CHAPTER 9

BRIDGES, STRAIN GAGES
AND SOME VARIABLE IMPEDANCE TRANSDUCERS

Many transducers translate a change in the quantity you wish to measure into a change in impedance, i.e., resistance, capacitance or inductance. For example a strain gage, changes strain ($\Delta l/l$) into a change in relative resistance ($\Delta R/R_0$).

We usually measure voltages and so would wish to change this impedance change into a voltage. There are many ways of doing this, e.g., use a known current source and measure the voltage across the impedance. Unfortunately the change in impedance is often very small compared to the total impedance of the transducer and the resulting signal will have a large DC shift. You are interested in the small deviations of the signal away from this large DC shift. In extreme cases, when measuring the total voltage with a data-acquisition system (ADC+PC) the quantization errors may be of the same order as the deviations in voltage you wish to measure.

BRIDGE CIRCUITS

A bridge circuit solves this problem by creating a voltage output that is proportional to the change in impedance rather than the absolute value of the impedance. Such a bridge circuit is shown below. Sometimes $Z_1$ is part of the transducer and we say that it is an active arm of the bridge, and the other impedances $Z_2, Z_3,$ and $Z_4$ are rest impedances. In other cases more of the impedances are active. For example, as we will see later, sets of two or four strain gages are used together in a bridge to increase sensitivity and also to compensate for temperature effects.

![Figure 1: A bridge circuit.](image-url)
The output voltage of this bridge may be calculated by using Kirchoff’s laws on circuits UPSRT and UPQRT. We will assume that no current flows to the next instrument, i.e., it is disconnected, or the next instrument has a very high input impedance. This output is, of course, the Thevenin voltage. The current flowing in RTUP is \( I \), and the currents flowing in PSR and PQR are \( I_1 \) and \( I_2 \), respectively.

\[
V_{\text{supply}} = I_2 Z_2 + I_2 Z_3 \quad (1)
\]
\[
V_{\text{supply}} = I_1 Z_1 + I_1 Z_4 \quad (2)
\]
\[
V_{\text{out}} = -I_1 Z_4 + I_2 Z_3 = +I_1 Z_1 - I_2 Z_2 \quad (3)
\]

Substituting expressions for \( I_1 \) and \( I_2 \), derived from the first two of these equations, into the third equation yields

\[
V_{\text{out}} = \left\{ \frac{Z_1}{Z_1 + Z_4} - \frac{Z_2}{Z_2 + Z_3} \right\} V_{\text{supply}} = \frac{Z_1 Z_3 - Z_2 Z_4}{(Z_1 + Z_4)(Z_2 + Z_3)} V_{\text{supply}}. \quad (4)
\]

We will use this equation many times in this chapter.

The output of this circuit is often a small voltage and so it is usually connected to some kind of amplifier. To examine what happens when the bridge is connected to the amplifier we need to generate the Thevenin equivalent circuit. The output voltage calculated above is the Thevenin voltage. We now need to calculate the Thevenin impedance or output impedance of the bridge. From this and knowledge of the input impedance, and transfer function of the amplifier we can generate an expression for the output of the amplifier.

The output impedance of this bridge is the impedance seen from the output terminal looking back into the bridge, when the voltage source is removed and replaced by its internal impedance. Let us assume that this internal voltage source impedance is zero. The output impedance that is seen is illustrated in Figure 2.

![Figure 2: Calculating the output impedance \( Z_{\text{out}} \) of a bridge. The output impedance is \( \{Z_1 \text{ in parallel with } Z_4\} \) combined in series with \( \{Z_2 \text{ in parallel with } Z_3\} \).](image-url)
Which gives

\[ Z_{\text{out}} = \frac{Z_1Z_4}{Z_1 + Z_4} + \frac{Z_2Z_3}{Z_2 + Z_3} \]. \quad (5)

Suppose that four strain gages are used in a bridge. These could be thin wires that stretch and contract as the structure, to which they are attached, moves. Let us suppose that this stretching and contracting leads to the following values for the impedances in the bridge:

\begin{align*}
Z_1 &= R_o + \Delta R, \quad Z_2 = R_o - \Delta R, \\
Z_3 &= R_o + \Delta R, \quad Z_4 = R_o - \Delta R.
\end{align*}

a. Calculate the output voltage of the bridge in terms of \( V_{\text{supply}}, R_o \) and \( \Delta R \).

b. The bridge circuit is connected to an analog to digital converter so that the output voltages could be stored on a PC. The input impedance of the amplifier is 1 K \( \Omega \), and \( R_o = 200 \Omega \).

