10^{-6}\ \mathrm{m}}$$

e) Equation 5.2.6: $p = \mathfrak{B}s$

Equation 5.2.7: $\mathfrak{B} = \gamma\mathcal{P}_0$

$\mathcal{P}_0 = 1.013\cdot 10^5\ \mathrm{Pa}$

$$|s| = \frac{P}{\mathfrak{B}} = \frac{P}{\gamma\mathcal{P}_0} = \frac{1.63\ \mathrm{Pa}}{1.402(1.013\cdot 10^5\ \mathrm{Pa})} = \boxed{1.14\cdot 10^{-5}}$$

Problem 7.1.5C (10 points)

Given

Pulsating sphere source

Find

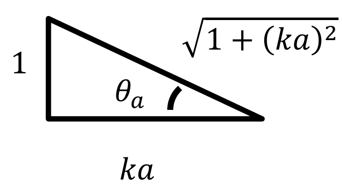
a) Show that the specific acoustic impedance at the surface can be approximated $\tilde{z}(a) \approx \rho_0 cka(j + ka)$ for the case $ka \ll 1$

b) Plot resistance, impedance, and magnitude of impedance as function of kr. Compare with expression from Equation 7.1.2 and find range of ka for which the approximation in a) has errors less than 10%

Solution

a) Equations 7.1.2 and 7.5.17: $\tilde{Z}_r = \rho_0 cS \cos\theta_a \, e^{j\theta_a}$ where $\cot\theta_a = ka$

$= \rho_0 cS \cos\theta_a (\cos\theta_a + j\sin\theta_a) = \rho_0 cS(\cos^2\theta_a + j\sin\theta_a\cos\theta_a)$



$$= \rho_0 cS\left[\left(\frac{ka}{\sqrt{1+(ka)^2}}\right)^2 + j\left(\frac{1}{\sqrt{1+(ka)^2}}\right)\left(\frac{ka}{\sqrt{1+(ka)^2}}\right)\right] = \rho_0 cS\frac{1}{1+(ka)^2}[(ka)^2 + jka]$$

If $ka \ll 1$, then $1 + (ka)^2 \approx 1$

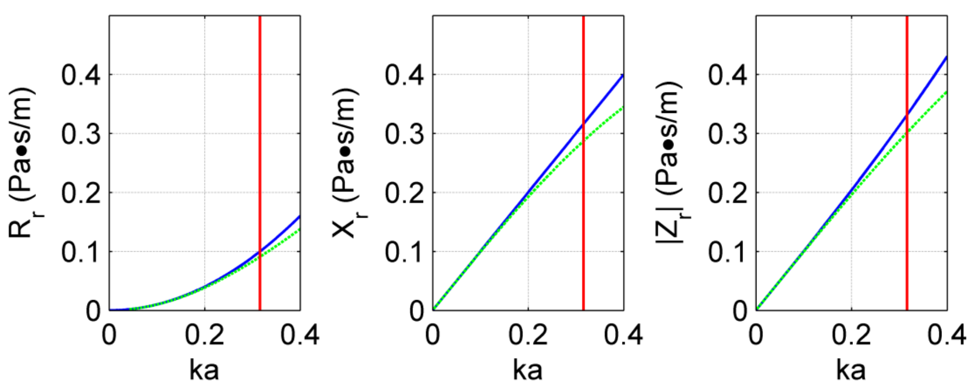
$= \rho_0 cS[(ka)^2 + jka]$

$\therefore$ $\boxed{\tilde{Z}_r = \rho_0 cSka(j + ka)}$

b) Errors are less than 10% for $ka \leq 0.316$

Approx Actual

$$\text{Error} = \frac{\text{actual} - \text{approx}}{\text{actual}}$$



Problem 7.4.1 (12 points)

Given

Baffled piston of radius a driven at angular frequency ω

Find

a) Smallest angle θ_1 for which $p = 0$ in far-field
b) Greatest finite distance for which $p = 0$ on acoustic axis (i.e. $p(r, 0) = 0$)
c) Discuss the possibility of obtaining $\theta_1 \ll 1$ and $r_1/a \ll 1$ simultaneously

Solution

a) Equation 7.4.17 (far-field equation): $p(r,\theta,t) = \frac{j}{2}\rho_0 c U_0 \frac{a}{r} ka \left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right] e^{j(\omega t - kr)}$

