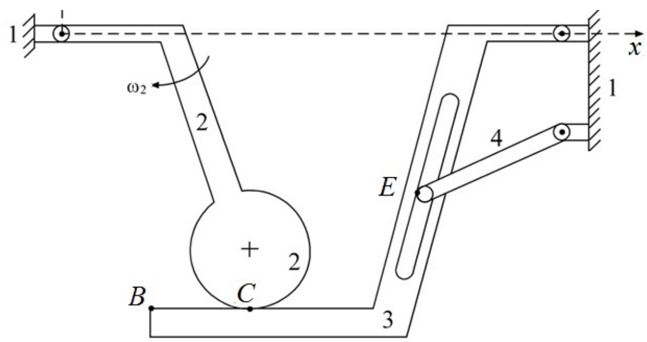
ME 459 Mechanism and Machine Theory Professor G. R. Pennock

Solution to Assignment 1 Posted on the Course Website on Saturday, September 2nd

Problem 1. The Kutzbach mobility criterion, see Eq. (1.1), page 12, can be written as

$$m = 3(n-1) - 2j_1 - 1j_2 \tag{1}$$



Case 1. If the point of contact C is a higher pair, that is, a j_2 joint, then there is rolling and slipping at point C. The number of links, the number of j_1 joints, and the number of j_2 joints, respectively, are

$$n = 4, \qquad j_1 = 3, \qquad \text{and} \qquad j_2 = 2 \tag{2}$$

Substituting Eq. (2) into Eq. (1), the mobility of the mechanism is then

$$m = 3(4-1) - 2(3) - 1(2) = 1 \tag{3}$$

Case 2. If the point of contact C is a lower pair, that is, a j_1 joint, then there is only rolling at point C. The number of links, the number of j_1 joints, and the number of j_2 joints, respectively, are

$$n = 4, \qquad j_1 = 4, \qquad \text{and} \qquad j_2 = 1 \tag{4}$$

Substituting Eq. (4) into Eq. (1), the mobility of the mechanism is then

$$m = 3(4-1) - 2(4) - 1(1) = 0 \tag{5}$$

This implies that the mechanism is a structure which does not agree with the problem statement. Therefore, Case 1 is correct; the point of contact C is a higher pair (that is, there is rolling and slipping at point C).
Aside. If the linkage is in the posture where point C lies on the x-axis then there is pure rolling at point C.
Problem 2 (Problem 1.32, page 46). The transmission angle of the four-bar linkage will be denoted here as ψ as shown in Figure 1. For the triangle O_2AO_4, the law of cosines can be written as

$$AO_4^{\,2} = O_2O_4^{\,2} + AO_2^{\,2} - 2(O_2O_4)(AO_2)\cos\theta_2 \tag{1}$$

Substituting the given dimensions into Eq. (1) gives

$$AO_4^{\,2} = (60\text{ mm})^2 + (20\text{ mm})^2 - 2(60\text{ mm})(20\text{ mm})\cos 30^\circ = 1922\text{ mm}^2 \tag{2}$$

Therefore, the distance between point A and point O_4 is

$$AO_4 = 43.8\text{ mm} \tag{3}$$

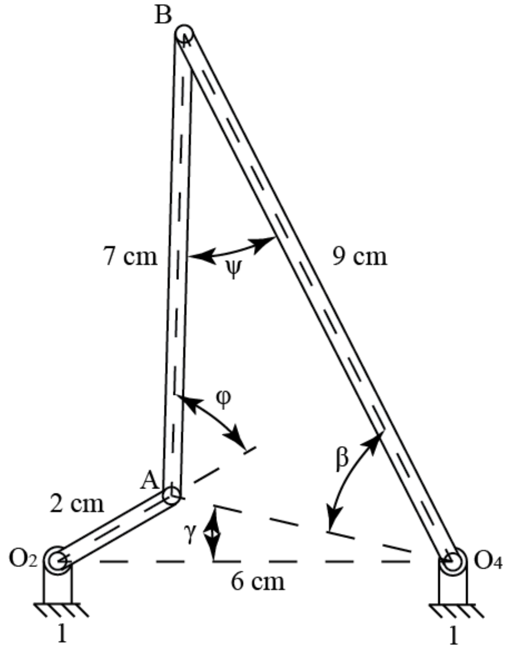


Figure 1. The transmission angle ψ of the four-bar linkage.

