ME 513 Homework No. 2

Solution

2.4.1.

Wave equation : $\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$

or $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$.

(a) $f_1(x-ct) = f_1(\xi)$

$LHS = \frac{\partial}{\partial x}\left(\frac{df_1}{d\xi}\frac{\partial \xi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{df_1}{d\xi}\right) = \frac{d^2 f_1}{d\xi^2}\frac{\partial \xi}{\partial x} = \frac{d^2 f_1}{d\xi^2}$.

$RHS = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{df_1}{d\xi}\frac{d\xi}{dt}\right) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(-c\frac{df_1}{d\xi}\right)$

$= \frac{1}{c^2}(-c)\frac{d^2 f_1}{d\xi^2}\frac{d\xi}{dt} = \frac{d^2 f_1}{d\xi^2}$.

$\therefore$ LHS = RHS. satisfied.

(b). $\ln[a(ct-x)]$.

$LHS = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{-a}{a(ct-x)}\right] = -\frac{1}{(x-ct)^2}$.

$RHS = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{ac}{a(ct-x)}\right] = \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{1}{t-\frac{x}{c}}\right]$

$= \frac{1}{c^2}\left[-\frac{1}{(t-\frac{x}{c})^2}\right] = -\frac{1}{(x-ct)^2}$.

$\therefore$ LHS = RHS. satisfied.

(c). $a(ct-x)^2$

$$LHS = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left[2a(x-ct)\right] = 2a.$$

$$RHS = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial}{\partial t}\left[2a(ct-x)\cdot c\right]$$

$$= \frac{1}{c^2} 2ac^2 = 2a.$$

$\therefore LHS = RHS.$ satisfied.

(d). $\cos[a(ct-x)]$.

$$LHS = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left[-\sin[a(ct-x)]\cdot(-a)\right]$$

$$= a\cos[a(ct-x)](-a)$$

$$= -a^2\cos[a(ct-x)].$$

$$RHS = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial}{\partial t}\left[-\sin[a(ct-x)]\cdot ac\right]$$

$$= \frac{1}{c^2}(-ac)\cos[a(ct-x)]\cdot ac$$

$$= -a^2\cos[a(ct-x)].$$

$\therefore LHS = RHS.$ Satisfied.

(e). $a(ct - x^2)$

$$LHS = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left[-a \cdot 2x\right] = -2a.$$

$$RHS = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial}{\partial t}\left[ac\right] = 0.$$

$LHS \neq RHS$. Wave equation not satisfied.

(f). $at(ct - x)$.

$$LHS = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left[-at\right] = 0.$$

$$RHS = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial}{\partial t}\left[2act - ax\right]$$

$$= \frac{1}{c^2} \cdot 2ac = \frac{2a}{c}.$$

$\therefore LHS \neq RHS$. wave equation not satisfied.

2.8.1.

(a). Amplitude 4 cm.

Phase speed 1.5 cm/s

Frequency $3/2\pi = 0.48$ Hz

Wavelength $\frac{1.5}{3} \cdot 2\pi = \pi$

Wavenumber $\frac{2\pi}{\pi} = 2$.

(b). $u = \frac{dy}{dt}\Big|_{x=0,\,t=0}$

$= -12\sin(3t-2x)\Big|_{x=0,\,t=0}$

$= 0$.

Since solution has general form

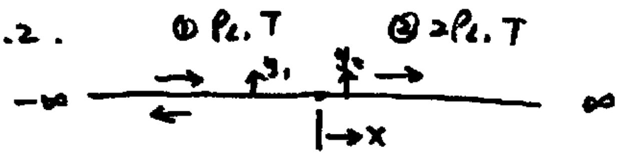
$y = A\cos(\omega t - kx)$

by identification $\omega = 3$ rad/s

$k = 2$ rad/cm

since $k = \omega/c$ $c = \omega/k = 1.5$

2.8.2.



Two wave equations:

$$\frac{\partial^2 y_1}{\partial x^2} + k_1^2 y_1 = 0.$$

$$\frac{\partial^2 y_2}{\partial x^2} + k_2^2 y_2 = 0.$$

The B.C.s are:

at $x=0$. $y_1 = y_2$ — (1)

Draw FBD for $x=0$.

we have:

$$T_1 \sin\theta_1 + T_2 \sin\theta_2 = 0$$

$$\Rightarrow -T_1 \frac{dy_1}{dx} + T_2 \frac{dy_2}{dx} = 0 \Rightarrow \frac{dy_1}{dx} = \frac{dy_2}{dx} \quad — (2).$$

Assume the solutions to the wave equations are:

$$y_1 = A e^{-jk_1 x} e^{j\omega t} + B e^{jk_1 x} e^{j\omega t}$$

$$y_2 = C e^{-jk_2 x} e^{j\omega t}.$$

where: $k_1 = \frac{\omega}{c_1} = \frac{\omega}{\sqrt{T/\rho_L}}$. $k_2 = \frac{\omega}{c_2} = \frac{\omega}{\sqrt{T/2\rho_L}} = \sqrt{2} k_1$.