What is the voltage drop when the bridge is connected to the amplifier, as a percentage of the output voltage when the circuit is not attached to the amplifier?

**Solution**

a. Substituting these values into equation (4), yields

\[ V_{\text{out}} = \left\{ \frac{R_o + \Delta R}{2R_o} - \frac{R_o - \Delta R}{2R_o} \right\} V_{\text{supply}}, \text{ and hence } \]

\[ V_{\text{out}} = \frac{2\Delta R}{2R_o} = \frac{\Delta R}{R_o} V_{\text{supply}}. \]

Here we can see that the output voltage is proportional to the change in impedance, \( \Delta R \).

b. The output voltage after the circuit has been connected to the amplifier is:

\[ \frac{Z_{\text{in}}}{Z_{\text{in}} + Z_{\text{out}}} \frac{\Delta R}{R_o} V_{\text{supply}}. \]

Here we have the additional loading term. Using equation (5) we have that:

\[ Z_{\text{out}} = 2 \frac{R_o^2 - \Delta R^2}{2R_o}. \]

Let's suppose that \( \Delta R \) is small compared with \( R_o \) so we can approximate \( Z_{\text{out}} \) by \( R_o \).

The loading term now becomes \( 1/[1 + R_o/Z_{\text{in}}] = 1/[1 + 200/1000] = 0.833. \)
The percent change in output voltage is \((1 - 0.833)100\% = 16.7\%\).

In strain gage bridges, \(\Delta R\) is often much smaller than \(R_o\), and the output voltage is very small, i.e., on the order of tens of millivolts. If you connect such a circuit to an analog to digital converter (ADC), make sure that the ADC input range is sufficiently small so that most of the bits are being used in the quantization process. If you cannot pre-set the ADC input range, amplify the signal to fill the fixed input range of the ADC.

**STRAIN GAGES**

There are many different types of strain gage, some are thin metallic wire and others are made of semiconductors, which are usually some kind of doped silicon. Strain gages are used extensively to measure vibrations in structures such as turbine blades, rotating shafts, and light structures made of thin panels or members. They are also used in investigations of crack initiation and propagation and the effects of fatigue in structures such as aircraft wings and fuselages.

Here we will concentrate on wire gages, though we should point out that silicon gages are commonly used in micro-transducers. Sometimes strain gages are used to directly measure strain, other times strain gages are used as part of a transducer. An example of this is a strain gage accelerometer, as illustrated in Figure 3.

![Figure 3: Strain gage accelerometer. Two gages above rest resistance \((R_o + \Delta R)\) and two gages below rest resistance \((R_o - \Delta R)\).](image)

The gages are placed in a bridge circuit as in the example at the end of the previous section. As the accelerometer experiences an acceleration the mass moves and the gages stretch or contract resulting in the change in resistance. The limitation on the measurement is the extent to which the gages can contract and the relationship between strain and resistance remain linear. The gages are pre-tensioned so that they remain in tension during contraction. This pre-tensioning will determine the maximum acceleration that can be measured. Usually the gages are set up so
that the maximum acceleration will induce a tension in the gage equal to half the initial pre-tensioning.

In many micro-transducers small beams are used to sense motions. These beams have doped silicon strain gages embedded in them. The motion induces resistance changes which are then converted to voltages by bridge circuits. You will exploit this principle in the last Laboratory of this course, using a cantilever beam plus wire gages as a means of measuring force.

**Advantages and Disadvantages of Strain Gages**

Wire strain gages have the advantage of being very light and therefore do not change the dynamics of the structure to which they are attached significantly. They are also very simple transducers and have a high frequency bandwidth. There is some skill required in pre-tensioning them and attaching them to the structure, and once attached it is not a simple task to detach and re-attach them elsewhere. If you stretch them too much, by putting them in a place where a lot of stretching and compressing occurs, they will pop off the structure. If you do not stretch them enough the signal output will be very small and embedded in the background electronic noise. A typical maximum operating range is -0.01 to +0.02 in/in. They are very sensitive to temperature (semiconductor gages are even more so) and should not be put in environments above 150° C. If you pass too much current through them they will melt. A typical maximum current is 50mA. Rest resistances are around 120 Ω and the change in resistance is from -2.4 to 4.8 Ω.

**How Strain Gages Work**

An illustration of a strain gage is shown in Figure 4. The long wires are aligned along the axis where the stretching and contracting of the surface will occur. This is called the *active* axis of the gage. The gage is relatively insensitive to stretching along the other axis, termed the *passive* axis.

