Homework 4 Solutions

65 Points

Problem 6.2.2 (10 points)

Given

A plane wave is reflected from the ocean floor at normal incidence with a level 20 dB below that of the incident wave

Find

Possible values of the specific acoustic impedance of the fluid bottom material

Solution

$$SPL_i = 20 + SPL_r$$
$$10\log_{10}\left(\frac{P_i^2}{P_{ref}^2}\right) = 20 + 10\log_{10}\left(\frac{P_r^2}{P_{ref}^2}\right)$$
$$\log_{10}\left(\frac{P_i^2}{P_{ref}^2}\right) = 2 + \log_{10}\left(\frac{P_r^2}{P_{ref}^2}\right)$$
$$\frac{P_i^2}{P_{ref}^2} = 10^{2+\log_{10}(P_r/P_{ref})} = 100\frac{P_r^2}{P_{ref}^2}$$
$$P_i^2 = 100P_r^2$$
$$P_r = \pm\frac{P_i}{10}$$
$$R = \pm 0.1$$

$\tilde{R} = \frac{r_2 - r_1}{r_2 + r_1}$ [KFCS, Equation 6.2.8]

$$\tilde{R}(r_2 + r_1) = r_2 - r_1$$
$$r_1\tilde{R} = r_2(1 - \tilde{R}) - r_1$$
$$r_2 = r_1\frac{1+\tilde{R}}{1-\tilde{R}}$$

Ocean water at 13°C: $r_1 = 1.54 \cdot 10^6\ \mathrm{Pa \cdot s/m}$ [KFCS, Table A10b]

$$\tilde{R} = 0.1\text{: } r_2 = r_1\frac{1+\tilde{R}}{1-\tilde{R}} = 1.54 \cdot 10^6\ \mathrm{Pa \cdot s/m}\left(\frac{1+0.1}{1-0.1}\right) = 1.88 \cdot 10^6\ \mathrm{Pa \cdot s/m}$$
$$\tilde{R} = -0.1\text{: } r_2 = r_1\frac{1+\tilde{R}}{1-\tilde{R}} = 1.54 \cdot 10^6\ \mathrm{Pa \cdot s/m}\left(\frac{1-0.1}{1+0.1}\right) = 1.26 \cdot 10^6\ \mathrm{Pa \cdot s/m}$$

$$\boxed{\begin{aligned} \tilde{R} = 0.1\text{:} \quad & r_2 = 1.88 \cdot 10^6\ \mathrm{Pa \cdot s/m} \\ \tilde{R} = -0.1\text{:} \quad & r_2 = 1.26 \cdot 10^6\ \mathrm{Pa \cdot s/m} \end{aligned}}$$

Problem 6.2.3, parts a) and b) (10 points)

Given
Plane wave is normally incident on the air/ocean water interface

Find
a) T, T_I if the initial wave is in the water
b) T, T_I if the initial wave is in the air

Solution
$T = \frac{2r_2}{r_2+r_1}$ [KFCS, Equation 6.2.9]

$T_I = \left(\frac{r_1}{r_2}\right)\left|\tilde{T}\right|^2 = \left(\frac{r_1}{r_2}\right)\left(\frac{2r_2}{r_2+r_1}\right)^2 = \frac{4r_1r_2}{(r_2+r_1)^2}$ [KFCS, Equation 6.2.11]