Substitute into (1)

$$\Rightarrow A + B = C. \quad — (3)$$

substitute into (2):

$$\Rightarrow -Ajk_1 + Bjk_1 = -Cjk_2.$$

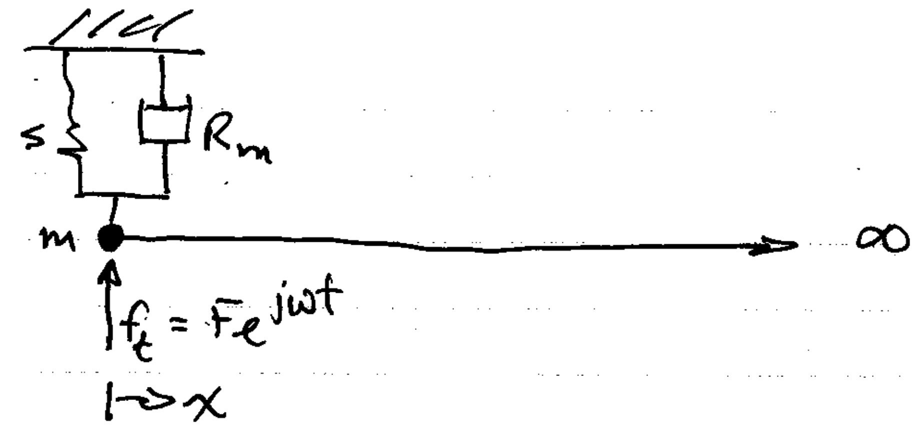
$$\Rightarrow (A-B)k_1 = \sqrt{2}\, C k_1$$

$$\Rightarrow A - B = \sqrt{2}\, C \qquad — (4).$$

Solve (3) & (4) for B, C in terms of A:

$$\Rightarrow \begin{cases} C = 2(\sqrt{2}-1)A \\ B = (2\sqrt{2}-3)A. \end{cases}$$

2.9.2

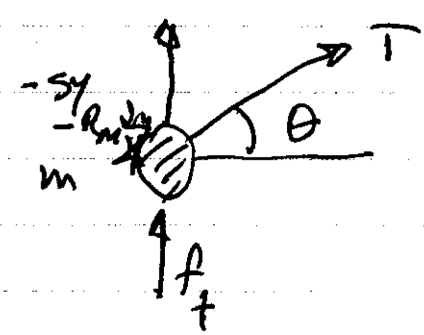


Semi-infinite string - nothing coming back from the +ve x-direction!
So, assumed solution

$$y(x,t) = Ae^{j(\omega t - kx)} \qquad k = \frac{\omega}{c} \quad c = \sqrt{\frac{T}{\rho_L}}$$

to determine A, apply b.c. at $x = 0$

FBD



$$\Sigma f_y = ma\big|_{x=0}$$

$$T\sin\theta\big|_{x=0} - sy\big|_{x=0} - R_m \frac{\partial y}{\partial t}\Big|_{x=0} + f_t = m\frac{\partial^2 y}{\partial t^2}\Big|_{x=0}$$

$$T\frac{\partial y}{\partial x}\Big|_{x=0} - sy\big|_{x=0} - R_m \frac{\partial y}{\partial t}\Big|_{x=0} + f_t = m\frac{\partial^2 y}{\partial t^2}\Big|_{x=0}$$

$$\frac{dy}{dx} = -jkA e^{j(\omega t - kx)}$$

$$\frac{dy}{dt} = j\omega A e^{j(\omega t - kx)}$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A e^{j(\omega t - kx)}$$

$$-jkTA\cancel{e^{j\omega t}} - sA\cancel{e^{j\omega t}} - j\omega R_m A\cancel{e^{j\omega t}} + F\cancel{e^{j\omega t}}$$

$$= -\omega^2 m A\cancel{e^{j\omega t}}$$

$$F = (jkT + s\cancel{A} + j\omega R_m - \omega^2 m)A$$

$$A = \frac{F}{jkT + s + j\omega R_m - \omega^2 m}$$

$$= \frac{F}{j\omega\left(\frac{k}{\omega}T - \frac{js}{\omega} + R_m + j\omega m\right)}$$

$$k = \frac{\omega}{c} \qquad c = \sqrt{T/\rho_L} \rightarrow T = c^2 \rho_L$$

$$A = \frac{F}{j\omega\left(\rho_L c + R_m + j\omega m - j\left(\frac{s}{\omega}\right)\right)}$$