For the triangle ABO_4, the law of cosines can be written as

$$AO_4{}^2 = AB^2 + O_4B^2 - 2(AB)(O_4B)\cos\psi \tag{4a}$$

Substituting Eq. (3) and the given link dimensions into Eq. (4a) gives

$$1922\ \text{mm}^2 = (70\ \text{mm})^2 + (90\ \text{mm})^2 - 2(70\ \text{mm})(90\ \text{mm})\cos\psi \tag{4b}$$

Rearranging Eq. (4b), the cosine of the transmission angle of the four-bar linkage is

$$\cos\psi = 0.8792 \tag{5}$$

Therefore, the transmission angle of the four-bar linkage is

$$\psi = 28.45° \tag{6}$$

The mechanical advantage of a four-bar linkage, see Section 1.10, pages 36-39. Using the notation shown in Figure 1, the mechanical advantage of the linkage can be written from Eq. (1.8), see page 36, as

$$MA = \frac{R_4 \sin\psi}{R_2 \sin\phi} \tag{7}$$

The angle between links 2 and 3 can be written by either extending link 2 or extending link 3. In Figure 1 link 2 is extended and the angle between links 2 and 3 can be written as

$$\phi = \theta_3 - 30° \tag{8a}$$

where θ_3 is the absolute posture of the coupler link measured from the horizontal X-axis. From the triangle ABO_4, the coupler angle can be written from Figure 1 as

$$\theta_3 = 180° - \gamma - \beta - \psi \tag{8b}$$

Now consider the triangle O_2AO_4. From the law of sines, the angle γ (that is, the angle between the ground link and the line AO_4) can be written as

$$\gamma = \sin^{-1}\left(\frac{20 \text{ mm} \sin 30°}{43.8 \text{ mm}}\right) = 13.19° \tag{9}$$

Also, from the triangle ABO_4, the angle β can be written from the law of sines as

$$\beta = \sin^{-1}\left(\frac{70 \text{ mm} \sin 28.45°}{43.8 \text{ mm}}\right) = 49.53° \tag{10}$$

Substituting Eqs. (6), (9), and (10) into Eq. (8b) and the result into Eq. (8a), the angle between links 2 and 3 is

$$\phi = 180° - 13.19° - 49.53° - 28.45° - 30° = 58.83° \qquad (\text{or } \phi = 121.17°) \tag{11}$$

The angle $\phi = 180° + 58.83° = 238.83°$ is also acceptable (since the mechanical advantage of a linkage is an absolute value). Substituting Eqs. (6) and (11), and the given link lengths, into Eq. (7), the mechanical advantage of the four-bar linkage (in the given posture) is

$$MA = \frac{90 \text{ mm} \sin 28.45^o}{20 \text{ mm} \sin 58.83^o} = \frac{42.8752 \text{ mm}}{17.1127 \text{ mm}} = 2.5 \tag{12}$$

To determine the type of four-bar linkage. Grashof's law for a four-bar linkage, see Section 1.9, that is, Eq. (1.6), page 34, states that in order for the linkage to be a crank-rocker four-bar linkage then

$$s + l \le p + q \tag{13}$$

The lengths of the links of the four-bar linkage are specified as

$$s = 20 \text{ mm}, \qquad l = 90 \text{ mm}, \qquad p = 60 \text{ mm}, \qquad \text{and} \qquad q = 70 \text{ mm} \tag{14}$$

Substituting the link lengths in Eq. (14) into Eq. (13) gives

$$20 \text{ mm} + 90 \text{ mm} < 60 \text{ mm} + 70 \text{ mm} \qquad \text{or} \qquad 110 \text{ mm} < 130 \text{ mm} \tag{15}$$