![Diagram](image)

Figure 4: (a) Wire strain gage configuration. (b) Often the *wire* is a metal film etched on a resin base, which is glued to the surface of the structure.

Suppose that we have an element of wire of length, \( l \), cross sectional area, \( A \) and resistivity, \( \rho \).

Then the resistance of the wire is:

\[
R = \frac{\rho l}{A} \quad (6)
\]
Expanding this in a Taylor series, and ignoring higher order terms we obtain:

\[ R = R_0 + \Delta R \]  

(7)

and \( \Delta R = \frac{\partial R}{\partial l} \Delta l + \frac{\partial R}{\partial A} \Delta A + \frac{\partial R}{\partial \rho} \Delta \rho \)

\[ = \frac{\rho}{A} \Delta l - \frac{\rho l}{A^2} \Delta A + \frac{l}{A} \Delta \rho \]

The relative change in resistance is therefore:

\[ \frac{\Delta R}{R} = \frac{\Delta l}{l} - \frac{\Delta A}{A} + \frac{\Delta \rho}{\rho} \]  

(8)

As the wire is stretched its cross sectional area gets smaller and its length gets longer. Suppose that the wire is rectangular of dimensions x and y, so that \( A = xy \). Using a truncated Taylor series expansion of this expression and dividing by \( A \) gives,

\[ \frac{\Delta A}{A} = \frac{\Delta x}{x} + \frac{\Delta y}{y} = 2 e_T \]  

(9)

where \( e_T \) is the transverse strain. \( e_L = \Delta l/l \) is the longitudinal strain. Poisson’s ratio, \( \nu \), relates the longitudinal strain to the transverse strain. Typically \( 0.25 < \nu < 0.4 \).

\[ e_T = -\nu e_L \]  

(10)

Using equations (9) and (10) in equation (8) gives:

\[ \frac{\Delta R}{R_0} = (1 + 2\nu) e_L + \frac{\Delta \rho}{\rho} \]  

(11)

The gage factor, \( G \), of a strain gage is defined to be:

\[ G = \frac{\Delta R/R_0}{e_L} = 1 + 2\nu + \frac{1}{e_L} \frac{\Delta \rho}{\rho} \]  

(12)

For most metals this results in a gage factor of around 2, and

\[ \frac{\Delta R}{R_0} = G e_L \]  

(13)
In the accelerometer example above, the output of the bridge circuit will be:

$$V_{out} = G e_L V_{supply}$$  \hspace{1cm} (14)

**Temperature Compensation**

The strain gages are very sensitive to temperature changes. If during your experiment the temperature of the gage changes, you may end up measuring changes in resistance due to temperature effects, rather than changes due to the stretching or compression of the surface of the structure as it moves. We deal with this problem by using two gages.

Figure 5: (a) Mounting strain gages on a thin beam to increase sensitivity to strain and eliminate sensitivity to temperature.

(b) Temperature compensation with a dummy gage

In some situations it is possible that one gage could be contracting while the other is stretching by the same amount. Gages mounted on the top and under surfaces of a thin beam, as illustrated in Figure 5, is a case where this is possible. The changes in the resistances due to the motions are $-\Delta R$ and $+\Delta R$, respectively. If it is possible to arrange the gages so that this is the case, the gages are placed at the $Z_1$ and the $Z_2$ positions in the bridge. There will be an additional change in resistance due to the temperature change, $\Delta R_T$. This will be the same sign for both gages. We now have the bridge configuration:

$$Z_1 = R_o + \Delta R + \Delta R_T, \quad Z_3 = R_o$$
$$Z_2 = R_o - \Delta R + \Delta R_T, \quad Z_4 = R_o$$  \hspace{1cm} (15)

Substituting these into equation (4) yields:

$$V_{out} = \left\{ \frac{R_o + \Delta R + \Delta R_T}{2R_o + \Delta R + \Delta R_T} - \frac{R_o - \Delta R + \Delta R_T}{2R_o - \Delta R + \Delta R_T} \right\} V_{supply}$$  \hspace{1cm} (16)

Combining the fractions, simplifying and dividing the top and bottom of the fraction by $R_o^2$, yields,
\[ V_{out} = \left( \frac{1 + \frac{\Delta R}{R_o} + \frac{\Delta R_T}{R_o}}{2 + \frac{\Delta R}{R_o} + \frac{\Delta R_T}{R_o}} \right) - \left( \frac{1 - \frac{\Delta R}{R_o} + \frac{\Delta R_T}{R_o}}{2 - \frac{\Delta R}{R_o} + \frac{\Delta R_T}{R_o}} \right) \times V_{supply} \] (17)

If we simplify the denominator using \( \frac{\Delta R + \Delta R_T}{R_o} \ll 2 \), this becomes,

\[ V_{out} = \frac{1}{2} \frac{\Delta R}{R_o} \times V_{supply} = \frac{1}{2} G e_L \times V_{supply} \] (18)

This simplification has removed the dependency on temperature.

