ME 375 System Modeling and Analysis

Section 3 – State Variable and Input-Output Modeling Frameworks

\[
\sum a_i \frac{d^i y}{dt^i} = \sum b_j \frac{d^j u}{dt^j} \quad \{\dot{q}\} = [A]\{q\} + [B]\{u\} \\
\{y\} = [C]\{q\} + [D]\{u\}
\]

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Key points to remember

Different frameworks for expressing models

- We are only observers of the system’s behavior:
  - We only measure a portion of the system dynamics.
  - **Input-output models are unique.**
  - **State variable models are not unique.**

- **Why choose state variable models?**
  - All integrators solve first order differential equations, and state variable models contain only first order equations.
  - State variable models provide an easy way to identify all the natural frequencies, time constants, etc. of the system by solving an eigenvalue problem.

- **Which model form to choose?**
  - Time domain-MIMO-numerical solution → **state-variable**
  - Frequency domain-SISO-analytical solution → **input-output**
State variable models

Specific example

1. Select state variables

- If we know ALL the state variables, \( q_i \), initially, we should be able to evaluate ALL variables at any later time.

- What does that mean?

Vibration examples (SDOF)

E.O.M. \[ M\ddot{y}_d + B\dot{y}_d + Ky_d = f(t) \]

- We need to know \( y_d \) because it determines the POTENTIAL ENERGY

- We need to know \( \dot{y}_d \) because it determines the KINETIC ENERGY

Conclusion

The initial energy in all elements must be known because the dynamics are determined by the transfer of this energy.

\( y_d \) and \( \dot{y}_d \) are independent energy storage elements
1. Select state variables (continued)
   - \( y_d \) and \( \dot{y}_d \) are independent, so they cannot be expressed as algebraic functions of one another.

2. Solve for the derivatives of state variables

\[
\begin{align*}
\frac{dy_d}{dt} &= \dot{y}_d \\
\frac{d\dot{y}_d}{dt} &= \frac{1}{M} (-B\ddot{y}_d - Ky_d - f(t)) \\
\text{OR with } q_1 &= y_d \text{ and } q_2 = \dot{y}_d, \text{ then} \\
\frac{dq_1}{dt} &= q_2 \\
\frac{dq_2}{dt} &= \frac{1}{M} (-Bq_2 - Kq_1 - f(t))
\end{align*}
\]

Note that:
- The right hand sides of equations do not contain derivatives.
- Equations are linear in \( y \) and \( \dot{y} \).
- Equations are linearly independent (i.e., one is not a combination of the others).
We usually write linear state variable equations in matrix form:

\[
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-K/M & -B/M
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix} + 
\begin{bmatrix}
0 \\
1
\end{bmatrix} f(t)
\]

Matrix form of state variable equations:

\[
\dot{q} = Aq + Bu
\]

3. Find equations for the outputs of interest:

Write the outputs as algebraic functions of the state variables (Note that the choice of states will determine if this is feasible):

\[
\begin{align*}
y_1 &= y_d \\
y_2 &= My_d
\end{align*}
\]

Position
Momentum

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = 
\begin{bmatrix}
1 & 0 \\
0 & M
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0
\end{bmatrix} f(t)
\]

\[y = Cq + Du\]

Matrix form of output equation.
State variable models

- Example: Two DOF system (MDOF)
  - How many state variables? (Energy)
  - What should they be? (Energy)

E.O.M.

\[
\begin{align*}
M_1 \ddot{x}_1 + (C_1 + C_2) \dot{x}_1 + (K_1 + K_2)x_1 - C_2 \dot{x}_2 - K_2x_2 &= 0 \\
M_2 \ddot{x}_2 + C_2 \dot{x}_2 + K_2x_2 - C_2 \dot{x}_1 - K_2x_1 &= f_2(t)
\end{align*}
\]

With \( q_1 \equiv x_1, q_2 \equiv \dot{x}_1 \) and \( q_3 \equiv x_2, q_4 \equiv \dot{x}_2 \) then

\[
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3 \\
\dot{q}_4 
\end{bmatrix} = 
\begin{bmatrix}
0 & \frac{1}{M_1} & 0 & 0 \\
\frac{K_1 + K_2}{M_1} & -\frac{C_1 + C_2}{M_1} & \frac{K_1}{M_1} & \frac{C_2}{M_1} \\
0 & 0 & 0 & 1 \\
\frac{K_2}{M_2} & \frac{C_2}{M_2} & -\frac{K_2}{M_2} & -\frac{C_2}{M_2}
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4 
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
f_2(t)
\end{bmatrix}
\]

\( \dot{q} = Aq + Bu \)
State variable models

- Example: Two DOF system (MDOF)

Output equations for: (a) positions of two masses, (b) kinetic energy of mass 1, and (c) transmitted force between DOFs 1 and 2.