Since the inequality holds then the linkage is a crank-rocker four-bar linkage. Also, since the shortest link is adjacent to the ground link then the shortest link is a crank, see Section 1.9, pages 33-36.
Problem 3 (Problem 1.34, page 46). The link lengths of the four-bar linkage are given as

$$r_2 = R_{AO_2} = 8 \text{ in}, \qquad r_3 = R_{BA} = 20 \text{ in}, \qquad r_4 = R_{BO_4} = 16 \text{ in}, \qquad \text{and} \qquad r_1 = R_{O_4O_2} = 16 \text{ in} \tag{1a}$$

Consider the first extreme posture where link 2 and link 3 are fully extended, see Figure P1.34, page 46, and also shown in Figure 1. The law of cosines for the triangle O_4O_2B can be written as

$$O_2B^2 = r_1^2 + r_4^2 - 2r_1r_4\cos O_2O_4B \tag{1b}$$

Substituting the link lengths in Eq. (1a) into Eq. (1b) gives

$$(28 \text{ in})^2 = (16 \text{ in})^2 + (16 \text{ in})^2 - 2(16 \text{ in})(16 \text{ in})\cos \text{O}_2\text{O}_4\text{B} \tag{2a}$$

Rearranging Eq. (2a) gives

$$\cos O_2O_4B = \frac{272 \text{ in}^2}{-512 \text{ in}^2} = -0.53125 \quad (2b)$$

Therefore, the angle is

$$O_2O_4B = 122.09^0 \quad (3)$$

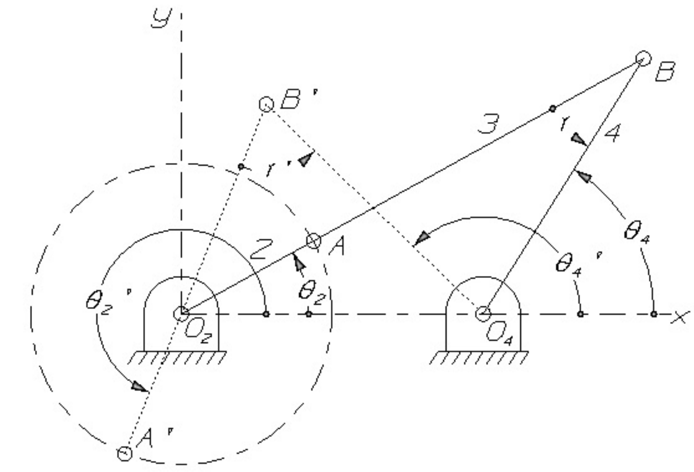


Figure 1. The two extreme postures of the four-bar linkage.

The posture of link 2 for the first extreme posture can be written from the isosceles triangle O_4O_2B as

$$\theta_2 = \cos^{-1} \frac{(r_2 + r_3)/2}{r_1} \quad (4a)$$

Substituting the link lengths given in Eq. (1a) into Eq. (4a), the posture of the crank (link 2) is

$$\theta_2 = \cos^{-1} \frac{8 \text{ in} + 20 \text{ in}}{2\,(16 \text{ in})} = 28.955° \quad (4b)$$

The posture of the coupler link in the first extreme posture is

$$\theta_3 = \theta_2 = 28.955° \quad (5)$$

The posture of the rocker (link 4) in the first extreme posture is

$$\theta_4 = 2\theta_2 = 2(28.955°) = 57.91° \quad (6)$$

Note that Eq. (6) is in agreement with Eq. (3), that is, $122.09^0 + 57.91^0 = 180^0$.

