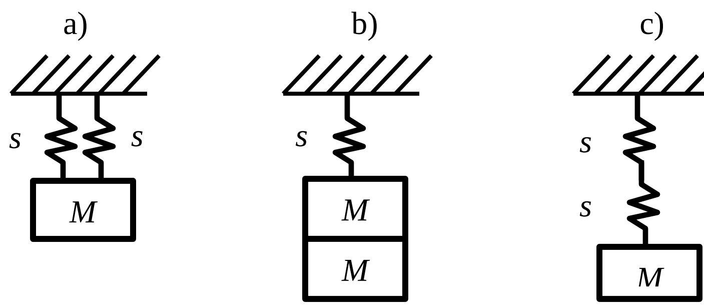
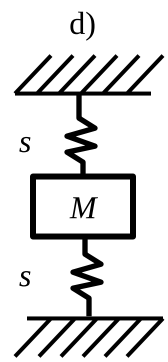
Homework 1 Solutions

Problem 1.2.1 (12 points)

Given





Find

Natural frequencies

Solution

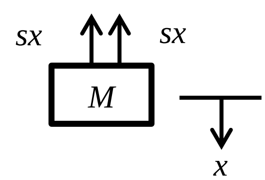
General equation of motion: $\ddot{x} + \omega_0^2 x = 0$

a) $\sum F_x = -2sx = M\ddot{x}$

$M\ddot{x} + 2sx = 0$

$\ddot{x} + \frac{2s}{M}x = 0$

$$\boxed{\omega_0 = \sqrt{\frac{2s}{M}}}$$

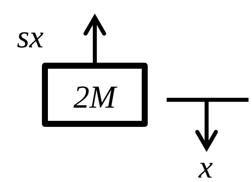


b) $\sum F_x = -sx = 2M\ddot{x}$

$2M\ddot{x} + sx = 0$

$\ddot{x} + \frac{s}{2M}x = 0$

$$\boxed{\omega_0 = \sqrt{\frac{s}{2M}}}$$



c) Two springs of coefficient s in series

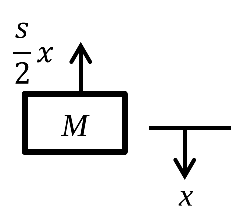
Total spring coefficient $s_{tot} = \frac{s_1 s_2}{s_1 + s_2} = \frac{s^2}{2s} = \frac{s}{2}$

$\sum F_x = -s_{tot}x = -\frac{s}{2}x = M\ddot{x}$

$M\ddot{x} + \frac{s}{2}x = 0$

$\ddot{x} + \frac{s}{2M}x = 0$

$$\boxed{\omega_0 = \sqrt{\frac{s}{2M}}}$$

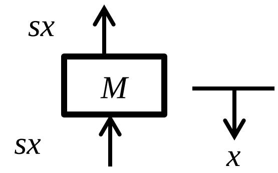


d) $\sum F_x = -2sx = M\ddot{x}$

$M\ddot{x} + 2sx = 0$

$\ddot{x} + \frac{2s}{M}x = 0$

$\boxed{\omega_0 = \sqrt{\frac{2s}{M}}}$



Problem 1.3.2 (5 points)

Given

Simple oscillator with natural frequency 5 rad/s
Displaced 0.03 m from equilibrium and released from rest

Find

a) The initial acceleration
b) The amplitude of the resulting motion
c) The maximum speed attained

Solution

$x_0 = 0.03 \text{ m}$

$u_0 = 0$

$\omega_0 = 5 \text{ rad/s}$

Equation 1.3.1: $x = x_0 \cos \omega_0 t + \frac{u_0}{\omega_0} \sin \omega_0 t = x_0 \cos \omega_0 t$

a) $a = \ddot{x} = \frac{d^2}{dt^2}(x_0 \cos \omega_0 t) = x_0 \frac{d^2}{dt^2}(\cos \omega_0 t) = -x_0 \omega_0^2 \cos \omega_0 t$

$a(0) = -x_0 \omega_0^2 \cos[\omega_0(0)] = -x_0 \omega_0^2 = -(0.03 \text{ m})(5 \text{ rad/s})^2 = -0.75 \text{ m/s}^2$

$\boxed{a(0) = -0.75 \text{ m/s}^2}$

b) Because $u_0 = 0$, therefore $\boxed{A = x_0 = 0.03 \text{ m}}$

c) $u = \dot{x} = \frac{d}{dt}(x_0 \cos \omega_0 t) = -x_0 \omega_0 \sin \omega_0 t$