In many cases it will not be convenient to mount strain gages to get \(-\Delta R\) from a second gage while the first gage is undergoing a change of \(+\Delta R\). We can, however, still remove the dependency on temperature by using two gages. The second gage is mounted so that along its active axis it is not being stretched. We sometimes term this a *dummy* gage. However, it is close enough to the other gage to experience the same temperature changes. This is illustrated in Figure 5(b). The impedances in the bridge are now:

\[ Z_1 = R_o + \Delta R + \Delta R_T \quad , \quad Z_3 = R_o \]
\[ Z_2 = R_o + \Delta R_T \quad , \quad Z_4 = R_o \] (19)

Substituting into equation (4) and simplifying will result in the following equation.

\[ V_{out} = \frac{1}{4} \frac{\Delta R}{R_o} \times V_{supply} = \frac{1}{4} G e_L \times V_{supply} \] (20)

Note that the sensitivity of this measurement system is half of the one above (4 in the denominator rather than 2), and is one quarter of the sensitivity of a bridge with four active arms. Increasing the number of active gages (in the correct configuration) increases the sensitivity. The output of the bridge is typically very small, on the order of tens of millivolts; this increase in sensitivity is desirable.

**Voltage Supply and Current through Gages**

The voltage supply to the bridge can be a DC source (battery) or an AC source. If the source is AC the output will be an amplitude modulated signal. The voltage source needs to be of a much higher frequency that those appearing in the strain signal, and the bridge must be followed by a demodulation circuit. Amplitude modulation and demodulation are described in the next Chapter. In bridges with other forms of impedance (capacitance and inductance) the voltage source must
be AC for the transducer to work. Many transducers fall into this category and the signal conditioning will include the bridge plus a demodulation circuit.

In the following discussion, if the voltage supply is AC then the output level is the rms level of the output at a particular time. The integration in the rms calculation is performed over several periods of the voltage source, a time over which the strain signal is deemed to be constant. If the source is DC the output level is the voltage coming out of the bridge.

The level of the current flowing through the bridge must not exceed \( I_{\text{max}} \), which will be specified by the strain gage manufacturer. Your choice of voltage supply and strain gage resistance will be guided by \( I_{\text{max}} \). In order to have a more flexible bridge arrangement there is often a variable resistance, \( R_p \), in series with the bridge, as illustrated in Figure 6.

![Figure 6: Four active arm strain gage bridge with variable resistor to control current flow through the gages.](image)

For simplicity, let's consider a four active arm bridge. Strain gages next to one another measure tension and compression. The total resistance on the left arm of the bridge is equal to the resistance on the right arm of the bridge, \( 2R_o \). Hence the current divides into two at junction P. The total resistance of the bridge is \( 2R_o \) in parallel with \( 2R_o \) which is equal to \( R_o \). Current, \( I \), flowing from the voltage source is therefore:

\[
I = 2I_{\text{max}} = \frac{V_{\text{supply}}}{R_p + R_o}
\]  

**Example**

The rest resistance of gages in a four active arm bridge is 120 \( \Omega \). The maximum current through the gages is 20mA. The voltage supply is 15 \( V_{\text{rms}} \). What is the minimum resistance required of the variable resistor placed in series with the voltage source?
Solution

From equation (21),

\[
R_p = \frac{V_{\text{supply}}}{2 I_{\text{max}}} - R_o = \frac{15}{0.04} - 120 = 255 \Omega
\]

Bridge Adjustment and Calibration

Bridges often contain other variable resistors, used to remove the effects of unequal resistors. Due to poor tolerance specifications in manufacturing, the rest resistance of the gages will not all be equal. The bridge is adjusted when all the gages are at rest so that the output voltage is zero. The calibration resistor, \( R_c \), can then be used to calibrate the bridge. The position of the calibration resistor is shown in Figures 6 and 7. \( R_c \) is placed in parallel with \( R_2 \), and in series with a switch. When the switch is closed and the gages are at rest, \( Z_2 \) has a resistance of \( R_o \) in parallel with \( R_c \). Setting this equal to \( R_o - \Delta R \) we obtain,

\[
R_o - \Delta R = \frac{R_o R_c}{R_o + R_c}
\]

and hence,

\[
\Delta R = \frac{R_o^2}{R_o + R_c}
\]