Air at 20°C: $r_{air} = 415\ \ \mathrm{Pa \cdot s/m}$ [KFCS, Table A.10c]
Ocean water at 13°C: $r_{water} = 1.54 \cdot 10^6\ \ \mathrm{Pa \cdot s/m}$ [KFCS, Table A.10b]

a) $r_1 = r_{water}\ ,\ r_2 = r_{air}$

$$T = \frac{2r_2}{r_2+r_1} = \frac{2r_{air}}{r_{air}+r_{water}} = \frac{2(415)}{415+1.54\cdot10^6} = \boxed{5.39 \cdot 10^{-4}}$$

$$T_I = \frac{4r_1r_2}{(r_2+r_1)^2} = \frac{4r_{water}r_{air}}{(r_{air}+r_{water})^2} = \frac{4(1.54\cdot10^6)(415)}{(415+1.54\cdot10^6)^2} = \boxed{1.08 \cdot 10^{-3}}$$

b) $r_1 = r_{air}\ ,\ r_2 = r_{water}$

$$T = \frac{2r_2}{r_2+r_1} = \frac{2r_{water}}{r_{water}+r_{air}} = \frac{2(1.54\cdot10^6)}{1.54\cdot10^6+415} = \boxed{2.00}$$

$$T_I = \frac{4r_1r_2}{(r_2+r_1)^2} = \frac{4r_{air}r_{water}}{(r_{water}+r_{air})^2} = \frac{4(415)(1.54\cdot10^6)}{(1.54\cdot10^6+415)^2} = \boxed{1.08 \cdot 10^{-3}}$$

Problem 6.2.6C (10 points)

Given

Plane wave normally incident on a fluid-fluid boundary

Find

a) R, T, R_I, T_I for $0 < r_1/r_2 < 10$
b) Comment on the results for $r_1/r_2 = 0$, $r_1/r_2 = 1$, and $r_1/r_2 \to \infty$

Solution

See Problem 6.2.3 for derivation

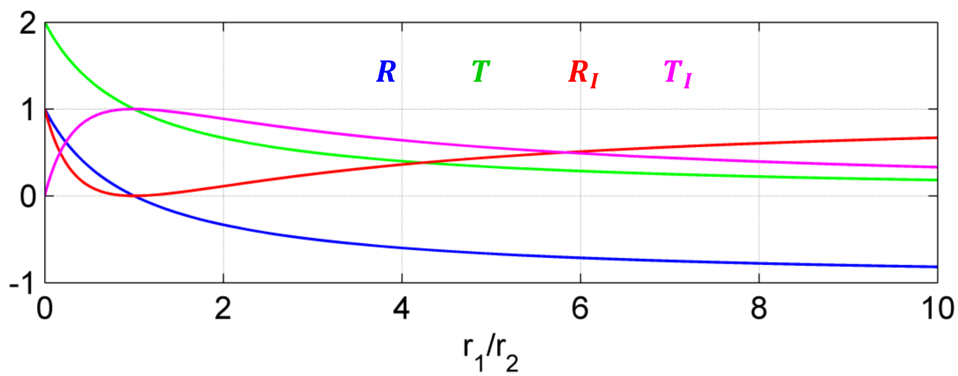
Equation 6.2.8: $\tilde{R} = \frac{r_2 - r_1}{r_2 + r_1}\left(\frac{r_2^{-1}}{r_2^{-1}}\right) = \frac{1 - r_1/r_2}{1 + r_1/r_2}$

Plug into Equation 6.1.4: $R_I = \left|\tilde{R}\right|^2 = \left(\frac{1 - r_1/r_2}{1 + r_1/r_2}\right)^2$

Equation 6.2.9: $\tilde{T} = \frac{2r_2}{r_2 + r_1}\left(\frac{r_2^{-1}}{r_2^{-1}}\right) = \frac{2}{1 + r_1/r_2}$

Plug into Equation 6.1.3: $T_I = \left(\frac{r_1}{r_2}\right)\left|\tilde{T}\right|^2 = \left(\frac{r_1}{r_2}\right)\left(\frac{2}{1 + r_1/r_2}\right)^2 = \frac{4r_1/r_2}{(1 + r_1/r_2)^2}$

a)



b) Special cases

i. $r_1/r_2 = 0$

$R = \frac{1-0}{1+0} = \boxed{1}$ $\quad R_I = \left|\tilde{R}\right|^2 = 1^2 = \boxed{1}$