For the second extreme posture, the input link 2 and the coupler link 3 are folded on top of each other (as shown in Figure 1). The posture of the crank can be written as

$$\theta_2' = \cos^{-1} \frac{r_3 - r_2}{2r_1} \quad (7a)$$

Substituting the link lengths given in Eq. (1a) into Eq. (7a), the posture of the crank (for this second extreme posture) is

$$\theta_2' = \cos^{-1} \frac{20 \text{ in} - 8 \text{ in}}{2(16 \text{ in})} = 247.98° \quad (7b)$$

The posture of the coupler link, for the second extreme posture of the linkage, is

$$\theta_3' = \theta_2' - 180^0 = 67.98° \quad (8)$$

Note from Eq. (7b) that the posture of the coupler link, for the second extreme posture, can be written as

$$O_4O_2B' = 247.98° - 180° = 67.98° \quad (9a)$$

Therefore, the posture of the rocker for the second extreme posture is

$$\theta'_4 = 2(O_4O_2B') = 2(67.98°) = 135.96° \quad (9b)$$

The total rocking angle of the rocker can be written as

$$\Delta\theta_4 = \theta'_4 - \theta_4 \quad (10a)$$

Substituting Eqs. (6) and (9b) into Eq. (10a), the total rocking angle of the rocker is

$$\Delta\theta_4 = 135.96° - 57.91° = 78.05° \quad (10b)$$

The transmission angle for the first extreme posture can be written from the figure as

$$\gamma = 180° - 28.955° - (180° - 57.91°) \quad (11a)$$

Therefore, the transmission angle for the first extreme posture is

$$\gamma = \theta_2 = 28.955° \quad (11b)$$

The transmission angle for the second extreme posture can be written as

$$\gamma' + (\theta'_2 - 180°) + (180° - \theta'_4) = 180° \quad (12a)$$

that is

$$\gamma' + \theta'_2 - \theta'_4 = 180° \quad (12b)$$

Substituting Eqs. (7b) and (9b) into Eq. (12b), the transmission angle for the second extreme posture can be written as

$$\gamma' + 247.98° - 135.96° = 180° \quad (13a)$$

Therefore, the transmission angle for the second extreme posture is

$$\gamma' = 180° - 247.98° + 135.96° = 67.98° \quad (13b)$$

Problem 4. Grashof's law, see Section 1.9, Eq. (1.6) on page 34, states that in order for a four-bar linkage to be a crank-rocker then

$$s + l \le p + q \quad (1)$$

where the link lengths are

$$s = 3 \text{ in}, \qquad l = 16 \text{ in}, \qquad p = 14 \text{ in}, \qquad \text{and} \qquad q = 5 \text{ in} \quad (2)$$

Substituting Eqs. (2) into Eq. (1) gives

$$3 \text{ in} + 16 \text{ in} \le 14 \text{ in} + 5 \text{ in} \quad (3a)$$

that is

$$19 \text{ in} \le 19 \text{ in} \quad (3b)$$

Since the equality holds then the four-bar linkage is a Grashof linkage. Also, since the shortest link is adjacent to the ground link then the shortest link is a crank, see Grashof's law, Section 1.9, pages 33-36.

According to Eq. (1.8), see page 36, the mechanical advantage is infinite when the angle $\beta = 0^o$, or when $\beta = 180°$. (The angle β is the angle between links 2 and 3 as shown in Figure 1). First, consider $\beta = 0^o$, that is, links 2 and 3 are folded on top of each other. The law of cosines for the triangle O_4O_2B can be written as

$$BO_4{}^2 = O_2O_4{}^2 + O_2B^2 - 2 \times O_2O_4 \times O_2B \times cos(180^o - \theta_2) \tag{4}$$

where

$$BO_2 = BA - AO_2 = 16 \text{ in} - 3 \text{ in} = 13 \text{ in} \tag{5}$$

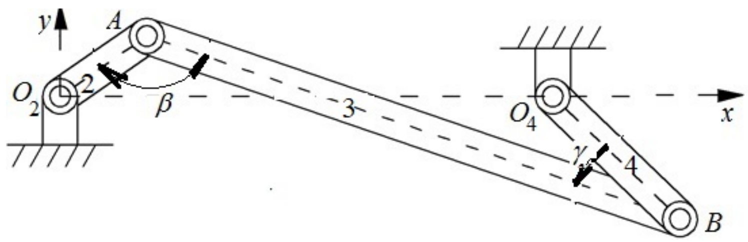


Figure 1. The angle β and the transmission angle γ.