This is equivalent to applying a strain of:

\[
e_L = \frac{1}{G \frac{R_o}{R_o}} \Delta R
\]

to one gage in the bridge. The output voltage should be:

\[
V_{\text{out}} = \frac{1}{4 R_o} \Delta R V_{\text{supply}}
\]

Generally we would choose \( R_c \) to produce a \( \Delta R \) typically found when the bridge is being used to take measurements.
Example
What should the calibration resistance be to simulate the output of a four active arm bridge where the strain sensed by each gage is 0.01 in/in? The gage factor is 2.0 and $R_o$ is 150 $\Omega$.

Solution
When the bridge is in calibration mode it is simulating a one active arm bridge. The output for a four active arm bridge is four times this. So we need to simulate a strain four times that sensed by each of the four active arms. Using equation (24),

$$4 \times 0.01 = \frac{1}{2.0} \frac{\Delta R}{150}$$

$$\Delta R = 150 \times 0.08 = 12 \Omega$$

Rearranging equation (23) yields:

$$R_c = \frac{R_o^2}{\Delta R} - R_o$$

and hence,

$$R_c = \frac{(150)^2}{12} - 150 = 1,725 \Omega$$

Figure 7: Bridge circuit with variable resistors to adjust bridge for zero output when gages are at rest.

OTHER VARIABLE IMPEDANCE BRIDGES
Bridges are used with other transducers that produce variations in inductance or capacitance. This often introduces a frequency dependence, though sometimes you can be clever in how you
place the capacitors or inductors in the bridge circuit and remove this frequency dependence. Let's look at a simple example to illustrate this.

**Example**

A push pull inductive device involves two electromagnetic circuits, as shown in the Figure below. The central plate is attached to the moving object. As the plate moves, the sizes of the air gaps in the two circuits change. When the object is at rest the air gaps are all d meters wide. When the object moves a distance x meters, the air gaps in the upper circuit become d-x and in the lower circuit d+x. This has an effect on the inductance of the coils. The relationship between the coils’ inductances and the distance x is:

\[
L_1 = \frac{P}{L_0 - kx}, \quad L_2 = \frac{P}{L_0 + kx},
\]

where \( L_0 \), \( P \) and \( k \) are constants determined by the electromagnetic properties of the circuit.

![Diagram of push pull inductive device](image)

Figure 8: Push-pull inductance device used to measure displacement (x).

\( L_1 \) and \( L_2 \) are placed in a bridge circuit in positions \( Z_1 \) and \( Z_4 \), respectively. \( Z_2 = Z_3 = R_0 \). Let's calculate the output voltage of the bridge. The supply voltage is an AC source, let's denote it by \( V_s \cos \omega t \).

\[
V_{\text{out}} = V_s \cos \omega t \frac{Z_1 Z_3 - Z_2 Z_4}{(Z_1 + Z_4)(Z_2 + Z_3)} = V_s \cos \omega t \frac{j\omega L_1 Z_3 - j\omega L_2}{(j\omega L_1 + j\omega L_2)(Z_2 + Z_3)}
\]

\[
= V_s \cos \omega t \frac{j\omega PR_0}{L_0 - kx} \frac{1}{L_0 - kx} \frac{j\omega P}{L_0 + kx} \frac{1}{L_0 + kx} 2R_0
\]

Notice that the \( j\omega \) terms in the numerator cancel the ones in the denominator. Multiplying top
and bottom by \((L_0 + kx)(L_0 - kx)\) gives:

\[
V_{\text{out}} = V_s \cos \omega_s t \cdot \frac{PR_0(L_0 + kx) - PR_0(L_0 - kx)}{P(L_0 + kx) + P(L_0 - kx)} \cdot \frac{k}{2R_0} = V_s \cos \omega_s t \cdot \frac{k}{2L_0} x
\]

This output is a high frequency cosine \((\omega_s)\) with an amplitude \(V_s kx / 2L_0\). This is called an amplitude modulated signal because, as \(x(t)\) changes with time, the amplitude of the cosine changes. \(V_{\text{out}}\) might look like the signal shown in Figure 9, if \(x(t) = A_0 + A_1 \sin \omega_1 t, A_0 > A_1\) and \(\omega_1 \ll \omega_s\).

Figure 9: The output of a push-pull inductance device plus bridge circuit

Usually, the push-pull device plus bridge system will be connected to a demodulation device that produces a signal proportional to \(x(t)\). Modulation and demodulation are discussed in the next chapter.