$T = \frac{2}{1+0} = \boxed{2}$ $\quad T_I = \frac{4(0)}{(1+0)^2} = \boxed{0}$

Rigid surface: perfect reflection and pressure doubling

ii. $r_1/r_2 = 1$

$R = \frac{1-1}{1+1} = \boxed{0}$ $\quad R_I = \left|\tilde{R}\right|^2 = 0^2 = \boxed{0}$

$T = \frac{2}{1+1} = \boxed{1}$ $\quad T_I = \frac{4(1)}{(1+1)^2} = \boxed{1}$

Perfect impedance match: zero reflection and perfect transmission

iii. $r_1/r_2 \to \infty$

$R = \frac{1-\infty}{1+\infty} = \boxed{-1}$ $\quad R_I = \left|\tilde{R}\right|^2 = |-1|^2 = \boxed{1}$

$T = \frac{2}{1+\infty} = \boxed{0}$ $\quad T_I = \frac{4\infty}{(1+\infty)^2} \approx \frac{4}{\infty} = \boxed{0}$

Pressure release boundary: out-of-phase reflection and zero transmission

Problem 6.3.4 (10 points)

Given

Your task is to maximize the transmission of sound waves from water into steel

Find

a) Optimum characteristic impedance of the material to be placed between the water and the steel
b) ρ and c of a layer (1 cm thick) so that $T = 1$ at 20 kHz

ASSUME NORMAL INCIDENCE

Solution

Characteristic impedance of sea-water at 13°C: $r_1 = 1.54 \cdot 10^6 \text{ Pa} \cdot \text{s/m}$ [KFCS, Table A10b]
Bulk characteristic impedance of steel: $r_3 = 47.0 \cdot 10^6 \text{ Pa} \cdot \text{s/m}$ [KFCS, Table A10a]

a) Equation 6.3.8: $T_I = \frac{4}{2 + (r_3/r_1 + r_1/r_3)\cos^2 k_2 L + (r_2^2/r_1 r_3 + r_1 r_3/r_2^2)\sin^2 k_2 L}$

In order for T_I to be maximum, denominator must be minimum. Hence, because we are trying to optimize for r_2, the term $(r_2^2/r_1 r_3 + r_1 r_3/r_2^2)$ must be minimum.

In generalized form: $y = \frac{x^2}{A} + Ax^{-2}$ (where $x = r_2$ and $A = r_1 r_3 =$ real positive).

We have $y =$ maximum or minimum if $\frac{dy}{dx} = 0$.

Thus, $\frac{dy}{dx} = \frac{2x}{A} - 2Ax^{-3} = 0$.

Solving, $x^4 = A^2$

Assuming x is real positive, we may simplify $x = \sqrt{A}$

Thus, $r_2 = \sqrt{r_1 r_3} = \sqrt{(1.54 \cdot 10^6 \text{ Pa} \cdot \text{s/m})(47.0 \cdot 10^6 \text{ Pa} \cdot \text{s/m})} = 8.51 \cdot 10^6 \text{ Pa} \cdot \text{s/m}$

$$\boxed{r_2 = \sqrt{r_1 r_3} = 8.51 \cdot 10^6 \text{ Pa} \cdot \text{s/m}}$$

b) The T_I formula in part a) has several special cases. The fourth special case is given in Equation 6.3.16: $T_I = \frac{4 r_1 r_3}{(r_2 + r_1 r_3/r_2)^2}$ if $k_2 L \approx \left(n - \frac{1}{2}\right)\pi$

This special case reduces to 1 if $r_2 = \sqrt{r_1 r_3}$, which we already established in part a).