$\therefore$

$$y(x,t) = A\, e^{j(\omega t - kx)}$$

Mechanical Impedance at drive point

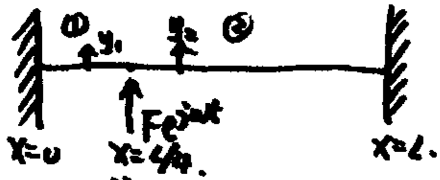
$$Z_{mo} = \left.\frac{\text{driving force}}{\text{complex velocity}}\right|_{x=0}$$

$$\frac{\partial y}{\partial t} = j\omega A\, e^{j(\omega t - kx)}$$

$$\therefore \quad Z_{mo} = \frac{F e^{j\omega t}}{\dfrac{j\omega\, F e^{j\omega t}}{j\omega\left(\rho_L c + R_m + j\omega m - j\left(\frac{s}{\omega}\right)\right)}}$$

$$= \rho_L c + R_m + j\omega m - j\left(\frac{s}{\omega}\right)$$

2.9.3.



The wave equations are:

(assume ρ_L, T in both segments).

$$\frac{\partial^2 y_1}{\partial x^2} + k^2 y_1 = 0$$

$$\frac{\partial^2 y_2}{\partial x^2} + k^2 y_2 = 0$$

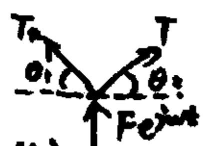
The B.C.s are:

At $x=0$ $\quad y_1(0,t)=0$. $\quad$ — (1)

At $x=l$ $\quad y_2(l,t)=0$ $\quad$ — (2)

At $x=l/4$. $\quad y_1 = y_2$ $\quad$ — (3)

Draw FBD.



$$T\sin\theta_1 + T\sin\theta_2 + Fe^{j\omega t} = 0$$

$$\Rightarrow T\left(\frac{dy_2}{dx} - \frac{dy_1}{dx}\right) + Fe^{j\omega t} = 0 \quad \text{— (4).}$$

Assume the solutions are in the form,

$$y_1(x,t) = (Ae^{-jkx} + Be^{jkx})e^{j\omega t}$$

$$y_2(x,t) = (Ce^{-jkx} + De^{jkx})e^{j\omega t}.$$

substitute into the B.C.s. we got:

(5) $A + B = 0$

(6) $Ce^{-jkl} + De^{jkl} = 0$

(7) $Ae^{-jkl/4} + Be^{jkl/4} = Ce^{-jkl/4} + De^{jkl/4}$

(8) $Ce^{-jkl/4} - De^{jkl/4} - Ae^{-jkl/4} + Be^{jkl/4} = \frac{F}{jkT}$

Express B. C. D in terms of A:

$$B = -A$$

$$D = -e^{-2jkl}C = \frac{-e^{-jk9l/4} + e^{-jk7l/4}}{e^{-jkl/4} - e^{-jk7l/4}} A$$

$$C = \frac{e^{-jkl/4} - e^{jkl/4}}{e^{-jkl/4} - e^{-jk7l/4}} A.$$

Then substitute into (2). solving for A:

$$A = \frac{F}{2jkT} \frac{e^{-jkl/4} - e^{-jk7l/4}}{e^{-jk2l} - 1}$$

$$\Rightarrow \quad B = -A$$

$$D = \frac{F}{2jkT} \frac{e^{-jk7l/4} - e^{-jk9l/4}}{e^{-jk2l} - 1}$$

$$C = \frac{F}{2jkT} \frac{e^{-jkl/4} - e^{jkl/4}}{e^{-jk2l} - 1}.$$

substitute back into $y_2(X,t)$:

$$y_2 = (Ce^{-jkx} + De^{jkx})e^{j\omega t}$$

$$= \frac{F}{2jkT}\,\frac{e^{j\omega t}}{e^{-jk2L}-1}\Big[(e^{-jkL/4} - e^{jkL/4})e^{-jkx}$$

$$+ (e^{-jk7L/4} - e^{-jk9L/4})e^{jkx}\Big].$$

$$= \frac{F}{2jkT}\,\frac{e^{-jkL/4} - e^{jkL/4}}{e^{-jk2L}-1}\,e^{j\omega t}\Big[e^{-jkx} - e^{-jk2L}e^{jkx}\Big].$$