Drive directional factor to zero: $J_1(x) = 0$ where $x = ka\sin\theta_1$

First zero of J_1: $j_{11} = 3.83$ [Appendix A5c]

$j_{11} = x_1 = ka\sin\theta_1 = 3.83$

$$\boxed{\theta_1 = \sin^{-1}\left(\frac{3.83}{ka}\right)}$$

b) Equation 7.4.4: $p(r,0,t) = \rho_0 c U_0\left\{1 - \exp\left[-jk\left(\sqrt{r^2+a^2} - r\right)\right]\right\} e^{j(\omega t - kr)}$

(axial response equation)

Drive $\left\{1 - \exp\left[-jk\left(\sqrt{r^2+a^2} - r\right)\right]\right\} = 0$

$\exp\left[-jk\left(\sqrt{r^2+a^2} - r\right)\right] = 1$

$k\left(\sqrt{r^2+a^2} - r\right) = 2\pi$

$\sqrt{r^2+a^2} - r = \frac{2\pi}{k}$

$r^2 + a^2 = \left(\frac{2\pi}{k} + r\right)^2 = \frac{4\pi^2}{k^2} + \frac{4\pi}{k}r + r^2$

$a^2 = \frac{4\pi^2}{k^2} + \frac{4\pi}{k}r$

$r = \frac{k}{4\pi}\left(a^2 - \frac{4\pi^2}{k^2}\right) = \frac{ka^2}{4\pi} - \frac{\pi}{k}$

$$\boxed{r_1 = \frac{ka^2}{4\pi} - \frac{\pi}{k}}$$

c) $\theta_1 = \sin^{-1}\left(\frac{3.83}{ka}\right) \ll 1 \quad \rightarrow \quad \frac{3.83}{ka} \ll 1 \quad \rightarrow \quad a \gg \frac{3.83}{k}$

$\frac{r_1}{a} = \frac{ka}{4\pi} - \frac{\pi}{ka} \ll 1$

If $a = \frac{3.83}{k}$, then $\frac{r_1}{a} = \frac{3.83}{4\pi} - \frac{\pi}{3.83} = -0.515$

If $a = 3\left(\frac{3.83}{k}\right) = \frac{11.49}{k}$, then $\frac{r_1}{a} = \frac{11.49}{4\pi} - \frac{\pi}{11.49} = 0.641$

a cannot get too much bigger than $\frac{3.83}{k}$ before $\frac{r_1}{a} \ll 1$ is impossible

$\therefore$ $\boxed{\text{We cannot have } \theta_1 \ll 1 \text{ and } \frac{r_1}{a} \ll 1 \text{ at the same time}}$

Problem 7.4.2 (10 points)

Given

Piston of radius a mounted on one side of an infinite baffle
Radiates into air
$\lambda = \pi a$

Find

a) Compute and plot the relative axial intensities produced by the piston over the range $0 \le r \le 3a$
   THIS WORDING IS UNCLEAR. INTERPRETATION: PLOT PRESSURE AMPLITUDE.
b) Range of distances over which the divergence is approximately spherical

Solution

$k = \frac{2\pi}{\lambda} = \frac{2\pi}{\pi a} = \frac{2}{a}$

Equation 7.4.5: $P(r,0) = 2\rho_0 c U_0 \left| \sin\left\{\frac{1}{2} kr \left[\sqrt{1+\left(\frac{a}{r}\right)^2} - 1\right]\right\}\right|$