$$k_2 = \frac{\omega}{c_2} \approx \left(n - \frac{1}{2}\right)\frac{\pi}{L}$$

$$c_2 \approx \frac{\omega L}{\left(n - \frac{1}{2}\right)\pi} = \frac{2\pi f L}{\left(n - \frac{1}{2}\right)\pi} = \frac{2fL}{n - \frac{1}{2}} = \frac{4fL}{2n-1} = \frac{4(2 \cdot 10^4 \text{ s}^{-1})(0.01 \text{ m})}{2n-1} = \frac{800}{2n-1} \text{ m/s}$$

$$\rho_2 = \frac{r_2}{c_2} = \frac{\sqrt{r_1 r_3}}{4fL}(2n-1) = \frac{(8.51 \cdot 10^6 \text{ Pa} \cdot \text{s/m})}{4(2 \cdot 10^4 \text{ s}^{-1})(0.01 \text{ m})}(2n-1) = 1.06 \cdot 10^4 (2n-1) \text{ kg/m}^3$$

$$\boxed{\rho_2 = \frac{\sqrt{r_1 r_3}}{4fL}(2n-1) = 1.06 \cdot 10^4 (2n-1) \text{ kg/m}^3}$$

Problem 6.6.3C (10 points)

Given
Sound wave obliquely incident on a normally-reacting solid

Find
Plot magnitude and phase of R as a function of θ for:

a) $r_n/r_1 = 2$, $x_n/r_1 = 0$
b) $r_n/r_1 = x_n/r_1 = 2$
c) $r_n/r_1 = x_n/r_1 = 4$

Comment on the conditions for minimum R

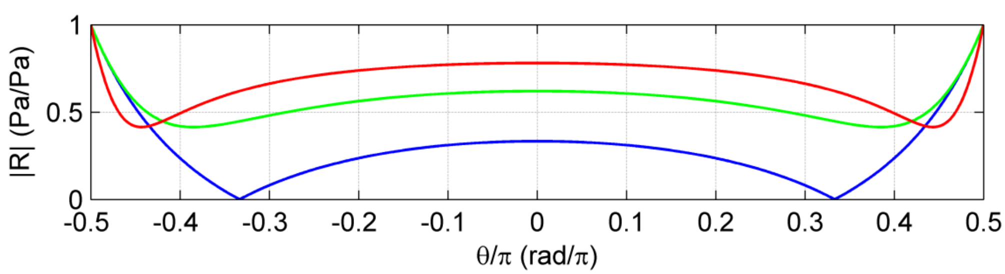
Solution
The expression for oblique-incidence R for a normally-reacting solid is given in Equation 6.6.5:

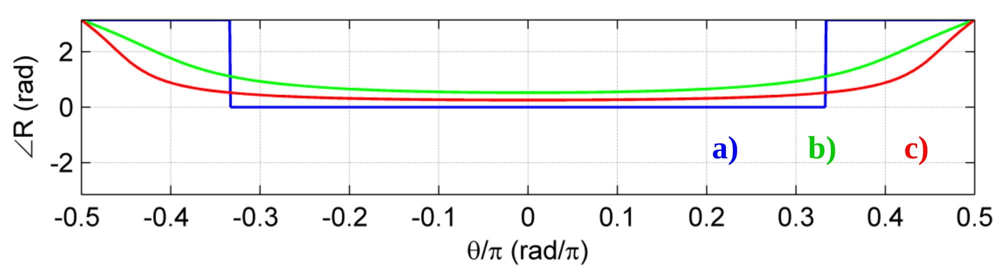
$$\tilde{R} = \frac{(r_n - r_1/\cos\theta_i) + jx_n}{(r_n + r_1/\cos\theta_i) + jx_n} \times \left(\frac{r_1^{-1}}{r_1^{-1}}\right) = \frac{(r_n/r_1 - 1/\cos\theta_i) + j\,x_n/r_1}{(r_n\,/\,r_1 + 1/\cos\theta_i) + j\,x_n/r_1}$$

a) $r_n/r_1 = 2$, $x_n/r_1 = 0$: $\tilde{R} = \frac{2-1/\cos\theta_i}{2+1/\cos\theta_i}$ minimum at $\cos\theta_i = \frac{1}{2}$, i.e. $\theta_i = \pm\frac{\pi}{3}$

b) $r_n/r_1 = x_n/r_1 = 2$: $\tilde{R} = \frac{(2-1/\cos\theta_i)+2j}{(2+1/\cos\theta_i)+2j}$

c) $r_n/r_1 = x_n/r_1 = 4$: $\tilde{R} = \frac{(4-1/\cos\theta_i)+4j}{(4+1/\cos\theta_i)+4j}$