(a). $$Z = \frac{Fe^{j\omega t}}{\text{Transverse velocity}}$$

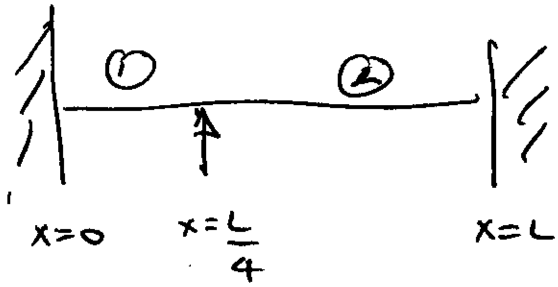
$$\frac{\partial y_2}{\partial t}\Big|_{x=L/4} = \frac{F}{2jkT}\,\frac{e^{-jkL/4} - e^{jkL/4}}{e^{-jk2L}-1}\Big[e^{-jkL/4} - e^{-jk7L/4}\Big] \cdot j\omega e^{j\omega t}$$

$$\Rightarrow Z = \frac{Fe^{j\omega t}}{\frac{\partial y_2}{\partial t}\Big|_{x=L/4}}$$

$$= \frac{2\rho_L c\,(e^{-jk2L}-1)}{(e^{-jkL/4} - e^{jkL/4})(e^{-jkL/4} - e^{-jk7L/4})}.$$

$$= -j\rho_L C\Big[\cot\frac{kL}{4} + \cot\frac{3kL}{4}\Big].$$

2.9.3 (Alternative)



① $y_1 = Ae^{-jkx} + Be^{jkx}$

at $x=0$ $y_1 = 0$

$$0 = A + B$$

$$B = -A$$

$$y_1 = A\left(e^{-jkx} - e^{+jkx}\right)$$

$$= -2jA \sin kx$$

$y_2 = Ce^{-jkx} + De^{jkx}$

② at $x = L$ $y_2 = 0$

$$0 = Ce^{-jkL} + De^{jkL}$$

$$De^{jkL} = -C^{-jkL}$$

$$D = -Ce^{-2jkL}$$

$$y_2 = Ce^{-jkx} - Ce^{-2jkL}e^{jkx}$$

$$= e^{-jkL}C\left(e^{-jkx}e^{jkL} - Ce^{-jkL}e^{jkx}\right)$$

$$= e^{-jkL}C\left(e^{jk(L-x)} - e^{-jk(L-x)}\right)$$

$$= 2je^{-jkL}C \sin k(L-x)$$

③ at $x = \frac{L}{4}$ $y_1 = y_2$

$$-\cancel{2j}A \sin k\frac{L}{4} = \cancel{2j}e^{-jkL} C \sin k(L - \frac{L}{4})$$

$$-A \sin \frac{kL}{4} = e^{-jkL} C \sin \frac{3kL}{4}$$

$$\frac{-A \sin \frac{kL}{4}}{\sin \frac{3kL}{4}} e^{+jkL} = C$$

$$C = -A \frac{\sin \frac{kL}{4}}{\sin \frac{3kL}{4}} e^{+jkL}$$

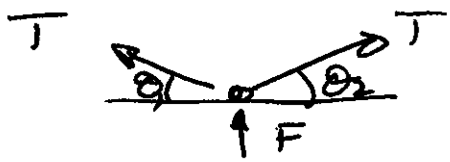
$$y_1 = -2jA \sin kx \qquad y_2 = -2j\cancel{e^{-jkL}} A \frac{\sin \frac{kL}{4}}{\sin \frac{3kL}{4}} \cancel{e^{+jkL}} \sin k(L-x)$$

$$= -2jA \frac{\sin \frac{kL}{4}}{\sin \frac{3kL}{4}} \sin k(L-x)$$

$$\frac{\partial y_1}{\partial x} = -2jkA \cos kx$$

$$\frac{\partial y_2}{\partial x} = +2jkA \frac{\sin \frac{kL}{4}}{\sin \frac{3kL}{4}} \cos k(L-x)$$