Substituting Eq. (5) and the given link lengths into Eq. (4) gives

$$5^2 = 14^2 + 13^2 - 2 \times 14 \times 13 \times \cos(180^o - \theta_2) \tag{6a}$$

Rearranging Eq. (6a) gives

$$\cos(180^o - \theta_2) = \frac{14^2+13^2-5^2}{2\times14\times13} = 0.9341 \tag{6b}$$

Therefore, the posture of the input link is

$$\theta_2 = 180^o - 20.92^o = 159.08^o \tag{7}$$

Also from the law of cosines, the transmission angle can be written as

$$O_2O_4^2 = O_2B^2 + BO_4^2 - 2\times O_2B\times BO_4 \times \cos\gamma \tag{8a}$$

Substituting Eq. (4) and the given link lengths into Eq. (8b) gives

$$14^2 = 13^2 + 5^2 - 2 \times 13 \times 5 \times cos\,\gamma \tag{8b}$$

Then rearranging Eq. (8b) gives

$$cos\,\gamma = \frac{13^2+5^2-14^2}{2\times13\times5} = -\ 0.0154 \tag{9a}$$

Therefore, the transmission angle is

$$\gamma = 90.88^o \tag{9b}$$

Check: From the law of sines (that is, the sine rule), the transmission angle can be written as

$$\frac{\sin\gamma}{R_{O_4O_2}} = \frac{\sin\phi}{R_{BO_4}} \tag{10a}$$

Rearranging Eq. (10a) and substituting the given dimensions into the resulting equation gives

$$\sin\gamma = \frac{\sin 20.92^\circ}{5 \text{ in}}(14 \text{ in}) = 0.999\ 78 \tag{10b}$$

Therefore, it appears that the transmission angle for this extreme posture (from the sine rule) is

$$\gamma = 88.80^\circ \tag{11}$$

Note, however, that this answer for the transmission angle is WRONG (that is, the wrong quadrant). The transmission angle is greater than 90°, that is, the angle is in the second quadrant. Also, note that the transmission angle is not a right angle (this can be proved from Pythagoras theorem).
Summary. For the open configuration, the input angle and the corresponding transmission angle are

$$\theta_2 = -159.08^\circ \qquad \text{and} \qquad \gamma = 90.88^\circ \tag{12a}$$

The crossed configuration is a vertical reflection of the open configuration. Therefore, the input angle and the corresponding transmission angle would be

$$\theta_2 = 159.08^\circ \qquad \text{and} \qquad \gamma = 90.88^\circ \tag{12b}$$

Note that the transmission angle is the same for both the open and the crossed configurations.

Now consider $\beta = 180^o$, that is, links 2 and 3 are fully extended. The law of cosines for the triangle O_4O_2B can be written as

$$BO_4{}^2 = O_2O_4{}^2 + O_2B^2 - 2 \times O_2O_4 \times O_2B \times cos(180^o - \theta_2) \tag{13a}$$

where

$$BO_2 = BA + AO_2 = 16 \text{ in} + 3 \text{ in} = 19 \text{ in} \tag{13b}$$

Substituting Eq. (13b) and the given link lengths into Eq. (13a) gives

$$(5 \text{ in})^2 = (19 \text{ in})^2 + (14 \text{ in})^2 - 2(19 \text{ in})(14 \text{ in})\cos\theta_2 \tag{14a}$$

Then rearranging Eq. (14a), the input angle can be written as

$$\cos\theta_2 = \frac{(19 \text{ in})^2 + (14 \text{ in})^2 - (5 \text{ in})^2}{2(19 \text{ in})(14 \text{ in})} = 1.000\ 00 \tag{14b}$$

Therefore, the input angle is

$$\theta_2 = 0.00^\circ \tag{15a}$$

The corresponding transmission angle is

$$\gamma = 0.00^\circ \tag{15b}$$

Note that the posture with links 2 and 3 fully extended is known to be correct (that is, a singular configuration) from intuition. Since all four links of the four-bar linkage lie along the fixed X-axis then smooth, uninterrupted motion of the linkage in this posture would be unlikely without further design modifications.