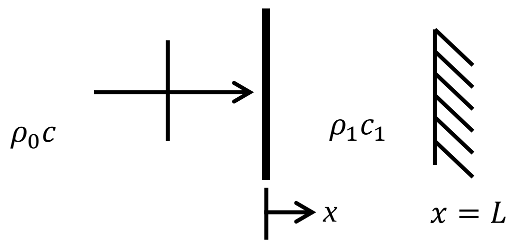


Minimum $\tilde{R}$ when $\frac{r_1}{\cos\theta} = z_n$

Problem 6 (15 points)

Given

Fluid layer over a perfectly hard backing



Find

a) Expression for surface normal impedance (at normal incidence), i.e. z_{n1} at $x = 0$
b) Sketch of surface normal impedance
c) Plane wave pressure reflection coefficient; show that in this case $|R| = 1$

Solution

$$\tilde{p}_0(x) = Ae^{-jk_0x} + Be^{jk_0x}$$
$$\tilde{p}_1(x) = Ce^{-jk_1x} + De^{jk_1x}$$

$$\tilde{u}_{0x}(x) = -\frac{1}{j\omega\rho_0}\frac{\partial\tilde{p}_0}{\partial x} = -\frac{1}{j\omega\rho_0}\left(-jk_0Ae^{-jk_0x} + jk_0Be^{jk_0x}\right) = \frac{1}{\rho_0 c}\left(Ae^{-jk_0x} - Be^{jk_0x}\right)$$
$$\tilde{u}_{1x}(x) = -\frac{1}{j\omega\rho_1}\frac{\partial\tilde{p}_1}{\partial x} = -\frac{1}{j\omega\rho_1}\left(-jk_1Ce^{-jk_1x} + jk_1De^{jk_1x}\right) = \frac{1}{\rho_1 c_1}\left(Ce^{-jk_1x} - De^{jk_1x}\right)$$

a) Velocity at perfectly hard backing: $\tilde{u}_{1x}(L) = 0$

$$\tilde{u}_{1x}(L) = \frac{1}{\rho_1 c_1}\left(Ce^{-jk_1L} - De^{jk_1L}\right) = 0$$
$$D = Ce^{-2jk_1L}$$

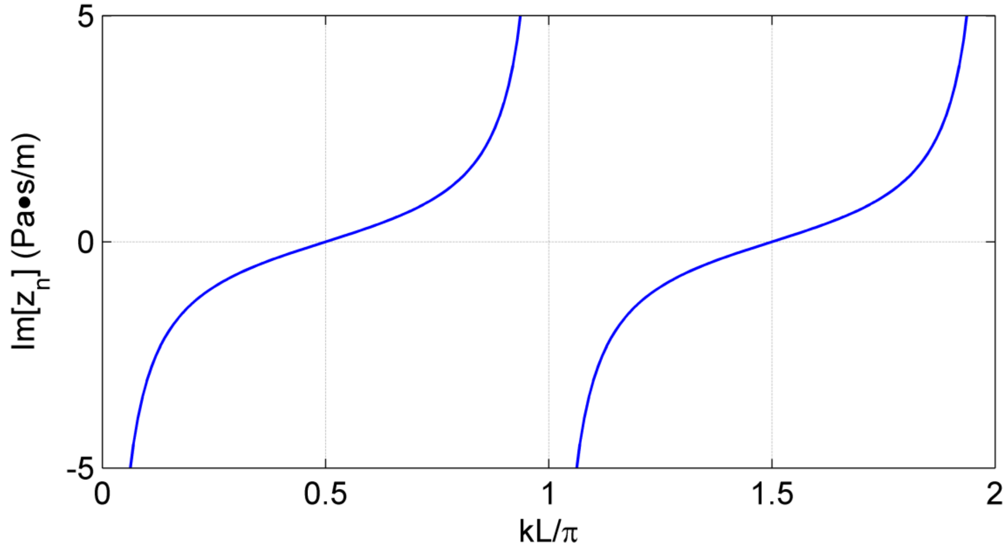
Thus, $\tilde{p}_1(x) = Ce^{-jk_1x} + \left(Ce^{-2jk_1L}\right)e^{jk_1x} = Ce^{-jk_1L}\left(e^{jk_1L}e^{-jk_1x} + e^{-jk_1L}e^{jk_1x}\right)$