(4) at $x = \frac{L}{4}$ $\quad \Sigma f_y = 0$



$$F + T\sin\theta_1 + T\sin\theta_2 = 0$$

$$F - T\frac{\partial y_1}{\partial x} + T\frac{\partial y_2}{\partial x} = 0$$

$$F + 2jkTA\cos\frac{kL}{4} + 2jkTA\frac{\sin\frac{kL}{4}}{\sin\frac{3kL}{4}}\cos\frac{3kL}{4} = 0$$

$$F + 2jkTA\left(\cos\frac{kL}{4} + \frac{\sin\frac{kL}{4}}{\sin\frac{3kL}{4}}\cos\frac{3kL}{4}\right) = 0$$

$$2jkTA\left(\cos\frac{kL}{4} + \frac{\sin\frac{kL}{4}}{\sin\frac{3kL}{4}}\cos\frac{3kL}{4}\right) = -F$$

$$A = -\frac{F}{2jkT\left[\cos\frac{kL}{4} + \frac{\sin\frac{kL}{4}}{\sin\frac{3kL}{4}}\cos\frac{3kL}{4}\right]}$$

(5) Drive point impedance

$$z_{mo} = \frac{F}{j\omega y_1|_{x=\frac{L}{4}}}$$

$$y_1 = + \cancel{2j} \frac{F}{\cancel{2j} kT \left[\cos\frac{kL}{4} + \frac{\sin\frac{kL}{4}}{\sin\frac{3kL}{4}}\cos\frac{3kL}{4}\right]} \sin kx$$

$$\frac{\partial y_1}{\partial t} = \frac{j\omega F}{kT} \frac{1}{\cos\frac{kL}{4} + \frac{\sin\frac{kL}{4}}{\sin\frac{3kL}{4}}\cos\frac{3kL}{4}} \sin kx$$

$$z_{mo} = \frac{F}{\frac{j\omega F}{kT}\frac{1}{\cos\frac{kL}{4} + \frac{\sin\frac{kL}{4}}{\sin\frac{3kL}{4}}\cos\frac{3kL}{4}}\sin\frac{kL}{4}}$$

$$= \frac{\cancel{F}}{\frac{j\omega \cancel{F}}{kT}\frac{1}{\frac{\cos\frac{kL}{4}}{\sin\frac{kL}{4}} + \frac{\cos\frac{3kL}{4}}{\sin\frac{3kL}{4}}}}$$

$$= \frac{1}{j\frac{\omega}{kT}\frac{1}{\cot\frac{kL}{4} + \cot\frac{3kL}{4}}}$$

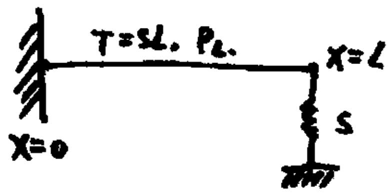
$c = \sqrt{\frac{T}{\rho_L}}$

$\rho_L c^2 = T$

$k = \frac{\omega}{c}$

$$= -j\rho_L c\left(\cot\frac{kL}{4} + \cot\frac{3kL}{4}\right)$$

2.11.1.

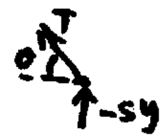


The wave equation is $\frac{\partial^2 y}{\partial x^2}+k^2 y=0$.

The B.C.s are:

at $x=0$. $y=0$ (1)

at $x=L$



$T\sin\theta - sy = 0$.

$\Rightarrow T\frac{dy}{dx}+sy=0$ (2).

Assume the solution is:

$$y(x,t)=(Ae^{-jkx}+Be^{jkx})e^{j\omega t}$$

substitute into B.C.s.

$$A+B=0 \qquad \text{—(3)}$$

$$T[-jkAe^{-jkL}+jkBe^{jkL}]+s[Ae^{-jkL}+Be^{jkL}]=0. \qquad \text{—(4)}$$

From (3) $\Rightarrow B=-A$. substitute into (4):

and rearrange, obtain:

$$\tan(kL) = -kL. \qquad \text{—(5)}$$

(5) is solved graphically, we get:

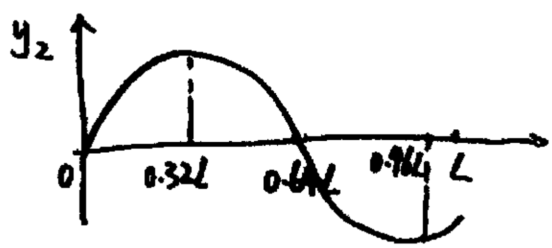
$(kL)_1 \simeq 0.65\pi$.

$(kL)_2 \simeq 1.57\pi$.

$(kL)_3 \simeq 2.54\pi$.

The fundamental is $\sin\frac{(kL)_1}{L}x = \sin\frac{0.65\pi}{L}x$.

First overtone: $\sin\frac{(kL)_2}{L}x = \sin\frac{1.57\pi}{L}x$.



2.11.1.