$u_{max} = x_0 \omega_0 = (0.03 \text{ m})(5 \text{ rad/s}) = 0.15 \text{ m/s}$

$\boxed{u_{max} = 0.15 \text{ m/s}}$

Problem 1.5.3 (KFCS) (10 points)

Given

Two complex numbers: $\boldsymbol{A} = A\exp\{j(\omega t + \theta)\}$, $\boldsymbol{B} = B\exp\{j(\omega t + \varphi)\}$

Find

a) $\mathrm{Re}\{\boldsymbol{AB}\}$
b) $\mathrm{Re}\{\boldsymbol{A}/\boldsymbol{B}\}$
c) $\mathrm{Re}\{\boldsymbol{A}\}\mathrm{Re}\{\boldsymbol{B}\}$
d) $\angle\{\boldsymbol{AB}\}$
e) $\angle\{\boldsymbol{A}/\boldsymbol{B}\}$

Solution

$$\boldsymbol{AB} = A\exp\{j(\omega t + \theta)\} \cdot B\exp\{j(\omega t + \varphi)\} = AB\exp\{j(2\omega t + \theta + \varphi)\}$$

$$\frac{\boldsymbol{A}}{\boldsymbol{B}} = \frac{A\exp\{j(\omega t+\theta)\}}{B\exp\{j(\omega t+\varphi)\}} = \frac{A}{B}\exp\{j(\omega t + \theta) - j(\omega t + \varphi)\} = \frac{A}{B}\exp\{j(\theta - \varphi)\}$$

a) $\mathrm{Re}\{\boldsymbol{AB}\} = AB\cos(2\omega t + \theta + \varphi)$

b) $\mathrm{Re}\{\boldsymbol{A}/\boldsymbol{B}\} = \frac{A}{B}\cos(\theta - \varphi)$

c) $\mathrm{Re}\{\boldsymbol{A}\}\mathrm{Re}\{\boldsymbol{B}\} = AB\cos(\omega t + \theta)\cos(\omega t + \varphi)$

d) $\angle\{\boldsymbol{AB}\} = 2\omega t + \theta + \varphi$

e) $\angle\{\boldsymbol{A}/\boldsymbol{B}\} = \theta - \varphi$

Problem 1.6.4 (KFCS) (10 points)

Given

Damped oscillator with general solution $x = A\exp(-\beta t)\cos(\omega_d t + \varphi)$

NOTE: typo in book. The cosine term omits the t.

Starts at rest with a positive speed u_0

NOTE: confusing language. System cannot start at rest (zero velocity) and have a positive speed at the same time. I interpret the problem statement to mean that the system starts at x = 0 with a positive speed.

Find

Value of A

Solution

$$u(t) = \dot{x}(t) = \frac{d}{dt}x(t) = \frac{d}{dt}[A\exp(-\beta t)\cos(\omega_d t + \varphi)]$$

$$= A[-\beta\exp(-\beta t)\cos(\omega_d t + \varphi) - \omega_d\exp(-\beta t)\sin(\omega_d t + \varphi)$$

$$= -A\exp(-\beta t)\,[\beta\cos(\omega_d t + \varphi) + \omega_d\sin(\omega_d t + \varphi)]$$

$$x(0) = A\exp\big(-\beta(0)\big)\cos(\omega_d(0) + \varphi) = A\cos\varphi = 0 \textbf{ (1)}$$

$$u(0) = -A\exp\big(-\beta(0)\big)\,[\beta\cos(\omega_d(0) + \varphi) + \omega_d\sin(\omega_d(0) + \varphi)]$$

$$= -A[\beta\cos\varphi + \omega_d\sin\varphi] = -A\beta\cos\varphi - A\omega_d\sin\varphi = u_0 \textbf{ (2)}$$

Remember that from **(1)**, $A\cos\varphi = 0$

Substituting,$= -\beta(0) - A\omega_d\sin\varphi = -A\omega_d\sin\varphi = u_0$

Solving, $A = -\frac{u_0}{\omega_d\sin\varphi}$

Note from **(1)**: If $A\cos\varphi = 0$, then either $A = 0$ or $\varphi = \pm\frac{\pi}{2}$. Obviously, $A \neq 0$ because then u_0 could not be positive. So therefore $\varphi = \pm\frac{\pi}{2}$. This means that $A = \mp\frac{u_0}{\omega_d}$ depending on whether φ is positive or negative.

Full solution: $\boxed{A = -\frac{u_0}{\omega_d\sin\varphi} \quad \text{for} \quad \varphi = \pm\frac{\pi}{2}}$