$$= Ce^{-jk_1L}\left[e^{jk_1(L-x)} + e^{-jk_1(L-x)}\right] = 2Ce^{-jk_1L}\cos[k_1(L-x)]$$

$$\tilde{u}_{1x}(x) = \frac{1}{\rho_1 c_1}\left[Ce^{-jk_1x} - \left(Ce^{-2jk_1L}\right)e^{jk_1x}\right] = \frac{Ce^{-jk_1L}}{\rho_1 c_1}\left[e^{jk_1L}e^{-jk_1x} - e^{-jk_1L}e^{jk_1x}\right]$$
$$= \frac{Ce^{-jk_1L}}{\rho_1 c_1}\left[e^{jk_1(L-x)} - e^{-jk_1(L-x)}\right] = \frac{2jCe^{-jk_1L}}{\rho_1 c_1}\sin[k_1(L-x)]$$
$$z_{n1} = \left.\frac{\tilde{p}_1(x)}{\tilde{u}_{1x}(x)}\right|_{x=0} = \rho_1 c_1 \left.\frac{2Ce^{-jk_1L}\cos[k_1(L-x)]}{2jCe^{-jk_1L}\sin[k_1(L-x)]}\right|_{x=0} = -j\rho_1 c_1 \cot[k_1(L-x)]_{x=0}$$

$$\boxed{z_{n1} = -j\rho_1 c_1 \cot(k_1 L)}$$

b) Plotted with $\rho_1 c_1 = 1$



c) Pressure continuity at fluid interface: $\tilde{p}_0(0) = \tilde{p}_1(0)$

Velocity continuity at fluid interface: $\tilde{u}_{0x}(0) = \tilde{u}_{1x}(0)$

Thus, $\frac{\tilde{p}_0(0)}{\tilde{u}_{0x}(0)} = \frac{\tilde{p}_1(0)}{\tilde{u}_{1x}(0)} = z_{n1}$

Substitute in $\tilde{p}_0(0)$ and $\tilde{u}_{0x}(0)$: $\rho_0 c \frac{A+B}{A-B} = z_{n1}$

Multiply left side by $\left(\frac{A^{-1}}{A^{-1}}\right)$: $\rho_0 c \frac{1+R}{1-R} = z_{n1}$

$\rho_0 c\ (1 + R) = z_{n1}(1 - R)$

$R(\rho_0 c + z_{n1}) = z_{n1} - \rho_0 c$

$R = \frac{z_{n1}-\rho_0 c}{z_{n1}+\rho_0 c} = \frac{-j\rho_1 c_1 \cot(kL)-\rho_0 c}{-j\rho_1 c_1 \cot(kL)+\rho_0 c} = \boxed{-\frac{\rho_0 c+j\rho_1 c_1 \cot(k_1 L)}{\rho_0 c-j\rho_1 c_1 \cot(k_1 L)}}$

Numerator and denominator are complex conjugates, so their magnitude is the same

$\therefore\ \boxed{|R| = 1}$