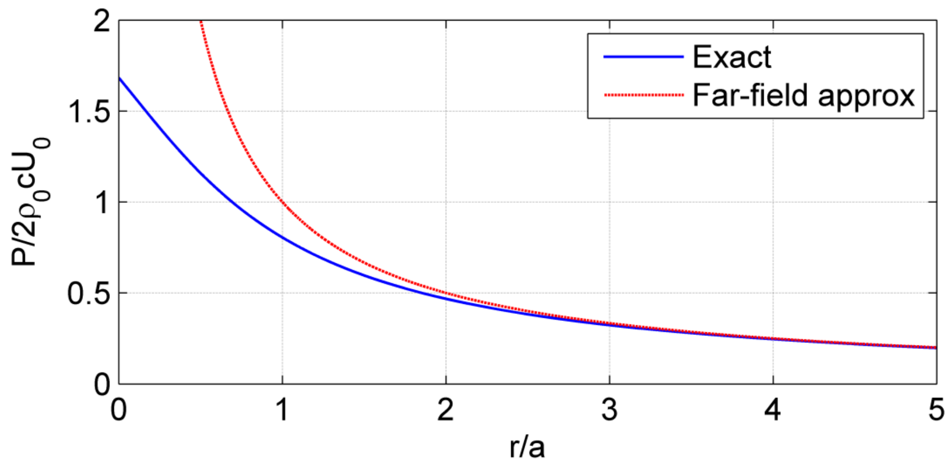
$= 2\rho_0 c U_0 \left| \sin\left\{\frac{1}{2}\left(\frac{2}{a}\right) r \left[\sqrt{1+\left(\frac{a}{r}\right)^2} - 1\right]\right\}\right| = 2\rho_0 c U_0 \left| \sin\left\{\left(\frac{r}{a}\right) \left[\sqrt{1+\left(\frac{r}{a}\right)^{-2}} - 1\right]\right\}\right|$

Equation 7.4.7: $P_{ax}(r) = \frac{1}{2}\rho_0 c U_0 \left(\frac{a}{r}\right) ka = \frac{1}{2}\rho_0 c U_0 \left(\frac{a}{r}\right)\left(\frac{2}{a}\right) a = \rho_0 c U_0 \left(\frac{r}{a}\right)^{-1}$

(for the far-field: $\frac{r}{a} > \frac{ka}{2} = \left(\frac{2}{a}\right)\left(\frac{a}{2}\right) = 1 \quad \rightarrow \quad \frac{r}{a} > 1$)

Divergence is roughly spherical

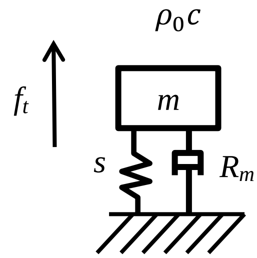
a)



b) Far-field condition for equation 7.4.7: as calculated above,

$\boxed{\frac{r}{a} > 1 \ \rightarrow \text{divergence is roughly spherical}}$

Problem 7.5.1 (10 points)



Given

Piston transducer radiating into fluid
Assume $ka \gg 2$

Find

a) ω_0
b) Sketch $\Pi(\omega)$ if the transducer is driven with a force of constant amplitude. Indicate mass-controlled and stiffness-controlled regions. (Assume ω_0 occurs well above the lower limit of the approximations implicit in $ka \gg 2$)

Solution

a) Equation 7.5.11: $\tilde{z}_r = \rho_0 c S[R_1(2ka) + jX_1(2ka)]$

See Figure 7.5.2: for $ka \gg 2$, then $R_1(2ka) \to 1$, $X_1(2ka) \to 0$

Thus, $\tilde{z}_r = \rho_0 c S$

$\tilde{z}_m = R_m + j\left(\omega m - \frac{s}{\omega}\right)$

Equation 7.5.3: $\tilde{u}_0 = \frac{\tilde{f}}{\tilde{z}_m + \tilde{z}_r} = \frac{F e^{j\omega t}}{R_m + j\left(\omega m - \frac{s}{\omega}\right) + \rho_0 c S}$

For maximum $\tilde{u}_0$, minimize denominator: $j\left(\omega m - \frac{s}{\omega}\right) = 0$

$\therefore \boxed{\omega_0 = \sqrt{\frac{s}{m}}}$

b) Equation 7.5.16: $\Pi \approx \frac{1}{2}\rho_0 c S U_0^2$ where $ka \gg 1$

$U_0 = |u_0| = \left|\frac{F e^{j\omega t}}{R_m + j\left(\omega m - \frac{s}{\omega}\right) + \rho_0 c S}\right| = \frac{F}{\sqrt{(R_m + \rho_0 c S)^2 + \left(\omega m - \frac{s}{\omega}\right)^2}}$

$\Pi \approx \frac{1}{2}\rho_0 c S U_0^2 = \frac{\rho_0 c S F^2}{2\left[(R_m + \rho_0 c S)^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right]}$