With \( y_1 \equiv x_1, y_2 \equiv \frac{1}{2} M_1 x_1^2, y_3 \equiv x_2, y_4 \equiv K_2 (x_2 - x_1) + C_2 (\dot{x}_2 - \dot{x}_1) \) then

\[
\begin{align*}
\{y_1\} &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \{q_1\} + \begin{bmatrix} 0 \end{bmatrix} \\
\{y_3\} &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \{q_2\} + \begin{bmatrix} 0 \end{bmatrix} \{f_2(t)\} \\
\{y_4\} &= \begin{bmatrix} -K_2 & -C_2 & K_2 & C_2 \end{bmatrix} \{q_3\} + \begin{bmatrix} 0 \end{bmatrix} \{f_3(t)\}
\end{align*}
\]

\[ y = Cq + Du \text{ and } y_2 = \frac{1}{2} M_1 q_2^2 \]
State variable models

- Example: Two DOF system (MDOF)
  - What happens when $M_1$ goes to zero?

E.O.M.

\[
\begin{align*}
(C_1 + C_2)\ddot{x}_1 + (K_1 + K_2)x_1 - C_2\ddot{x}_2 - K_2x_2 &= 0 \\
M_2\ddot{x}_2 + C_2\ddot{x}_2 + K_2x_2 - C_2\ddot{x}_1 - K_2x_1 &= f_2(t)
\end{align*}
\]

We lost an energy storage element

With $q_1 \equiv x_1$, and $q_2 \equiv x_2$, $q_3 \equiv \dot{x}_2$ then

\[
\begin{bmatrix} 
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3 
\end{bmatrix} = 
\begin{bmatrix} 
\frac{K_1 + K_2}{C_1 + C_2} & \frac{K_2}{C_1 + C_2} & \frac{C_2}{C_1 + C_2} \\
0 & 0 & 1 \\
\frac{K_2}{M_2} & \frac{K_1 + K_2}{C_1 + C_2} & \frac{K_2}{M_2} \\
-\frac{K_2}{M_2} & -\frac{K_1 + K_2}{C_1 + C_2} & -\frac{K_2}{M_2} \\
-\frac{C_2}{M_2} & -\frac{C_1 + C_2}{C_1 + C_2} & -\frac{C_2}{M_2} \\
\end{bmatrix}
\begin{bmatrix} 
q_1 \\
q_2 \\
q_3 
\end{bmatrix} + 
\begin{bmatrix} 
0 \\
0 \\
1
\end{bmatrix} \{f_2(t)\}
\]

\[ \dot{q} = Aq + Bu \]

For output $\dot{x}_1$, we have to use a state equation:

\[
y_1 = \dot{x}_1 = q_1 = 
\begin{bmatrix} 
-\frac{K_1 + K_2}{C_1 + C_2} & \frac{K_2}{C_1 + C_2} & \frac{C_2}{C_1 + C_2} \\
\end{bmatrix}
\begin{bmatrix} 
q_1 \\
q_2 \\
q_3 
\end{bmatrix} + [0] \{f_2(t)\} \]
State variable models

- Example: Two DOF system (MDOF)
  - What about when $C_1,C_2$ go to zero? (i.e. in the undamped approximation)

E.O.M.

\[
\begin{align*}
(K_1 + K_2)x_1 - K_2x_2 &= 0 \\
M_2\ddot{x}_2 + K_2x_2 - K_2x_1 &= f_2(t)
\end{align*}
\]

Did we lose another energy storage element?

With $q_1 \equiv x_2$, and $q_2 \equiv \dot{x}_2$, then

\[
\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} K_1 \\ -\frac{K_2}{M_2} + \frac{K_2^2}{M_2(K_1 + K_2)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M_2} \end{bmatrix} \{f_1(t)\}
\]

$\dot{q} = Aq + Bu$

What does this mean?

For output $\dot{x}_1$, we have to use an algebraic equation and the first state equation:

\[
y_1 = \dot{x}_1 = \begin{bmatrix} 0 & 0 \\ \frac{K_2}{K_1 + K_2} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \{f_2(t)\}
\]

Does this remind you of something (Volt. Div.)?

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State variable models

General form

- The general form of the state variable model contains a series of first order differential equations:
  - Possibly nonlinear and/or time varying
  - Linear, time invariant

\[
\dot{q}_{n+1} = f(q_{n+1}, u_{m+1}, t)
\]
\[
y_{p+1} = g(q_{n+1}, u_{m+1}, t)
\]

\[
q = Aq + Bu
\]
\[
y = Cq + Du
\]